Splitting of hypercube into \( k \)-faces and DP-colorings of hypergraphs *

Vladimir N. Potapov  
*Sobolev Institute of Mathematics, Novosibirsk, Russia; email: vpotapov@math.nsc.ru*

Abstract  
We develop a connection between DP-colorings of \( k \)-uniform hypergraphs of order \( n \) and coverings of \( n \)-dimensional hypercube by pairs of antipodal \((n - k)\)-dimensional faces. Bernshteyn and Kostochka established that the minimum number of edges in a non-2-DP-colorable \( k \)-uniform hypergraph is \( 2^{k-1} \). In this paper, we use the fact that this bound is attained if and only if there exists a splitting of the \( n \)-dimensional Boolean hypercube into \( 2^{k-1} \) pairs of antipodal \((n - k)\)-dimensional faces. We present an example of such antipodal splitting for \( k = 5 \). Moreover, based on examples for \( k = 3, 5 \), we give a construction of such splittings for \( k = 3^5 \). We prove that there exists a unique non-2-DP-colorable 5-uniform hypergraph with 16 edges.

Keywords: hypergraph coloring, DP-coloring, splitting of hypercube, unitrade.

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1 Introduction

Let \( Q_n^k \) be an \( n \)-dimensional Boolean hypercube. We consider a splitting of \( Q_n^k \) into \( m \)-dimensional axis-aligned planes or \( m \)-faces. If \( m = 1 \) then such splitting is equivalent to a perfect matching in the Boolean hypercube. Two \( m \)-faces are called parallel if they have the same directions and a pair of parallel faces is called antipodal if for each vertex from one face there exists antipodal vertex in another face. It is clear that each covering of \( Q_n^k \) consists of \( 2^k \) or more \((n - k)\)-faces. If a covering \( C \) of \( Q_n^k \) consists of exactly \( 2^k \) \((n - k)\)-faces then \( C \) is a splitting of \( Q_n^k \) into \((n - k)\)-faces. So, a covering of Boolean hypercube is called an antipodal \( k \)-splitting if it consists of exactly \( 2^k \) \((n - k)\)-faces and it does not contain pairs of parallel non-antipodal faces.

Concept of a graph DP-coloring was developed by Dvorak and Postle \cite{3} in order to generalize the notation of a proper coloring. In \cite{1} Bernshteyn and Kostochka considered a problem to estimate the minimum number of edges in non-2-DP-colorable \( k \)-uniform hypergraphs. The existence of a non-2-DP-colorable \( k \)-uniform hypergraph with \( e \) edges and \( n \) vertices is equivalent to the existence of a covering of \( Q_n^k \) by \( e \) pairs of antipodal \((n - k)\)-faces. If the hypergraph have not multiple edges then the definition of DP-coloring implies that this covering does not contain pairs of parallel non-antipodal faces. If \( e = 2^{k-1} \) then a non-2-DP-colorable \( k \)-uniform hypergraph with \( e \) edges generates an antipodal \( k \)-splitting and vice versa.

A splitting of hypercube into \((n - k)\)-faces is a special case of A-designs. In \cite{9} there were given constructions of A-design with additional properties such as a lack of adjacent parallel faces. For \( n - 2k + 2 \geq 0 \) we construct \( k \)-splitting of \( Q_n^k \) with at most two \((n - k)\)-faces of any fixed direction (Proposition \cite{9}).

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The main results of this paper are a construction of antipodal $k$-splittings for $k = 3^i5^p$ (Corollary 1), an example (Proposition 3) of non-2-DP-colorable 5-uniform hypergraph with 16 edges and a proof of its uniqueness (Theorem 2). It is known (see [1]) that for even $k$ each $k$-uniform hypergraph with $2^{k-1}$ edges has 2-DP-coloring. The following problem remains open.

Do there exist non-2-DP-corolable $k$-uniform hypergraphs with $2^{k-1}$ edges for any odd $k$?

The proof of uniqueness of non-2-DP-colorable 5-uniform hypergraphs with 16 edges is based on a concept of $k$-unitrades. A trade, broadly speaking, is the difference between two combinatorial structures with the same parameters. Trades, bitrades and unitrades are using for investigation and construction via switching method of combinatorial designs, latin squares, error-correcting codes and other structures [1, 6]. The notion of unitrades was introduced in [8]. Here we prove that every antipodal $k$-splitting of a Boolean hypercube or non-2-DP-colorable $k$-uniform hypergraph with $2^{k-1}$ edges creates a $k$-unitrade with cardinality $2^{k-1}$. In Theorem 1 we give a classification of 5-unitrades with cardinality 16.

2 Splitting of hypercube

We denote a $(n-k)$-face of $Q^n_2$ by a $n$-tuple $(a_1, a_2, \ldots, a_n)$ of symbols 0, 1, * where the symbol * is used $n-k$ times. In more details, $(a_1, \ldots, a_n) = \{(x_1, \ldots, x_n) : x_i = a_i \text{ if } a_i = 0 \text{ or } a_i = 1\}.$

If $A = \{(a_1, \ldots, a_n)\}$ is an antipodal $k$-splitting then $A_{\tau} = \{(a_{\tau 1}, \ldots, a_{\tau n})\}$ is an antipodal $k$-splitting for any permutation $\tau$. Let us agree to $* \oplus 0 = * \oplus 1 = *$. We define Boolean addition of $n$-tuples to act coordinate-wise. Then $a \oplus b$ is a $(n-k)$-face of $Q^n_2$ for any $(n-k)$-face $a$ and any $b \in Q^n_2$. It is clear that if $A$ is an antipodal $k$-splitting then $A \oplus b = \{a \oplus b : a \in A\}$ is an antipodal $k$-splitting for each $b \in Q^n_2$. We will refer to the above operations as isometries of a Boolean hypercube. $A$ and $A'$ are called equivalent antipodal $k$-splitting if $A'$ is obtained from $A$ by an isometry.

Proposition 1. If there exists an antipodal $k$-splitting of $Q^n_2$ then there exists an antipodal $k$-splitting of $Q^n_{2+1}$.

Proof. If $A$ is an antipodal $k$-splitting of $Q^n_2$ then $B = \{(a_1, \ldots, a_n, *) : (a_1, \ldots, a_n) \in A\}$ is an antipodal $k$-splitting of $Q^n_{2+1}$. △

Proposition 2. If there exist an antipodal $k_1$-splitting of $Q^{n_1}_2$ and an antipodal $k_2$-splitting of $Q^{n_2}_2$ then there exists an antipodal $k_1 k_2$-splitting of $Q^{n_1+n_2}_2$.

