On two-fold packings of radius-1 balls in Hamming graphs

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Abstract—A λ-fold r-packing in a Hamming metric space is a code C such that the radius-r balls centered in C cover each vertex of the space by not more than λ-times. The well-known r-error-correcting codes correspond to the case λ = 1. We propose asymptotic bounds for q-ary 2-fold 1-packings as q grows, find that the maximum size of a binary 2-fold 1-packing of length 9 is 96, and derive upper bounds for the size of a binary λ-fold 1-packing.

Index Terms—Hamming graph, multifold ball packings, two-fold ball packings, t-list decodable codes, completely regular codes, linear programming bound.

I. INTRODUCTION

The Hamming distance d_H(x, y) between two words x and y of the same length is the number of coordinates in which x and y differ. The Hamming graph H(n, q) (if q = 2, the n-cube H(n)) is a graph whose vertices are the words of length n over the alphabet {0, ..., q – 1}, two words being adjacent if and only if they differ in exactly one position. The weight wt(x) of a word x is the number of nonzeros in x.

We will say that a set C of vertices of H(n, q) is a λ-fold r-packing if for every vertex x of H(n, q) the number of elements of C at distance at most r from x does not exceed λ. The concept of 1-fold r-packing coincides with the well-known concept of r-error-correcting code. The sphere-packing bound for error-correcting codes is generalized to the obvious bound

\[ |C| \leq \frac{\lambda q^n}{|B_r|} \]  

on the cardinality of an λ-fold r-packing, where B_r is a radius-r ball in H(n, q).

In the literature, the λ-fold r-packings are also known as the \(<l\)-list decodable codes with radius r, where l = λ + 1, see e.g. [4].

Blinovsky [4], [5] proved that there exists a sharp bound \(\tau(\lambda + 1, q)\) such that if \(r = \tau n\), \(\tau < \tau(\lambda + 1, q)\) then the largest possible λ-fold r-packing code is exponentially large in n. He obtained in [6] some formulae for this bound. For example, \(\tau(\lambda + 1, 2) = \frac{1}{2} - \frac{2^{\lambda+1}}{2^{\lambda+1}}\), where \(k = \left\lfloor \frac{\lambda+1}{2} \right\rfloor\). How large can C be when \(\tau\) is just above the threshold \(\tau(\lambda + 1, q)\)?

In [1], [10] it is proved that the maximum possible size of 2-fold packing with radius \((\tau(3, 2) + \varepsilon)n = \left(\frac{1}{2} + \varepsilon\right)n\) is \(\Theta(\frac{1}{2} n)\) as \(\varepsilon \to 0\). For λ-fold packing with radius \((\tau(\lambda + 1, 2) + \varepsilon)n\), there is a bound \(O(\frac{1}{\sqrt{\lambda}})\) for the size of C as \(\lambda\) is odd (1).

In this paper we propose asymptotic bounds for 2-fold 1-packing in H(n, q) as q grows (Section II), constructions of the optimal 2-fold 1-packings in H(9) (Section III), and upper bounds for λ-fold 1-packing in H(n) (Section IV).

II. TWO-FOLD PACKING OF BALLS IN q-ARY HAMMING GRAPH

Let \(f_r(m, v, e)\) be the maximum number of edges in an r-uniform hypergraph on m vertices which does not contain e edges spanned by v vertices. We are interested in only r-partite hypergraph which has parts of equal cardinality q, i.e. \(m = rq\). This restriction is not important for asymptotic results which we will use below.

Define all parts of an n-partite hypergraph as copies of \(\{0, \ldots, q - 1\}\) corresponding to the values of coordinates of vertices in H(n, q). Let the edges of the hypergraph be the codewords of a code \(C\) in H(n, q). If 3 codewords belong to the same ball of radius 1, then 3 edges of the hypergraph are spanned by \(n + 3\) vertices.

