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MATHEMATICS

Distribution of Finite-Dimensional and Separable Stalks of an Ample Banach Bundle

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Abstract—The topological characteristics are studied of the set of points at which the stalks of an ample Banach bundle are finite-dimensional or separable. A connection is established between the property of the stalks of a bundle to be finite-dimensional or separable with the analogous property of the stalks of the ample hull of the bundle. A new criterion is obtained for existence of the dual bundle.

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As is known (see [1]), every Banach–Kantorovich space $\mathcal{U}$ is isomorphic to an ideal of the space $C_\infty(Q, \mathcal{X})$ of extended continuous sections of a suitable ample Banach bundle $\mathcal{X}$ over an extremally disconnected compact space Q. Furthermore, the properties of the bundle $\mathcal{X}$ or those of its single stalks reflect the analogous global or local properties of the space $\mathcal{U}$. In particular, the property of $\mathcal{U}$ to be locally finite-dimensional or order separable is closely connected with the property of the stalks of $\mathcal{X}$ to be finite-dimensional or separable.

We study the topological characteristics of the set of points at which the stalks of an ample Banach bundle are finite-dimensional or separable, examine the connection between the property of the stalks of a bundle to be finite-dimensional or separable with the analogous property of the stalks of the ample hull of the bundle, and obtain a new criterion for existence of the dual bundle in the separable case.

Throughout the paper, $\mathcal{X}$ is an arbitrary continuous Banach bundle over an extremally disconnected compact space Q, $\overline{\mathcal{X}}$ is the ample hull of $\mathcal{X}$, $\omega = \{0, 1, 2, \ldots\}$, $\overline{\omega} = \omega \cup \{\infty\}$. We use the terminology and notation adopted in [1, 2].

The dimension of $\mathcal{X}$ is the function $\dim\mathcal{X}\colon Q \to \overline{\omega}$ which maps each point $q \in Q$ into the dimension $\dim\mathcal{X}(q) \in \omega$ of the stalk $\mathcal{X}(q)$ in case the latter is finite-dimensional, and takes the value $(\dim\mathcal{X})(q) = \infty$ otherwise. The dimension of $\mathcal{X}$ is bounded (on $P \subset Q$) whenever $\dim\mathcal{X} \le n$ (on P) for some $n \in \omega$. The dimension of $\mathcal{X}$ is locally bounded on $P \subset Q$ if it is bounded on a neighborhood of each point $p \in P$. Given $F\colon Q \to \overline{\omega}$ and $d \in \overline{\omega}$, denote $\{F \le d\} := \{q \in Q\colon F(q) \le d\}$. The symbols $\{F < d\}$, $\{F = d\}$, etc., are introduced similarly. From [3, 18.1] it is clear that the set $\{\dim\mathcal{X} \ge n\}$ is open for all $n \in \omega$.

Theorem 1. *Suppose that $\mathcal{X}$ is ample.*

(1) *The set $\{\dim\mathcal{X} = n\}$ is clopen for every $n \in \omega$. In particular, $\{\dim\mathcal{X} < \infty\}$ is open and σ-closed, $\{\dim\mathcal{X} = \infty\}$ is closed and σ-open.*

(2) *The following are equivalent:* (a) *$\{\dim\mathcal{X} < \infty\}$ is clopen;* (b) *$\{\dim\mathcal{X} = \infty\}$ is clopen;* (c) *the set of values of $\dim\mathcal{X}$ is finite.*

Theorem 2. *If all stalks of $\mathcal{X}$ are finite-dimensional then the following are equivalent:*

(1) *$\mathcal{X}$ is ample;*

(2) *$\{\dim\mathcal{X} = n\}$ is open for every $n \in \omega$;*

(3) *$\{\dim\mathcal{X} = n\}$ is closed for every $n \in \omega$ and the dimension of $\mathcal{X}$ is bounded;*

(4) *there is a finite partition of Q into clopen subsets such that the dimension of $\mathcal{X}$ is constant on each element of the partition.*

Theorem 3. *All stalks of $\overline{\mathcal{X}}$ are finite-dimensional if and only if the dimension of $\mathcal{X}$ is bounded. In this case, the dimension of $\overline{\mathcal{X}}$ is also bounded and* $\max\dim\overline{\mathcal{X}} = \max\dim\mathcal{X}$.