Proof. Let $A$ be an antipodal $k_1$-splitting of $Q^{n_1}_2$ and $B = B_0 \cup B_1$ be an antipodal $k_2$-splitting of $Q^{n_2}_2$ where sets $B_0$ and $B_1$ do not contain parallel $(n_2 - k_2)$-faces. Consider $(a_1, \ldots, a_{n_1}) \in A$. For all $a_i$, if $a_i = 0$ we replace $a_i$ by arbitrary $b \in B_0$; if $a_i = 1$ then we replace $a_i$ by arbitrary $b \in B_1$; if $a_i = *$ then we replace $a_i$ by $(*, \ldots, *)$. So, we obtain a set $C$ of $|A|(|B|/2)^{k_1} = \frac{2^k \cdot 2^{(k_2-1)k_1}}{n_2}$ tuples corresponding to $m$-faces, where $m = (n_1 - k_1) n_2 + k_1 (n_2 - k_2) = n_1 n_2 - k_1 k_2$. It is not difficult to verify that 1) $C$ is a covering of $Q^{n_1+n_2}_2$; 2) all tuples of $C$ are disjoint and, consequently, $C$ is $k_1 k_2$-splitting; 3) $C$ contains pairs of antipodal faces because $A$ and $B$ contain pairs of antipodal faces; 4) $C$ does not contain parallel non-antipodal faces because $A$ and $B$ do not contain such faces. □

Proposition 3. There exist an antipodal 3-splitting of $Q^3_2$ and an antipodal 5-splitting of $Q^5_2$.

1 Earlier, I mistakenly argued that such an example does not exist, i.e., each 5-uniform hypergraph with 16 edges is 2-DP-colorable (see [1]).
Proof. The following antipodal 3-splitting of $Q_2^4$ correspond to the well-known antipodal perfect matching in $Q_2^4$.

\[
\begin{array}{cccc}
0 & 0 & 0 & 0, \\
0 & 0 & 0 & 1, \\
0 & 1 & * & 0, \\
0 & 0 & 1 & *, \\
1 & 1 & 0 & *. \\
\end{array}
\]

is even (or zero). Since odd-weighted vertices in \(a\) face number of $\text{parity because } k\) is even. But for all other $\text{Proposition 5.}\)

The following antipodal 5-splitting of $Q_2^5$ is constructed by a method described in Sections 4 and 5.

\[
\begin{array}{cccccc}
00 & 1* & 00 & **, & 11 & 0* & 11 & **, \\
00 & *1 & 10 & **, & 11 & *0 & 01 & **, \\
0* & 01 & 00 & **, & 1* & 10 & 11 & **, \\
*0 & 10 & 01 & **, & *1 & 01 & 10 & **, \\
01 & 1* & 0* & 0*, & 10 & 0* & 1* & 1*, \\
00 & *0 & 1* & 1*, & 11 & *1 & 0* & 0*, \\
0* & 10 & 1* & 0*, & 1* & 01 & 0* & 1*, \\
*0 & 00 & 0* & 1*, & *1 & 11 & 1* & 0*, \\
01 & 0* & *1 & 0*, & 10 & 1* & *0 & 1*, \\
1* & 10 & *0 & 0*, & 0* & 01 & *1 & 1*, \\
*1 & 00 & *0 & *0, & *0 & 11 & *1 & 1*, \\
*0 & 00 & ** & 00, & *1 & 11 & ** & 11, \\
*0 & 0* & *1 & 01, & *1 & 1* & *0 & 10, \\
*0 & 1* & *1 & 00, & *1 & 0* & 0* & 11, \\
** & 00 & *0 & 01, & ** & 11 & *1 & 10. \\
\end{array}
\]

\[\square\]

\textbf{Corollary 1.} There exists an antipodal $3^t5^p$-splitting of $Q_2^{2^t+3^p}$ for every positive integers $t$ and $p$.

Proofs of the following statement can be found in [1]. Here it is rewritten in notations of this article.

\textbf{Proposition 4 ([1]).} If $k$ is even then an antipodal $k$-splitting of $Q_n^2$ does not exist.

Proof. Let $A$ be an antipodal $k$-splitting and $k$ is even. Let us consider an $(n-k)$-face $a \in A$, the $(n-k)$-face $\overline{a} \in A$ antipodal to $a$ and a $k$-face $a^\perp$ orthogonal (dual) to $a$, i.e., positions of asterisks in $a$ and $a^\perp$ are complementary. By the definitions, we obtain that $x = a \cap a^\perp$ and $\overline{x} = \overline{a} \cap a^\perp$ are antipodal vertices within the face $a^\perp$. For example, $a = (0, 1, 1, 0, *, *)$, $a^\perp = (*, *, *, *, 1, 0)$, $x = (0, 1, 1, 0, 1, 0)$, $\overline{x} = (1, 0, 0, 1, 1, 0)$. The vertices $x$ and $\overline{x}$ have the same parity because $k$ is even. But for all other $b \in A$ we obtain that $b \cap a^\perp$ has the same number of even-weighted and odd-weighted vertices (or $b \cap a^\perp$ is an empty set). Since $A$ is a splitting, the set $\{b \cap a^\perp : b \in A\}$ is a splitting of $a^\perp$ as well. Because the numbers of even-weighted and odd-weighted vertices in $a^\perp$ is equal, we have a contradiction. \[\square\]

\textbf{Proposition 5.} For any $k$-splitting $A$ of $Q_n^2$ ($n > k > 0$) and any fixed direction of faces the number of $(n-k)$-faces of this direction in $A$ is even.

Proof. Suppose $a \in A$ and $A$ contains $m$ $(n-k)$-faces with the same direction as $a$. Consider a face $a^\perp$. If $b \in A$ has the same direction as $a$ then $|b \cap a^\perp| = 1$, otherwise the number $|b \cap a^\perp|$ is even (or zero). Since $|a^\perp| = 2k = \sum_{b \in A} |b \cap a^\perp|$, $m$ is even. \[\square\]

\textbf{Proposition 6.} There exits $k$-splitting of $Q_2^2$ with at most two $(n-k)$-faces of any fixed direction if $n - 2k + 2 \geq 0$, $0 < k < n$. 

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Proof. By induction on $n$. For $n = 2, k = 1$ and $n = 3, k = 2$ it is easy to verify this statement directly. The case $n = 4, k = 3$ follows from Proposition 3. Let $B$ be a $k$-splitting of $Q^n_2$ with at most two $(n-k)$-faces of any fixed direction. By Proposition 5, $B$ contains two or zero faces of any direction. Let $B = B_0 \cup B_1$ where sets $B_0$ and $B_1$ do not contain parallel $(n-k)$-faces. Consider the set $A = \{bh, b* : b \in B_0\} \cup \{b * 1, b * 0 : b \in B_1\}$. By the construction, the set $A$ is a $(k+1)$-splitting of $Q^{n+2}$ with at most two $(n-k+1)$-faces of any fixed direction. Besides, the set $C = \{b* : b \in B\}$ is a $k$-splitting of $Q^{n+1}$.

Using Corollary 1 and Propositions 6, we can construct a $k$-splitting of $Q^n_2$ without parallel $(n-k)$-faces on a distance less than $d$ for sufficiently large $n$.

3 2-DP-coloring

Let $G$ be a $r$-hypergraph on $n$ vertices. For each $e \in E(G)$ we consider two antipodal 2-colorings $\varphi_e : e \rightarrow \{0,1\}$ and $\varphi_e = \varphi_e \oplus 1$. Let $\Phi$ be a collection of $\varphi_e$, $e \in E(G)$. We say that a 2-coloring $f : V(G) \rightarrow \{0,1\}$ avoids $\Phi$ if $f|_e \neq \varphi_e$ and $f|_e \neq \varphi_e$ for each $e \in E(G)$.