It is known that \(m^{2-o(1)} \leq f_r(m, 6, 3) = o(m^2)\) [12]. In [2], this bounds was generalized: \(m^{k-o(1)} \leq f_r(m, 3(r-k) + k + 1, 3) = o(m^k)\) for \(2 \leq k < r\). Finding the value of k from the equation \(3(n-k) + k + 1 = n + 3\), we get \(k = n - 1\). Consequently, the following theorem is true.

Theorem 1: If a set \(C\) of vertices of H(n, q) is largest 2-fold packing of radius-1 balls, then \(q^{n-1-o(1)} \leq |C| \leq o(q^{n-1})\) where \(q \to \infty\).

Similar bounds for n-fold packings are rather simple.

Proposition 1: If a set \(C\) of vertices of H(n, q) is a largest n-fold packing of radius-1 balls, then

\[ q^{n-1} \leq |C| \leq \frac{q^n}{q-1+1/n}. \]

Proof: An example of n-fold 1-packing is an MDS code with code distance 2. Such code exits for any n and q and its cardinality equals \(q^{n-1}\). Since \(|B_1| = \lambda(q-1) + 1\), we conclude from (1) that \(|C|/\lambda(q-1) + 1) \leq nq^n\).

It is easy to see that for \(n = 3\) the largest 1-fold 1-packing is the repetition code of size q; the largest 3-fold 1-packing is an MDS code code of size \(q^2\). For an arbitrary n, a largest λ-fold packing have the cardinality about \(q^{n-1}\) as \(\lambda = 2, \ldots, n\).

III. UNITRADES AND EQUITABLE PARTITIONS

A. Unitrades and equitable partitions

The halved n-cube \(\frac{1}{2} H(n)\) is a graph whose vertices are the even-weight (or odd-weight) binary n-words, two words being adjacent if and only if they differ in exactly two positions.
A set $T$ of vertices of $H(n, q)$ such that $|B \cap T| \in \{0, 2\}$ for every radius-1 ball $B$ in $H(n)$ is called a 1-perfect unitrade. A set $T$ of vertices of $\frac{1}{2}H(n)$ such that $|B \cap T| \in \{0, 2\}$ for every maximum clique $B$ in $\frac{1}{2}H(n)$ is called an extended 1-perfect unitrade.

Given an extended 1-perfect unitrade in $\frac{1}{2}H(n+1)$, removing the last coordinate results in a 1-perfect unitrade in $H(n)$. It is straightforward from the definition that every 1-perfect unitrade is a 2-fold 1-packing. In particular, as a result of the counting unitrades by computer (the algorithm is the same as described in [8] with the only difference that we do not request the bipartiteness of unitrades, in contrast with [8]), we find 2-fold 1-packings of cardinality 96, which is very close to the sphere-packing bound $[2 \cdot 2^9/(1 + 9)] = 102$. To compare, the largest 1-error correcting code in $H(9)$ has 40 codewords [3], and the union of two disjoint such codes is a 2-fold 1-packing of cardinality 80 only.

The connection with such good packings (in the next section, we will see that they are optimal) motivates to study properties of these unitrades of cardinality 96. It was found that these unitrades can be described in terms of equitable partitions, which in fact is interesting enough to present it here, but needs to introduce some additional concepts.

A partition $\pi = (C_0, C_1, \ldots, C_m)$ of the vertices of a graph (in our case, $H(10)$) is called equitable if for every $i$ and $j$ from $\{0, 1, \ldots, m\}$ there is an integer $s_{i,j}$ such that each vertex $v$ in $C_i$ has exactly $s_{i,j}$ neighbors in $C_j$. The matrix $(s_{i,j})_{i,j=0}^{m}$ is called the intersection matrix (sometimes, the quotient matrix of the graph with respect to the partition). If it is triadiagonal (equivalently, $\pi$ is a distance partition with respect to $C_0$, then $C_0$ is called a completely regular code with intersection matrix $(s_{i,j})_{i,j=0}^{m}$). In this case, $C_m$ is a completely regular code with intersection matrix $(s_{m-i,m-j})_{i,j=0}^{m}$.