Theorem 4. (1) *If $q \in Q$ and $\dim\mathcal{X}(q) = n < \infty$ then the equality $\overline{\mathcal{X}}(q) = \mathcal{X}(q)$ is equivalent to the containment $q \in \operatorname{int}\{\dim\mathcal{X} = n\}$.*

(2) *The equalities* $\{\dim\overline{\mathcal{X}} = n\} = \operatorname{cl}\operatorname{int}\{\dim\mathcal{X} = n\}$ $(n \in \omega)$ *and* $\{\dim\overline{\mathcal{X}} = 0\} = \operatorname{int}\{\dim\mathcal{X} = 0\}$ *hold.*

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The following assertion ensues from the above facts with [2, 3.2.9(1)] taken into account:

Corollary 1. *Suppose that all stalks of $\mathcal{X}$ are finite-dimensional.*

(1) *The stalks of $\mathcal{X}$ and $\overline{\mathcal{X}}$ coincide on a dense subset of Q.*

(2) *The set* $\{\dim\overline{\mathcal{X}} < \infty\}$ *is open, σ-closed, and dense in Q; the equality* $\{\dim\overline{\mathcal{X}} < \infty\} = Q$ *holds if and only if* $\dim\mathcal{X}$ *is bounded.*

Theorem 5. *Suppose that $\mathcal{X}$ is ample and a point $q \in Q$ is not σ-isolated. Then the stalk $\mathcal{X}(q)$ is separable if and only if it is finite-dimensional.*

Corollary 2. *If $\mathcal{X}$ is ample then the following are equivalent:*

(1) *the stalks of $\mathcal{X}$ are separable at all points which are not σ-isolated;*

(2) *the stalks of $\mathcal{X}$ are finite-dimensional at all nonisolated points;*

(3) *the set* $\{\dim\mathcal{X} = \infty\}$ *is finite;*

(4) *there is a partition of Q into clopen subsets Q_0, Q_1, …, Q_n, $n \in \omega$, such that the dimension of $\mathcal{X}$ is constant and finite on each of the sets Q_1, Q_2, …, Q_n, and Q_0 is a finite set of isolated points.*

Recall that a bundle $\mathcal{X}$ is separable whenever $C(Q, \mathcal{X})$ includes a countable subset which is stalkwise dense in $\mathcal{X}$.

Theorem 6. *If $\mathcal{X}$ is ample then the following are equivalent:*

(1) *$\mathcal{X}$ is separable;*

(2) *all stalks of $\mathcal{X}$ are separable;*

(3) *the stalks of $\mathcal{X}$ are separable at all isolated points and at all points which are not σ-isolated;*

(4) *the stalks of $\mathcal{X}$ are finite-dimensional everywhere except a finite set of isolated points at which the stalks are separable;*

(5) *there is a partition of Q into clopen subsets Q_0, Q_1, …, Q_n, $n \in \omega$, such that the dimension of $\mathcal{X}$ is constant and finite on each of the sets Q_1, Q_2, …, Q_n, and Q_0 is a finite set of isolated points at which the stalks of $\mathcal{X}$ are separable.*

Corollary 3. *The following are equivalent:*

(1) *$\overline{\mathcal{X}}$ is separable;*

(2) *the stalks of $\mathcal{X}$ are separable at each point of a finite set $S \subset Q$ and the dimension of $\mathcal{X}$ is bounded on $Q \backslash S$;*

(3) *the stalks of $\mathcal{X}$ are separable at each point of a finite set $S \subset Q$ and the dimension of $\mathcal{X}$ is locally bounded on $Q \backslash S$.*

Theorem 7. *If the dual bundle $\mathcal{X}'$ exists, a point $q \in Q$ is not σ-isolated, and the stalk $\mathcal{X}(q)$ is separable, then $\mathcal{X}(q)$ is finite-dimensional and coincides with $\overline{\mathcal{X}}(q)$.*

Theorem 8. *If the stalks of $\mathcal{X}$ are separable at all points which are not σ-isolated then the following are equivalent:*

(1) *the dual bundle $\mathcal{X}'$ exists;*

(2) *$\mathcal{X}$ is ample;*

(3) *there is a partition of Q into clopen subsets Q_0, Q_1, …, Q_n, $n \in \omega$, such that the dimension of $\mathcal{X}$ is constant and finite on each of the sets Q_1, Q_2, …, Q_n, and Q_0 is a finite set of isolated points.*

Furthermore, (1)–(3) *imply* $\mathcal{X}'(q) = \mathcal{X}(q)'$ *for* $q \in Q$.

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