A hypergraph $G$ is called proper 2-colorable if there exists 2-coloring $f$ avoiding $\Phi_0$, where $\Phi_0$ consists of constant maps. A hypergraph $G$ is called 2-DP-colorable if for every collection $\Phi$ there exists a 2-coloring $f$ avoiding $\Phi$.

A 2-coloring of $k$-uniform hypergraph on $n$ vertices is on-to-one correspond to $n$-tuple over alphabet $\{0,1\}$ ($f \in Q^n_2$). Each $k$-hyperedge corresponds to $(n-k)$-faces of $Q^n_2$ of some direction. For example, $k$-hyperedge consisting of $i_1$th,...,$i_k$th vertices corresponds to faces $(*,*,*,*,*,*,*,*,*,*,*,*)$. A 2-coloring $f$ avoids $\varphi_e = (*)$ iff $f \not\in \varphi_e$. A 2-coloring $f$ avoids $\Phi$ if $f \not\in \varphi_e \cup \varphi_e^*$ for each $\varphi_e \in \Phi$.

Consider a table of size $n \times \ell$ where every column correspond to a $(n-k)$-face of an antipodal covering of $Q^n_2$. Let us replace in the table symbols 1 by 0 and symbols $*$ by 1. By definition, the resulting table is the incidence matrix of non-2-DP-colorable $k$-uniform hypergraph with $\ell$ edges. Consequently, we have the following statement.

**Proposition 7.** A $k$-uniform hypergraph with $\ell$ edges and $n$ vertices is non-2-DP-colorable if and only if its incidence matrix corresponds to a covering of $Q^n_2$ by $\ell$ pairs of antipodal $(n-k)$-faces.

Proposition 7 implies the following statement.

**Corollary 2.** There exists a non-2-DP-colorable $k$-uniform hypergraph with $2^{k-1}$ edges if and only if there exists an antipodal $k$-splitting of $Q^n_2$.

A non-2-DP-colorable 3-uniform hypergraph with 4 edges that corresponds to the antipodal 3-splitting from Proposition 3 is presented in [1]. By Corollaries 1 and 2 we obtain the following statement.

**Corollary 3.** There exist non-2-DP-colorable $k$-uniform hypergraphs with $2^{k-1}$ vertices, where $k = 3^25^2$. Since a union of $2\ell$ $(n-k)$-faces contains $\ell 2^{n-k+1}$ vertices at most, we obtain the following corollary.

**Corollary 4.** Any $k$-uniform hypergraph with $\ell < 2^{k-1}$ edges is 2-DP-colorable. A non 2-proper colorable hypergraph corresponds to a covering consisting of faces which contain vertices 0 or 1. Therefore, any $k$-uniform hypergraph with $2^{k-1}$ edges is proper 2-colorable. Moreover, by similar arguments we obtain that any $k$-uniform hypergraph with $s^{k-1}$ or less edges is proper $s$-colorable. There is a better bound for the case of proper colorings.
Cherkashin and Kozik [2] Radhakrishnan and Srinivasan [10] (for $s = 2$) showed that any $k$-uniform hypergraph with $c(s)(\frac{k}{\ln k})^{\frac{s-1}{s}}k^{-1}$ or less edges is proper $s$-colorable, where $c(s) > 0$ does not depend on $k$ and $k$ is large enough.

4 Trades

A pair $\{T_0, T_1\}$ of disjoint collections of $k$-subsets (blocks) of a set $V$, $|V| = n$, is called a bitrade (more specifically, a $(k-1)-(n,k)$ bitrade) if every $(k-1)$-subset of $V$ is contained in the same number of blocks of $T_0$ and $T_1$. A bitrade corresponds to a possible difference between two Steiner designs. A collection $U$ of $k$-subsets (blocks) of a set $V$ is called a $k$-unitrade if every $(k-1)$-subset of $V$ is contained in even number of blocks from $U$. If every $(k-1)$-subset is contained in 2 or 0 blocks only then a unitrade is called simple. It is easy to see that if $\{T_0, T_1\}$ is a bitrade then $T_0 \cup T_1$ is a unitrade. A set $\text{supp}(U) = \bigcup_{u \in U} u \subset V$ is called a support of $U$.

Further, we use symbols $1, 2, \ldots$ for elements of the set $V$. Denote $U'_a = \{u \setminus \{a\} : a \in u, u \in U\}$ for $a \in V$.

**Proposition 8 (elementary properties of unitrades).**

1) A set of indicators of $k$-unitrades on $V$ is a vector space over $GF(2)$.

2) If $U$ is a $k$-unitrade on an $n$-element set then it is a $k$-unitrade on an $(n+1)$-element set.

3) Every $k$-unitrade $U$ is a $k$-unitrade on $|\text{supp}(U)|$-element set.

4) If $U$ is a $k$-unitrade then $U'_a$ is a $(k-1)$-unitrade.

**Proof.** Items 1)-3) follow directly from the definition of unitrades. Let us prove 4). If a $(k-1)$-block $w \in U'_a$ covers a $(k-2)$-block $u$ then the $k$-block $w \cup \{a\} \in U$ covers $(k-1)$-block $u \cup \{a\}$. The converse is also true. Then each $(k-2)$-block $u$ is contained in blocks of $U'_a$ with the same multiplicity as the $(k-1)$-block $u \cup \{a\}$ is contained in blocks of $U$. $\square$

For convenience, we identify unitrades and their indicators. Denote by $\mathbb{V}(k,n)$ the vector space of $k$-unitrades on an $n$-element set.

Two unitrades $U_1$ on a set $V_1$ and $U_2$ on a set $V_2$ said to be equivalent if there exists an injection $f : V_1 \to V_2$ such that $U_2 = \{f(u) : u \in U_1\}$. If for any two blocks $u_1, u_2 \in U$ there exists a bijection $f : V \to V$ such that $f(U) = U$ and $f(u_1) = u_2$ then $U$ is called transitive.

Let $V$ be a $(k+1)$-element set. Denote $W_k = \{V \setminus \{a\} : a \in V\}$.

**Proposition 9.** If $U$ is a $k$-unitrade then $|U| \geq k + 1$ and if $|U| = k + 1$ or $|\text{supp}(U)| = k + 1$ then $U$ is equivalent to $W_k$.

**Proof.** If both $k$-blocks $u_1$ and $u_2$ cover two different $(k-1)$-blocks then $u_1 = u_2$. Each $k$-block $u$ covers $k$ different $(k-1)$-blocks. So each $k$-unitrade contains not less than $k + 1$ blocks. If $|\text{supp}(U)| \geq k + 2$ then there exist $u_1, u_2 \in U$ such that $|u_1 \cap u_2| \leq k - 2$. In this case $u_1$ and $u_2$ contain at least $k + 2$ different $(k-1)$-blocks. Therefore $|U| \geq k + 2$. We prove that $|\text{supp}(U)| = k + 1$ when $|U| = k + 1$, i.e., $U$ is equivalent to $W_k$. $\square$

We will denote by $U^i$ an arbitrary unitrade equivalent to $U$. For example, $k$-unitrades equivalent to $W_k$ are denoted by $W_k^i$.