It occurs that each of the three nonequivalent extended 1-perfect unitrades of length 10 and cardinality 96 can be represented as the cell $C_4$ of an equitable partition $\pi = (C_0, C_1, C_2, C_3, C_4)$ with intersection matrix

$$
\begin{pmatrix}
0 & 10 & 0 & 0 \\
1 & 0 & 9 & 0 \\
0 & 6 & 0 & 2 \\
0 & 0 & 10 & 0 \\
0 & 0 & 0 & 10
\end{pmatrix}
$$

(2)

It is not difficult to see the inverse: for every equitable partition with intersection matrix (2), the cell $C_4$ (as well as $C_3$) is an extended 1-perfect unitrade by definition, and its cardinality is determined from the intersection matrix because of the obvious relation $|C_i|b_{i,j} = |C_j|b_{i,j}$, $i, j \in \{0, 1, \ldots, m\}$.

Unifying the cells $C_3$ and $C_4$, we obtain an equitable partition with tridiagonal intersection matrix. So, $C_0$ is a completely regular code. One such code was already known [11, Theorem 1(2)]; it is linear of dimension 5 and equivalent to $C_0$ when $C_4$ is the unitrade

$$(0001111011, 0010101010, 0100110100, 1000110111)$$

from our classification. The other two unitrades of cardinality 96 correspond to nonlinear completely regular codes with the same intersection matrix, of ranks 6 and 7.

In this section, we describe the three inequivalent unitrades of cardinality 96 (we denote them $C_4', C_4''$, $C_4'''$) and the corresponding completely regular codes ($C_0', C_0'', C_0'''$, respectively). We describe each of the completely regular codes together with some group structure, which shows some automorphisms of the code and of the unitrade. In particular, we see that the automorphism group acts transitively on the code (so, the code is transitive) and, moreover, has a subgroup that acts regularly on the code (so, the code is propelinear). We do not give proofs that the described sets have the stated properties and refer all the results of this section as computational. However, for the first two cases, some properties can be manually checked using check matrices.

We note that our classification results do not guarantee nonexistence of other length-10 completely regular codes with the intersection array $(10, 9, 2; 1, 6, 10)$. They only guarantees that if such code exist, it is either equivalent to one of $C_0', C_0'', C_0'''$, or the set of vertices at distance 3 from it cannot be split into two cells inducing an equitable partition with quotient matrix (2).

### B. The linear code

The linear completely regular code $C_0'$ with intersection array $(10, 9, 2; 1, 6, 10)$ can be defined by the generator matrix

$$
\begin{pmatrix}
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

(3)

or by the check matrix

$$
\begin{pmatrix}
1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0
\end{pmatrix}
$$

(4)

It can be easily seen that these two matrices are obtained from each other by a permutation of columns. This means that the dual code is permutably equivalent to the code itself (however, the code is not self-dual, in the usual sense). To obtain an equitable partition $(C_0', C_1', C_2', C_3', C_4')$ with intersection matrix (2), we should define $C_1'$ and $C_2'$ as the sets of vertices at distance 1 and two from $C_0'$, respectively. However, the last two cells split the set of vertices at distance 3 from $C_0'$, and such splitting is not unique. One of the ways is to define $K'$ as the span of the last four rows of (3) and set $C_4'$ to be the union of the 6 cosets of $K'$ with the representatives

$$(0001001000, 0010011000, 0001101100, 0000110110, 0000111010, 0000010011).$$
Alternatively, the same code $C'_0$ can be described as a $Z_2 Z_4$-linear code whose $Z_2 Z_4$-additive preimage has the generator matrix

$$
\begin{pmatrix}
0 & 2 & 2 & 1 & 1 & 0 & 0 \\
2 & 0 & 2 & 1 & 0 & 1 & 0 \\
2 & 2 & 0 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 & 1
\end{pmatrix}
$$

(note that the rows of this matrix are the Gray preimages of the first four rows of (3)) and the check matrix

$$
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 0 & 1
\end{pmatrix}
$$

(for the sake of symmetry, we presented the check matrix in a redundant form; actually, any of its order-4 rows can be replaced by the row $[2 2 2 1 1 1 1]$ of order 2).