**Proposition 10.** $\mathbb{V}(k,n)$ is the linear closure of unitrades equivalent to $W_k$.

**Proof.** By induction on $k$ and $n$. The space $\mathbb{V}(k,k+1)$ consists of a unique nonzero element $W_k$. Suppose the proposition holds for $\mathbb{V}(2,n)$. Consider $U \in \mathbb{V}(2,n+1)$. For each $v \in V$ there are $2t$ elements $u \in U$ such that $v \in u$. Then we can choose $t$ $2$-unitrades $W_2^i$ (all of them are equivalent to $W_2$) such that $W \cap \{v\} = \emptyset$, where $W = U \oplus \sum_{i=1}^t W_2^i$. Consequently,
$U = W \oplus \sum_{i=1}^{t} W_{i}^{2}$, $W \in \mathbb{V}(2, n)$ and $W$ is a linear combination of 2-unitrades that are equivalent to $W_{2}$ by inductive assumption.

Suppose the proposition holds for $\mathbb{V}(k+1, n)$ and $\mathbb{V}(k, m)$ for any positive integer $m$. Consider a $(k + 1)$-unidade $U \in \mathbb{V}(k + 1, n + 1)$. Then $U_{1} \in \mathbb{V}(k, n)$ is a linear combination $\sum_{i=1}^{t} W_{i}^{2}$ by inductive assumption. It is easy to see that $W_{k}^{1} = (W_{k+1}^{1})_{a}$. Then the $(k + 1)$-unidade $U \oplus \sum_{i=1}^{t} W_{i}^{2}$ belongs to the $\mathbb{V}(k + 1, n)$.  □

We will count $k$-unitrades with small cardinality for $k = 3, 4, 5$. As proved below (Lemma 1), we need a list of 5-unidade with cardinality 16 for a finding of an antipodal 5-splitting. Since the set of unitrades is a vector space over $GF(2)$, an enumeration of unitrades with small cardinality is equivalent to an enumeration of codewords with a small weight. It is a typical problem of coding theory.

**Proposition 11.** If $U$ is a $k$-unidade and $k$ is odd then $|U|$ is even.

Proof. By double counting the number of pairs $(v, u)$ such that $u \in U$ and $v$ is a $(k - 1)$-sub block of $u$ we get $|U| = \sum_{v} |\{(v, u) : u \in U, v \subset u\}|$. Since $k$ is odd and the sum on the right side is a sum of even numbers, $|U|$ is even.  □

**Proposition 12.** There are not exist $k$-unitrades of cardinalities between $k + 1$ and $2k$. Every $k$-unidade of cardinality $2k$ is a symmetric difference of two $k$-unitrades equivalent to $W_{k}$ with non-empty intersection.

Proof. Let $U$ be a $k$-unidade and let $|U| > k + 1$. There exist $u_{1}, u_{2} \in U$ such that $|u_{1} \cap u_{2}| < k - 1$ because otherwise $U$ is equivalent to $W_{k}$. If there exist $u_{1}, u_{2} \in U$ such that $|u_{1} \cap u_{2}| < k - 2$ then all $(k - 1)$-sub blocks of $u_{1}$ and $u_{2}$ are different. So, we obtain that $|U| \geq 2k + 2$ similarly to Proposition 9. Consider blocks $u_{1}, u_{2} \in U$ such that $|u_{1} \cap u_{2}| = k - 2$. There are exactly four $k$-blocks intersecting both blocks $u_{1}$ and $u_{2}$ in $k - 1$ elements. There four $k$-blocks cover two $(k - 1)$-subblocks of $u_{1}$ and two $(k - 1)$-subblocks of $u_{2}$. But we are able to choose only two from four $k$-blocks because another pair coes the same $(k - 1)$-subblocks. So, counting the minimal possible number of $k$-blocks which cover all $(k - 1)$-blocks included in $u_{1}$ and $u_{2}$, we obtain that $|U| \geq 2(k + 1) - 2$.

It is easy to see that the symmetric difference of two $k$-unitrades equivalent to $W_{k}$ and having a non-empty intersection consists of $2k$ blocks. Let $|U| = 2k$. Counting pairs $(a, u)$ such that $u \in U$, $a \in U$ we find an element $a$ which occurs not greater than $|U|/k/\text{supp}(U) \leq 2k/(k + 2) < 2k - 2$ times in blocks of $U$. We have that $|U|/a < 2k - 2$, consequently $U_{a}^{1}$ is equivalent to $W_{k-1}$. If $W_{k} \subset U$ then $|U| \geq 2k + 2$ by Propositions 8(1) and 9. Thus there exist a block $v \in W_{k}$ such that $W_{k} \setminus \{v\} \subset U$ and $|U \triangle W_{k}| = k + 1$. By Propositions 8(1) and 9 $U$ is a symmetric difference of two $k$-unitrades equivalent to $W_{k}$.  □

We will denote by $R_{k}$ a $k$-unidade of cardinality $2k$. Examples of 5-unitrades.

1. $W_{5} = \{(1, 2, 3, 4, 5), \{2, 3, 4, 5, 6\}, \{1, 3, 4, 5, 6\}, \{1, 2, 4, 5, 6\}, \{1, 2, 3, 4, 6\}\}$.

2. $R_{5} = \{(2, 3, 4, 5, 6), \{1, 3, 4, 5, 6\}, \{1, 2, 4, 5, 6\}, \{1, 2, 3, 5, 6\}, \{1, 2, 3, 4, 6\}\}$ \cup \{(2, 3, 4, 5, 7), \{1, 3, 4, 5, 7\}, \{2, 4, 5, 7\}, \{1, 2, 3, 5, 7\}, \{1, 2, 3, 4, 7\}\}.

Notice that all coordinates of $W_{k}$ are equivalent, i.e., any permutation of $\{1, \ldots, k + 1\}$ preserves $W_{k}$. But only the first five coordinates of $R_{5}$ are equivalent.

**Proposition 13.** There exists a unique 4-unidade of cardinality 9 up to equivalence: $P = \{(1, 2, 5, 6), \{1, 3, 5, 6\}, \{2, 3, 5, 6\}, \{1, 2, 4, 6\}, \{1, 3, 4, 6\}, \{2, 3, 4, 6\}, \{1, 2, 4, 5\}, \{1, 3, 4, 5\}, \{2, 3, 4, 5\}\}$.

Proof. Let $U$ be a 4-unidade of cardinality 9. There exists an element (for example, 1) which occurs $i$ times in all 4-blocks of $U$, where $i \leq 9 \cdot 4/|\text{supp}(U)|$. By Proposition 12 $|\text{supp}(U)| \geq 6$. By Proposition 8 $U_{a}^{1}$ is a 3-unidade. By Proposition 11 $i$ is even. So, we have that $i = 4$ or $i = 6$. If $i = 4$ then $U_{a}^{1} = W_{3}$ and $U = W_{4}^{1} \triangle W_{4}^{2}$, i.e., $|U| = 8$. If $i = 6$ then $U_{a}^{1} = R_{3}$. It can be
verified directly $U$ is equivalent to $P$ in this case. □

**Proposition 14.** Every 5-unitrade of cardinality 12 is either a union of two disjoint 5-unitrades $W_5^1$ and $W_5^2$ or up to equivalence equal to $S = \{\{1, 2, 3, 5, 6\}, \{1, 2, 4, 5, 6\}, \{1, 3, 4, 5, 6\}, \{2, 3, 4, 5, 6\}, \{1, 2, 3, 5, 7\}, \{1, 2, 4, 5, 7\}, \{1, 3, 4, 5, 7\}, \{2, 3, 4, 5, 7\}, \{1, 2, 3, 6, 7\}, \{1, 2, 4, 6, 7\}, \{1, 3, 4, 6, 7\}, \{2, 3, 4, 6, 7\}\}.

Proof. Let $U$ be an element of cardinality 12. Then there exists an element (for example, 1) which occurs minimal times $i$, $i \leq 12 \cdot 5/|\text{supp}(U)|$. By Proposition 9 we get $|\text{supp}(U)| \geq 7$ and $i \leq 8$. By Proposition 8(4), $U'_1$ is a 4-unitrade. By Propositions 9 and 12, we have that $i = 5$ or $i = 8$. If $i = 5$ then $U'_1 = W_4$. If there exists $W_5^3 \subset U$ then $U$ is a union of two disjoint 5-unitrades $W_5^1$ and $W_5^2$, otherwise $U = W_5^1 \triangle B$, where $B$ is a 5-unitrade by Proposition 8(1) and $|B| = 12 - 6 + 2 = 8$. The second case is impossible by Proposition 12. If $i = 8$ then $U'_1 = R_4$. In this last case it is not difficult to verify directly that $U$ is equivalent to $S$. □

**Proposition 15.** If $U$ is a 5-unitrade and $|U| = 16$ then there exists an element that belongs to 5 or 8 blocks of $U$.

Proof. Firstly, we prove that $|\text{supp}(U)| \geq 8$. If $|\text{supp}(U)| = 7$ then there are $\binom{5}{7} = 35$ 4-blocks of 7 elements. Since $(5 \cdot 16)/35 > 2$ there exists a 4-block contained in four 5-blocks. But over the set of cardinality 7 there exist only three 5-blocks covering a 4-block.

Secondly, we prove that there are no 5-unitrades $U$ such that 8 elements occur 10 times each of them in blocks of $U$. Any triplet of elements is contained in at least three (or zero) blocks of $U$ by Propositions 8(4) and 9. The number of all possible triplets of 8 elements is equal to $\binom{8}{3} = 56$. The number of triplets contained in blocks of $U$ is equal to $16 \cdot \binom{8}{3} = 160$. Then, there exists a triplet of elements that is not included in blocks of $U$. Each pair of elements is contained in 0, 4, 6 or more blocks of $U$ by Propositions 8(4), 9 and 11. If each element occurs 10 times in blocks of $U$ then each pair of elements in this triplet is contained in four or six blocks. It is easy to verify directly that one pair is contained in six blocks and each of the two other pairs is contained in four blocks. Consider these three subsets $O_1$, $O_2$ and $O_3$ of $U$, $|O_1| + |O_2| + |O_3| = 14$. By Propositions 9 and 12, $((O_1')')'$ is equivalent to $W_3$ or $R_3$, where $a, b$ are from the triplet. By direct calculations, we find elements that are contained at most in seven blocks of $O_1 \cup O_2 \cup O_3$ and consequently, these elements belong to less than ten blocks of $U$. We have a contradiction.

So, if $|\text{supp}(U)| = 8$ then there exists an element occurring $i \leq 9$ times in blocks of $U$. By Propositions 8(4) and 9, $i$ can be equal to 5, 8 or 9. If $|\text{supp}(U)| > 8$ then elements occur less than $5 \cdot 16/|\text{supp}(U)| < 10$ times in average. If some element belongs to nine blocks of $U$ then by Proposition 13 there exists another element belonging to at most $16 - 9 = 7$ blocks of $U$ because $|\text{supp}(U)| = 8 > |\text{supp}(P)| + 1$. By Propositions 8(4) and 12 there exists an element occurring 5 times. □

**Theorem 1** (characterization of 5-unitrade of cardinality 16).

Up to equivalence all 5-unitrades of cardinality 16 are exhausted by the following list:

1) disjoint union of non-intersecting $W_5$ and $R_5$;
2) $E = S \triangle W_5 = \{\{1, 2, 3, 5, 6\}, \{1, 2, 4, 5, 6\}, \{1, 3, 4, 5, 6\}, \{2, 3, 4, 5, 6\}, \{1, 2, 3, 5, 7\}, \{1, 2, 4, 5, 7\}, \{1, 3, 4, 5, 7\}, \{2, 3, 4, 5, 7\}, \{1, 2, 3, 6, 7\}, \{1, 2, 4, 6, 7\}, \{1, 3, 4, 6, 7\}\} ∪ \{\{2, 3, 4, 6, 8\}, \{2, 3, 4, 7, 8\}, \{2, 3, 6, 7, 8\}, \{2, 4, 6, 7, 8\}, \{3, 4, 6, 7, 8\}\};
3) $F = S^1 \triangle S^2 = \{\{1, 2, 3, 5, 7\}, \{1, 2, 4, 5, 7\}, \{1, 3, 4, 5, 7\}, \{2, 3, 4, 5, 7\}, \{1, 2, 3, 6, 7\}, \{1, 2, 4, 6, 7\}, \{1, 3, 4, 6, 7\}, \{2, 3, 4, 6, 7\}, \{1, 2, 3, 6, 8\}, \{1, 2, 4, 6, 8\}, \{1, 3, 4, 6, 8\}, \{2, 3, 4, 6, 8\}$.
\{1, 2, 3, 5, 8\}, \{1, 2, 4, 5, 8\}, \{1, 3, 4, 5, 8\}, \{2, 3, 4, 5, 8\}\}

Proof. Let \(U\) be a 5-unitrade of cardinality 16. By Proposition 15, some element (without loss of generality, 1) belongs to \(m = 5\) or \(m = 8\) blocks of \(U\).

If \(m = 5\) then \(U'_1\) is equivalent to \(W_4\) by Proposition 9. There exists \(W_5\) such that \((W_5)'_1 = U'_1\). By Proposition 8(1), the symmetric difference \(W_5 \Delta U\) is a unitrade. Since \(|W_5 \cap U| = 5\) or \(|W_5 \setminus U| = 6\), we obtain that \(|W_5 \Delta U| = 12\) or \(W_5 \subseteq U\). In the case \(|W_5 \Delta U| = 12\), the set \(W_5 \Delta U\) is equivalent to \(S\) or a union of two disjoint unitrades \(W_5^1\) and \(W_5^2\) by Proposition 14. In the second case, we have again that \(W_5^1 \subset U\) or \(W_5^2 \subset U\). In this case the set \(W_5 \Delta U\) is equivalent to \(R_5\) by Proposition 12. Therefore, in the first case, \(U\) is the symmetric difference of \(S\) and \(W_5\) and in the second case, \(U\) is a union of disjoint unitrades \(W_5\) and \(R_5\). Using condition \(|W_5 \cap S| = 1\) and transitivity of \(W_5\) and \(S\), we can verify directly that such symmetric differences \(S \Delta W_5\) are equivalent to \(E\).