**C. The non-linear $Z_2 Z_4$-linear code**

The second completely regular code $C''_0$ can be defined as $Z_2 Z_4$-dual to the first one, i.e., as having the generator matrix (6) and the check matrix (5). Alternatively, the same code $C'_0$ can be described as a $Z_2 Z_4$-linear code whose $Z_2 Z_4$-additive preimage has the generator matrix

$$
\begin{pmatrix}
1 & 1 & 2 & 2 & 2 & 2 \\
1 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 3 & 1
\end{pmatrix}
$$

and the check matrix

$$
\begin{pmatrix}
1 & 1 & 2 & 2 & 2 & 2 \\
1 & 0 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 3 & 0 & 1
\end{pmatrix}
$$

(obviously, these matrices are permutably equivalent). To describe $C''_0$, we again define $K''$ as the module generated by the last two rows of (7), and set $C''_0$ to be the Gray image of the union of the following 6 cosets of $K''$ with the representatives

$$
01 \, 1300, \quad 10 \, 1300, \quad 00 \, 2030, \\
10 \, 1030, \quad 01 \, 0330, \quad 11 \, 3330.
$$

**D. The non-$Z_2 Z_4$-linear code**

The last of the three completely regular codes does not have any $Z_2 Z_4$-linear structure. However, it is also a propelinear code, that is, its automorphism group has a subgroup acting regularly on the code. Such subgroup can be defined by three generators $\xi_0, \xi_1, \xi_2$, where $\xi_0(x) = 1111111111 + x$, $\xi_1(x) = 0101001111 + x_1$, $\xi_2(x) = 0001010011 + x_2$, and $\pi_1$, $\pi_2$ are the coordinate permutations (01)(23), (23)(01), respectively. The completely regular code $C_0$ is the orbit $\text{Orb}_{(\xi_0, \xi_1, \xi_2)}(0000000000)$, and the unitrade $C_4$ can be defined as the union of the orbits $\text{Orb}_{(\xi_0, \xi_1, \xi_2)}(000000000111)$, $\text{Orb}_{(\xi_0, \xi_2)}(0000011001)$, $\text{Orb}_{(\xi_1, \xi_2)}(0000011011)$, $\text{Orb}_{(\xi_0, \xi_1, \xi_2)}(0000101001)$, $\text{Orb}_{(\xi_0, \xi_1, \xi_2)}(0000110001)$, $\text{Orb}_{(\xi_0, \xi_1, \xi_2)}(0000110100)$, $\text{Orb}_{(\xi_0, \xi_1, \xi_2)}(0000101010)$, $\text{Orb}_{(\xi_0, \xi_1, \xi_2)}(0000101101)$ under the subgroup $\langle \xi_1, \xi_2 \rangle$ of order 16.

**IV. Upper bounds**

The weight distribution of a code $C$ of length $n$ is the sequence $\{A_i\}_{i=0}^n$, where $A_i$ is the number of the codewords of weight $i$ in $C$. The weight distribution $\{A_i\}_{i=0}^n$ of $C$ with respect to a word $x$ is the weight distribution of the code $C + x$. The distance distribution $\{B_i\}_{i=0}^n$ of $C$ is defined as the average weight distribution of $C$ with respect to all its codewords: $B_i = \frac{1}{|C|} \sum_{x \in C} A_i(x)$.