If \(m = 8\) then \(U'_1\) is equivalent to \(R_4\) by Proposition 12. By direct calculation, we obtain that \(S'_1\) and \(S'_2\) is equivalent to \(R_4\). Then \(|U \setminus S'| \leq 12\) for some unitrade \(S'\). By Proposition 8(1), the set \(A = U \setminus S'\) is a unitrade. By Proposition 11, the cardinality of \(A\) is even. By Propositions 9, 12 and 14 \(|A| = 6, 10\) or 12 and the set \(A = U \setminus S'\) is a unitrade equivalent to \(W_5\) (case (a)), \(R_5\) (case(b)), \(S\) (case (c)) or it is a union of two disjoint copies of \(W_5\) (case (d)).

In the case (a) we get \(W_5 = U \setminus S^1\), consequently \(U = W_5 \setminus S^1\). Then \(U\) is equivalent to \(E\). In the case (c) we have that \(S^2 = U \setminus S^1\), consequently \(U = S^2 \setminus S^1\). Using condition \(|S^1 \setminus S^2| = 4\) and transitivity of the unitrade \(S\), we can verify directly that such symmetric differences \(S^1 \setminus S^2\) are equivalent to \(F\). In the case (d) \(U\) includes \(W_5\) (it can be verified directly). Therefore, \(U\) is a union of disjoint \(W_5\) and \(R_5\). In the case (b) \(|U \setminus S'\| = 9\) and one of eight elements, without loss of generality, the element 8 does not belong to support of \(S'\). Then element 8 occurs in \(U\) at most 7 times. By Propositions 8, 9, 12 it occurs 5 times. The case \(m = 5\) is considered above. □

There are some nonequivalent 5-unitrades of type (1), three of which will be used below:

\[H^1 = W^3_2 \cup R^1_3 = \{1, 2, 3, 4, 5, 6, 7\}, \{1, 3, 4, 5, 8\}, \{1, 2, 4, 5, 8\}, \{1, 2, 3, 5, 8\}, \{1, 2, 3, 4, 8\}\}
\[H^2 = W^3_2 \cup R^1_3 = \{1, 2, 3, 4, 6, 7\}, \{1, 2, 4, 5, 6\}, \{1, 2, 3, 5, 6\}, \{1, 2, 3, 4, 6\}\]
\[H^3 = W^3_2 \cup R^1_3 = \{1, 2, 3, 4, 7\}, \{1, 3, 4, 5, 7\}, \{1, 2, 4, 5, 7\}, \{1, 2, 3, 5, 7\}\]

5 Uniqueness of non-2-DP-colorable 5-uniform hypergraph with 16 edges

Let \(A\) be a collection of pairs of antipodal \((n - k)\)-faces. For \(a \in A\) we define \(k\)-block \(\beta(a) = \{i_j : a_{i_j} \in \{0, 1\}\}\) and \(\beta(A) = \{\beta(a) : a \in A\}\).

**Lemma 1.** If \(A\) is an antipodal \(k\)-splitting of \(Q_n^k\) then \(\beta(A)\) is a \(k\)-unitrade with cardinality \(2^{k-1}\) over an \(n\)-element set.

Proof. Let \(A\) be an antipodal \(k\)-splitting of \(Q_n^k\). Consider the indicator \(\chi_a\) for \(a \in A\). \(\chi_a\) can be defined by the monomial \(\chi_a(x) = (x_{i_1} \oplus a_{i_1}) \cdots (x_{i_k} \oplus a_{i_k})\), where \(a_{i_j} \in \{0, 1\}\). Analogously,
we have that \( \chi_\pi(x) = (x_{i_1} \oplus a_{i_1} \oplus 1) \cdots (x_{i_k} \oplus a_{i_k} \oplus 1) \). Then we obtain

\[
\chi_a(x) \oplus \chi_\pi(x) = x_{i_1} x_{i_2} \cdots x_{i_{k-1}} \oplus x_{i_1} x_{i_2} \cdots x_{i_{k-2}} x_{i_k} + \cdots + x_{i_1} x_{i_3} \cdots x_{i_k} \oplus h_a(x),
\]

where \( \deg(h_a) < k - 1 \).

For each block \( b = \{i_1, \ldots, i_m\} \subset \{1, \ldots, n\} \) we define a monomial \( x_b = x_{i_1} \cdots x_{i_m} \). Each pair of antipodal \((n - k)\)-faces \( a \) and \( \overline{a} \) correspond to the sum of monomials \( \sum_{b \in \beta(a), |b| = k-1} x_b \) of degree \( k - 1 \). Since \( A \) is a splitting, the equality \( \sigma = \sum_{a \in A} \chi_a(x) = 1 \) holds. Consequently, the sum of all monomials of degree \( k - 1 \) in \( \sigma \) is equal to zero, i.e., \( \sum_{a \in A} \sum_{b \in \beta(a), |b| = k-1} x_b = 0 \). So, each monomial \( x_b \) occurs an even number of times in \( \sigma \) and the set \( \beta(A) \) is a \( k \)-unitrade.

Note that the table which columns are indicators of blocks of \( \beta(A) \) is the incidence matrix of non-2-DP-colorable \( 5 \)-uniform hypergraph corresponding to an antipodal \( k \)-splitting \( A \). By Propositions 3 and 9 it follows

**Corollary 5.** There is unique non-2-DP-colorable 3-uniform hypergraph with 4 edges.

The antipodal 5-splitting from Propositions 3 corresponds to 5-unitrade \( E \) from Theorem 1. At the last part of the article, we prove that the non-2-DP-colorable 5-uniform hypergraph is unique.

The following statement is proved in [5] and [7].

**Proposition 16** (see [5]). If \( \{T_0, T_1\} \) is a \((k-1) - (n,k)\) bitrade then \( |T_0| = |T_1| \) and \( 2|T_0| \geq 2^k \).

From Lemma 1 and Proposition 16 it follows that all unitrades corresponding to antipodal splittings are not bitrades.

**Proposition 17.** An antipodal \( k \)-splitting of \( Q^n_k \) does not exist for all \( n \leq k + 2 \), \( k \geq 5 \).

Proof. Let \( n = k + 2 \). There exist only three \( k \)-element blocks containing a fixed \((k - 1)\)-element block. Consequently each \((k - 1)\)-block is subset of 0 or 2 \( k \)-blocks of the \( k \)-unitrade corresponding to a splitting. The cardinality of the \( k \)-unitrade corresponding to an antipodal \( k \)-splitting is equal to \( 2^{k-1} \). The number of \((k - 1)\)-element blocks, which are covered by \( k \)-element blocks, is \( k \cdot 2^{k-1} > 2 \cdot \binom{k+2}{k-1} \). Then there exists a \((k - 1)\)-element block covered more than twice. We have a contradiction. The case \( n = k + 1 \) is similar to the case \( n = k + 2 \). □

**Proposition 18.** Let \( A \) be an antipodal \( k \)-splitting. Then \( |u_1 \cap u_2| \geq 2 \) for all \( k \)-blocks \( u_1, u_2 \in \beta(A) \).