**Theorem 2:** Every even-weight $\lambda$-fold 1-packing of $C$ of length $n$, where $\lambda \equiv \sigma \mod 2$, $\sigma \in \{0, 1\}$, satisfy

\begin{align*}
(a) \quad |C| &\leq \frac{2^{n-1}(\lambda n + 2\lambda - 4 + \sigma)}{(n-1)(n+3)} \quad \text{if} \ n \equiv 1 \mod 4, \\
(b) \quad |C| &\leq \frac{2^{n-1}(\lambda n - 2)}{(n-2)(n+2)} \quad \text{if} \ n \equiv 2 \mod 4, \\
(c) \quad |C| &\leq \frac{2^{n-1}(\lambda n - 2 + \sigma)}{(n-1)(n+1)} \quad \text{if} \ n \equiv 3 \mod 4, \\
(d) \quad |C| &\leq 2^{n-1}\lambda n \quad \text{if} \ n \equiv 0 \mod 4.
\end{align*}

**Proof:** Let $\{B'_i\}_{i=0}^n$ be the MacWilliams transform of the distance distribution $\{B_i\}_{i=0}^n$ of $C$; that is,

$$
|C| B'_k = \sum_{i=0}^{n} B_i K_k(i),
$$

$$
2^n B_k = |C| \sum_{i=0}^{n} B'_i K_k(i),
$$

where

$$
K_k(i) = \sum_{j=0}^{k} (-1)^j \binom{i}{j} \binom{n-i}{k-j}
$$

is a Krawtchouk polynomial; in particular,

$$
K_0(i) = 1, \quad K_2(i) = \frac{1}{2}(n-2i)^2 - \frac{1}{2}n, \quad K_{n-1}(i) = (-1)^i(n-2i), \quad K_n(i) = (-1)^i.
$$

It is well known that $B'_0 = 1$ and $B'_i \geq 0$ for $1 \leq i \leq n$ [7]. As $C$ is an even-distance code, $B_i = 0$ for odd $i$, and, since $K_{n-k}(i) = (-1)^k K_k(i)$, we have

$$
B'_k = B'_{n-k}.
$$

(a) Let $n \equiv 1 \mod 4$. Define $\alpha(i) = (n-3)K_0(i) + 2K_2(i) + 2K_{n-1}(i)$. Direct calculations now show that

$$
\alpha(i) = (n-2i-2 + (-1)^i)(n-2i+2 + (-1)^i).
$$

From (11) and $n = 2^n - 3 \equiv 1 \mod 4$ we derive

$$
\alpha(i) = 0 \quad \text{for} \ i = \frac{n-3}{2}, \frac{n-1}{2}, \frac{n+1}{2}, \frac{n+3}{2},
$$

$$
\alpha(i) > 0 \quad \text{for any other integer} \ i.
$$

From the packing condition we have $B_{n-1} \leq \lambda$ and, moreover,

$$
(n-3)B_0 + 2B_2 \leq \lambda n - 4 + \sigma,
$$

with equality only if there are no codewords of multiplicity more than 1. (Indeed, for a vertex $x$ of multiplicity $a_0(x) = 1$,}
the number \( a_2(x) \) of a codewords at distance 2 from \( x \) is at most \( [n(\lambda - 1)/2] = (n(\lambda - 1) + \sigma - 1)/2 \); so,
\[(n-3)a_0(x) + 2a_2(x) \leq n - 3 + n(\lambda - 1) + \sigma - 1 = \lambda n - 4 + \sigma.
\]
For a larger multiplicity, we have
\[(n-3)a_0(x) + 2a_2(x) \leq (n-3)a_0(x) + n(\lambda - a_0(x)) \leq \lambda n - 6,
\]
which is stronger than (13).

Utilizing (10), we then get
\[
2\alpha(0)B'_0 = \alpha(0)B'_0 + \alpha(n)B'_n \leq \sum_i \alpha(i)B'_i
= \frac{2^n((n-3)B_0 + 2B_2 + 2B_{n-1})}{|C|}
\leq \frac{2^n(\lambda n - 4 + \sigma + 2\lambda)}{|C|} = \frac{2^n(n-1)}{|C|},
\]
and thereby
\[
|C| \leq \frac{2^n(\lambda n - 2 + \sigma)}{2\alpha(0)B'_0} = \frac{2^{n-1}(\lambda n - 2 + \sigma)}{(n-1)(n+3)}.
\]
(b) Let \( n \equiv 2 \mod 4 \). Define \( \beta(i) = (n-2)K_0(i) + 2K_2(i) - 2K_n(i) \). Straightforwardly,
\[
\beta(i) = (n-2)i^2 - 2 - 2(-1)^i.
\]
From (15) and \( n = 2^n - 3 \equiv 1 \) (mod 4) we see that
\[
\beta(i) = 0, \quad \text{if } i \in \{n/2-1, n/2, n/2+1\};
\]
for any other integer \( i \), we have \( \beta(i) > 0 \). From the packing condition we have
\[(n-2)B_0 + 2B_2 \leq \lambda n - 2\]
with equality if and only if \( B_0 = 1 \) (i.e., there are no codewords of multiplicity more than 1) and \( B_2 = (\lambda - 1)n/2 \).

Then, we get
\[
2\beta(0)B'_0 = \beta(0)B'_0 + \beta(n)B'_n \leq \sum_i \beta(i)B'_i
= \frac{2^n((n-2)B_0 + 2B_2 + 2B_{n-1})}{|C|}
\leq \frac{2^n(\lambda n - 2)}{|C|},
\]
and thereby
\[
|C| \leq \frac{2^n(\lambda n - 2)}{2\beta(0)B'_0} = \frac{2^{n-1}(\lambda n - 2)}{(n-2)(n+2)}.
\]
(c) Let \( n \equiv 3 \mod 4 \). Define \( \gamma(i) = (n-1)K_0(i) + 2K_2(i) \). Straightforwardly,
\[
\gamma(i) = (n-2)i^2 - 1.
\]
Obviously,
\[
\gamma(i) = 0 \quad \text{for } i \in \{(n-1)/2, (n+1)/2\}
\]
and \( \gamma(i) > 0 \) for any other integer \( i \).

With an argument similar to that for (13), we have
\[(n-1)B_0 + 2B_2 \leq \lambda n - 2 + \sigma
\]
Then, we get
\[
2\gamma(0)B'_0 = \gamma(0)B'_0 + \gamma(n)B'_n \leq \sum_i \gamma(i)B'_i
= \frac{2^n((n-1)B_0 + 2B_2)}{|C|}
\leq \frac{2^n(\lambda n - 2 + \sigma)}{|C|},
\]
and hence
\[
|C| \leq \frac{2^n(\lambda n - 2 + \sigma)}{2\gamma(0)B'_0} = \frac{2^{n-1}(\lambda n - 2 + \sigma)}{(n-1)(n+1)}.
\]
(d) Let \( n \equiv 0 \mod 4 \). Define \( \delta(i) = nK_0(i) + 2K_2(i) \). Straightforwardly, \( \delta(i) = (n-2i)^2 \geq 0 \), and \( \delta(i) = 0 \iff i = n/2 \).

From the \( \lambda \)-fold packing condition, we have
\[nB_0 + 2B_2 \leq \lambda n
\]
Then, we get
\[
2\delta(0)B'_0 = \delta(0)B'_0 + \delta(n)B'_n \leq \sum_i \delta(i)B'_i
= \frac{2^n(nB_0 + 2B_2)}{|C|}
\leq \frac{2^n(\lambda n)}{|C|},
\]
and hence
\[
|C| \leq \frac{2^n\lambda n}{2\delta(0)B'_0} = \frac{2^{n-1}\lambda}{n}.
\]

**Corollary 1:** Assume that \( C \) is an even-weight \( \lambda \)-fold 1-packing of length \( n \), and assume that one of the equations (a)–(c) in Theorem 2, in respect to \( n \mod 4 \), is satisfied with equality. Then the weight distribution of \( C \) with respect to every its codeword is uniquely determined by the parameters \( n \) and \( \lambda \). In particular, \( C \) is an ordinary set (there are no codewords with multiplicity more than 1) and there are exactly \( n(\lambda - 1)/2 \) codewords at distance 2 from every codeword.