Proof. If \( u_1 \cap u_2 = \emptyset \) then the \((n - k)\)-face corresponding to \( u_1 \) and the \((n - k)\)-face corresponding to \( u_2 \) have a non-empty intersection. Otherwise without loss of generality we suppose that \( u_1 \cap u_2 = \{1\} \). Let \( a \in A \), \( \beta(a) = u_1 \) and \( a_1 = 0 \). Then there exist \( b, \overline{b} \) in \( A \) such that \( \beta(b) = \beta(\overline{b}) = u_2 \). If \( b_1 = 0 \) then \( a \cap b \neq \emptyset \). Otherwise \( \overline{b}_1 = 0 \) and \( a \cap \overline{b} \neq \emptyset \). □

Let \( a = (a_1, \ldots, a_n) \) be a \((n - k)\)-face of \( Q^n_k \). Define a projection \( P_{1,\ldots,k}[a] : Q^n_k \to \mathbb{N} \) onto the first \( k \) coordinates by the equation \( P_{1,\ldots,k}[a](x_1, \ldots, x_k) = \{ (x_1, \ldots, x_k, y_{k+1}, \ldots, y_n) \in a \} \). The weight spectrum of the projection \( P_{1,\ldots,k}[a] \) is the tuple \( w(P_{1,\ldots,k}[a]) = (z_0, z_1, \ldots, z_k) \), where \( z_i = \sum_{|x|=i} P_{1,\ldots,k}[a](x) \). It is easy to see that \( \sum_{a \in A} w(P_{1,\ldots,k}[a]) = (2^{n-k}, k2^{n-k}, \ldots, (k)2^{n-k}, \ldots, 2^{n-k}) = 2^{n-k}w(Q^n_k) \), where \( A \) is a splitting of \( Q^n_k \). Denote by \( \varrho(a, b) \) the distance between faces \( a \) and \( b \), i.e., a number of positions \( i \) such that \( a_i = 0 \) and \( b_i = 1 \) or vice versa. Obviously, isometries of the hypercube preserve this distance.

**Lemma 2.** Let \( A \) be an antipodal 5-splitting, \( u = \beta(a) \), \( a \in A \). If there exist only five \( v \in \beta(A) \) such that \( |\text{supp}(u) \cap \text{supp}(v)| = 4 \) then \( \varrho(a, b) \) is odd for any \( b \in A \) such that
\(|\text{supp}(u) \cap \text{supp}(\beta(b))| = 4.\)

Proof. Without loss of generality, we suppose that \(a^0 = (0, 0, 0, 0, 0, *, *, \ldots) \in A.\) Consider a sum of weight spectra of projections of two antipodal \((n - 5)\)-faces \(a, \pi \in A\) onto the first five coordinates. The sum depends on the number of asterisks in the first five coordinates of \(a.\) If the number of asterisks in the first five coordinates of \(a\) is zero \((a = a^0)\) then the sum of weight spectra of two antipodal \((n - 5)\)-faces is \((2n^5, 0, 0, 0, 0, 2n^5);\) if both numbers of asterisks and 1s (or 0s) in the first five coordinates of \(a\) are equal to one then the sum of weight spectra is \((0, 2n^6, 2n^6, 2n^6, 0)(\text{type I});\) if the number of asterisks equals one and the number of 1s equals two then the sum of weight spectra is \((0, 0, 2n^5, 2n^5, 0, 0)(\text{type II});\) if the number of asterisks is equal to 2 or 3 then the sum of weight spectra is \((0, 2n^7, 3 \cdot 2n^7, 3 \cdot 2n^7, 0)(\text{type III}).\) The cases of corresponding to 4 or 5 asterisks in the first five coordinates of \(a\) are impossible by Proposition 18. The sum of all weight spectra of the projection of the whole splitting \(A\) is equal to \((2n^5, 5 \cdot 2n^5, 5 \cdot 2n^5, 5 \cdot 2n^5, 2n^5, 2n^5).\) The sum of weight spectra of two pairs of types I and II is equal to the sum of weight spectra of two pairs of type III. It holds that

\[(0, 5 \cdot 2n^5, 10 \cdot 2n^5, 10 \cdot 2n^5, 5 \cdot 2n^5, 0) = 5(0, 2n^6, 3 \cdot 2n^6, 3 \cdot 2n^6, 2n^6, 0) + 5(0, 2n^6, 2n^6, 2n^6, 2n^6, 0).\]

Consequently, the difference between the number of pairs of \((n - 5)\)-faces of type I and the number of pairs of \((n - 5)\)-faces of type II is equal to 5.

By hypothesis of the lemma the total number of \((n - 5)\)-faces of types I and II is 5. Then all of them have type I. If \(b\) is an \((n - 5)\)-faces of type I then \(g(a, b) = 1\) and \(g(a, \overline{b}) = 3.\) □

**Corollary 6.** Let \(A\) be an antipodal 5-splitting. If \(\beta(A)\) is simple then \(g(a, b)\) is odd for any \(a, b \in A\) such that \(\text{supp}(\beta(a)) \cap \text{supp}(\beta(b)) = 4.\)

**Proposition 19.** Suppose

(a) \(A\) and \(B\) are sets consisting of three antipodal pairs of faces,

(b) \(\text{supp}(\beta(a)) \cap \text{supp}(\beta(b)) \subset \{1, 2, 3, 4\}\) for any \(a \in A\) and \(b \in B,\)

(c) each \(a \in A\) and \(b \in B\) contains one asterisk and one 1 (or one asterisk and two 1 for antipodal faces) in the first four coordinates.

Then \(\bigcup_{a \in A} a \cap \bigcup_{b \in B} b \neq \emptyset.\)

Proof. Denote by \(A_4\) and \(B_4\) projections of \(A\) and \(B\) onto the first four coordinates. By condition (c) \(A_4\) and \(B_4\) consist of 1-faces (edges) of 4-dimensional Boolean cube. By condition (b) \(a \in A\) and \(b \in B\) are disjoint if and only if their projections \(A_4\) and \(B_4\) are disjoint. It is easy to see that the union of three pairs of antipodal edges in \(Q_n^3\) contains 8 vertices at least, i.e., \(|\bigcup_{a \in A_4} a| \geq 8.\) By condition (c) \(\overline{0,1} \notin (\bigcup_{a \in A_4} a) \cup (\bigcup_{b \in B_4} b).\) Therefore sets \(\bigcup_{a \in A_4} a\) and \(\bigcup_{b \in B_4} b\) are intersected. □

A cycle in a Boolean hypercube is called antipodal if each pair of edges of the same direction is antipodal. The length of an antipodal cycle in \(Q_n^3\) is \(2n\) because the distance between antipodal vertices is equal to \(n.\) It is easy to see that all antipodal cycles are equivalent.

**Proposition 20.** There are no three disjoint antipodal cycles of length 10 in \(Q_n^3 \setminus \{0,1\}\).

Proof. Suppose that such three cycles exist. There are exactly 5 vertices of \(Q_n^3\) with weight 1. If a cycle in \(Q_n^3 \setminus \{0,1\}\) contains three vertices with weight 1 then it contains four vertices with weight 2. Such cycle is not antipodal. Then one of the three antipodal cycles has only one vertex with weight 1. Without loss of generality (because all antipodal cycles are equivalent) we suppose that this cycle consists of vertices

\[1000, 11000, 11010, 01010, 01011, 01111, 00111, 00101, 10101, 10100.\]
For vertices 01000 and 00110 there are only three unused neighbors with weight 2: 00110, 01001, 01100. Then one of two remaining cycles contains vertices 01000, 00110, 00111, 00011, 10001 and antipodal vertices 01101, 11101, 11100, 11110, 01110. But vertices 10010 and 01110 (or 01101) are not adjacent. We have a contradiction. □

**Theorem 2.** If $A$ is antipodal 5-splittings of $Q_5^2$ then $\beta(A) = E$.

Proof. By Lemma [1] and Theorem [1] it is sufficient to prove that any collection $A$ of $(n - 5)$-faces of $Q_5^2$ corresponding to other types of 5-unitrades $\beta(A)$ is not splitting.