**Proof:** Assume that \( C \) is an even-weight \( \lambda \)-fold 1-packing of length \( n \), \( n \equiv 1 \mod 4 \), and assume that the inequality (a) in Theorem 2 is satisfied with equality. This means that we have equalities everywhere in (14). As follows from (13) and the note after it, the equality in (14) implies \( B_0 = 1, B_2 = n(\lambda - 1)/2 \), and \( B_{n-1} = \lambda \). Since \( A_0(x) \geq 1 \), \( A_2(x) \leq n(\lambda - 1)/2 \), and \( A_{n-1}(x) \leq \lambda \) for every codeword \( x \), we also have \( A_0(x) = 1, A_2(x) = n(\lambda - 1)/2 \), and \( A_{n-1}(x) = \lambda \). Remind also that \( B_1 = A_1(x) = 0 \) for every odd \( i \).

Next, consider the dual distance distribution \( \{B'_1\}_i \). From (12) and the equality in (14) we find that \( B'_1 = 0 \) for all \( i \) except \( 0, (n-3)/2, (n-1)/2, (n+1)/2, (n+3)/2, n \). Moreover, we know that \( B'_i = B'_{n-i} \) for all \( i \) and \( B'_n = 1 \). So, for complete determining \( \{B'_i\}_i \), it remains to know \( B'_{(n-3)/2} \) and \( B'_{(n-1)/2} \). These two values can be found from
two equations (9), \( k = 0, 2 \). So, the dual distance distribution and, hence, the distance distribution are uniquely determined.

The same arguments can be applied to the dual weight distribution \( \{ A'_i(x) \}_{i=0}^n \) calculated from \( \{ A_i(x) \}_{i=0}^n \) by the same formulas as \( \{ B'_i \}_{i=0}^n \) from \( \{ B_i \}_{i=0}^n \) (9). Indeed, by [9, Theorem 7(b) in Ch.5, §5], \( B'_i = 0 \) implies \( A'_i(x) = 0 \). We also have \( A'_i(x) = A'_{2-i}(x) \) and \( A'_0(x) = A'_1(x) = 1 \), and we know \( A_0(x) \) and \( A_2(x) \). So, we can completely determine \( \{ A'_i(x) \}_{i=0}^n \) and then \( \{ A_i(x) \}_{i=0}^n \).

For \( n = 2, 3, 0 \mod 4 \), the proof is similar.

Substituting \( n = 10 \) and \( \lambda = 2 \) (case (b)), we find that the 2-fold 1-packings found in Section III are optimal. Moreover, utilizing Corollary 1, we can conclude that every even-weight 2-fold 1-packings of length 10 is an extended 1-perfect unitrade (equivalently, every 2-fold 1-packings of length 9 is a 1-perfect unitrade), so, the three equivalence classes described in Sections IIIB–IID exhaust the class off such packings.

Substituting \( n = 9 \) and \( \lambda = 2 \) (case (a)), we find that the 2-fold 1-packings of size 48 obtained by shortening a packing from Section III is also optimal. Further analyzing the local structure with help of Corollary 1, it is possible to prove that every 2-fold 1-packings of size 8 and size 48 can be lengthened to a 2-fold 1-packings of length 9 and size 96. So, all optimal 2-fold 1-packings are characterized up to length 9 (in the even-weight case, length 10).

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References