1. Let $\beta(A)$ be a union of non-intersecting two 5-unitrades. Without loss of generality the first of them $W_2^3$ is equivalent to $R_5$ and the second one is equal to $R_5$, supp($R_5$) = {1, ..., 7}. Denote by $C$ the support of $W_2^3$. If $C \subset \{1, ..., 7\}$ then $A$ is not an antipodal 5-splitting by Proposition [17].

If $|C \cap \{1, ..., 7\}| \leq 3$ it can easily be checked by direct calculations that there exist $u_1 \in W_2^3$ and $u_2 \in R_5$ such that $|u_1 \cap u_2| \leq 1$. Then $A$ is not an antipodal 5-splitting by Proposition [18]. Let $|C \cap \{1, ..., 7\}| = 4$ and let there be not $u_1 \in W_2^3$ and $u_2 \in R_5$ such that $|u_1 \cap u_2| \leq 1$. Then we have $|C \cap \{1, ..., 7\}| = 4$ by direct calculation. Since the first five coordinates of $R_5$ are equivalent and all coordinates of $W_2^3$ are equivalent, $\beta(A)$ is equivalent to $H^3$ (see an example below Theorem [1]). By definition $\{\{1, 2, 4, 8, 9\}, \{2, 3, 4, 8, 9\}, \{1, 3, 4, 8, 9\}\} \cup \{\{1, 2, 4, 5, 6\}, \{2, 3, 4, 5, 6\}, \{1, 3, 4, 5, 6\}\} \subset H^3$ and $\{1, 2, 3, 4, 7\} \in H^3$. Denote by $B \subset A$ the set of faces corresponding to blocks $\{1, 2, 4, 8, 9\}, \{2, 3, 4, 8, 9\}, \{1, 3, 4, 8, 9\}$ and by $D \subset A$ the set of faces corresponding to blocks $\{1, 2, 4, 5, 6\}, \{2, 3, 4, 5, 6\}, \{1, 3, 4, 5, 6\}$. Without loss of generality we suppose that $a = (0, 0, 0, 0, *, 0, *, 0, 0) \in A$. Since each $b \in B$ and $d \in D$ do not intersect $a, B$ and $D$ satisfy the hypothesis of Proposition [19]. Then there exist $b \in B$ and $d \in D$ with nonempty intersection and $A$ is not splitting.

Consider the case $|C \cap \{1, ..., 7\}| = 5$. If $\{1, ..., 5\} \subset C$ then $\beta(A)$ is equivalent to $H^1$ (see an example below Theorem [1]). Indeed, in this case $\beta(A)$ is unique up to equivalence because all coordinates of $W_2^3$ are equivalent. Consider the projection of $H^1$ onto coordinates $\{1, ..., 5\}$.

It is clear that $H^1$ consists of 3 subsets with equal projections corresponding to $\{2, 3, 4, 5\}, \{1, 3, 4, 5\}, \{1, 2, 4, 5\}, \{1, 2, 3, 5\}, \{1, 2, 3, 4\}$ and a pair of antipodal vertices $v, \bar{v} \in Q_5^2$. Projections of each of 3 parts are cycles that contain 5 pairs of antipodal edges (1-faces). These cycles are pairwise disjoint and do not contain vertices $v, \bar{v}$. Then $A$ is not an antipodal 5-splitting by Proposition [20].

In other cases we have $|C \cap \{1, ..., 5\}| = 4$ and $|C \cap \{6, 7\}| = 1$ (*) or $|C \cap \{1, ..., 5\}| = 3$ and $|C \cap \{6, 7\}| = 2$ (**). By direct calculations we obtain that the case (*) is impossible because such 5-unitrade $W_2^3$ and $R_5$ are intersected. In the case (**) $\beta(A)$ is equivalent to $H^2$ (see examples after Theorem [1]). By definition blocks $u_1 = \{2, 3, 6, 7, 8\}, u_2 = \{1, 3, 6, 7, 8\}, u_3 = \{1, 2, 6, 7, 8\}$ belong to $H_2$. It is easy verify that each face $b \in A$ such that $\beta(b) = u_i, i = 1, 2, 3, 5$ satisfy the hypothesis of Lemma [2]. Moreover, $\{1, 2, 3, 6, 7\} \in H^2$ then without loss of generality we suppose that $a = (0, 0, 0, *, 0, 0, *, 0) \in A$. We have supp$(\beta(a)) \cap \text{supp}(u_i) = 4$. Then by Lemma [2] their exist $b^i_1, b^i_2, b^i_3, b^i_4 \in A$ such that $\beta(b^i) = u_i$ and $\text{proj}(a, b^i) = 1$ for $i = 1, 2, 3$. By direct calculation we can verify that $b_1^i = 0$ for $i = 1, 2, 3$ or $b_2^i = 0$ for $i = 1, 2, 3$. For example, let $b_1^1 = 1, b_2^1 = 1$ and $b_3^1 = 1$. By Lemma [2] $\text{proj}(b^i, b^j) = 3$ for $i \neq j$ then $b_1^i \neq b_1^j$ for $i \neq j$. But $b_1^i$ can take only two values 0 or 1. Without loss of generality we assume that $b_1^i = 0$ for $i = 1, 2, 3$. Denote by $B \subset A$ the set of faces corresponding to blocks $u_1, u_2, u_3$ and by $D \subset A$ the set of faces corresponding to blocks $\{1, 2, 4, 5, 6\}, \{2, 3, 4, 5, 6\}, \{1, 3, 4, 5, 6\}$. Since each $b \in B$ and $d \in D$ do not intersect $a$, after renaming letters $4 \leftrightarrow 6$ $B$ and $D$ satisfy the hypothesis of Proposition [19]. Then there exist $b \in B$ and $d \in D$ with nonempty intersection and $A$ is not splitting.
2. Let $\beta(A)$ be equal to $F$. Suppose that $A$ is an antipodal 5-splitting. Consider the projection of $A$ onto the first six coordinates. We obtain that the first part $D$ of the projection corresponds to

$$\beta(D) = \{\{1, 2, 3, 5\}, \{1, 2, 4, 5\}, \{1, 3, 4, 5\}, \{2, 3, 4, 5\}, \{1, 2, 3, 6\}, \{1, 2, 4, 6\}, \{1, 3, 4, 6\}, \{2, 3, 4, 6\}\}$$

and the second part $D'$ of the projection corresponds to the same unitrade $\beta(D') = \beta(D)$. Moreover, $D$ and $D'$ are double coverings because each $a \in A$ has only one asterisk in the eighth (in case $D$) or in the seventh (in case $D'$) position. A doubled cardinality of each parts of the projection equals $2|\beta(D)| \cdot 2^3 = 2^7$. Then both parts $D$ and $D'$ are antipodal 4-splittings of $Q_6^2$, that contradicts to Proposition [17].

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References


