On abelian versions of Critical Factorization Theorem *

Extended abstract

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Abstract

In the paper we study abelian versions of the Critical Factorization Theorem. We investigate both similarities and differences between the abelian powers and the usual powers. The results we obtained show that the constraints for abelian powers implying periodicity should be quite strong, but still natural analogies exist.

1 Introduction

One of the main results of combinatorics on words, the Critical Factorization Theorem, relates local periodicities of a word to its global periodicity. It was first proved by Y. Cèsari and M. Vincent, and in the present form it is due to J. Duval [2, 3]. This theorem states, roughly speaking, a connection between local and global periods of a word; the local period at any position of the word is defined as the shortest repetition centered in this position. In [8] Mignosi, Restivo and Salemi proposed a different notion of a local period: a local period is defined as the length of the shortest repetition to the left from this position. In such a definition of local periods square is not enough to ensure the global periodicity, but the threshold is

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surprisingly given by the golden ratio $\varphi$. Namely, if every sufficiently long prefix of an infinite word has a $\varphi^2$-repetition as a suffix, then the sequence is ultimately periodic. The quantity $\varphi^2$ is optimal in the sense that the claim does not hold true for any real number less than $\varphi^2$. In a related paper [5] and PhD thesis of A. Lepistö [6] bounded periods were considered. The authors established strict borderline for periods of squares distinguishing non-periodicity and ultimate periodicity. Powers immediately to the right from each position were recently studied by K. Saari [10].

In combinatorics of words, abelian analogs of classical problems are often considered, such as abelian complexity, avoidance, abelian powers and their generalizations [1, 9]. The purpose of our research is to find abelian analogues of Critical Factorization Theorem. We seek for constraints for abelian powers enforcing a word to be periodic. Our results show that such constraints should be quite strong, but still natural analogies exist. In particular, for every integer $k$ we construct non-periodic words having bounded $2k$-powers centered at every position. We provide bounds for periods in abelian squares enforcing periodicity. In addition, we study abelian powers in Sturmian and Thue-Morse words.

2 Preliminaries

2.1 Central Factorization Theorem and its variations

In this section we present some necessary preliminary definitions and results concerning classical Critical Factorization Theorem.

Given a finite non-empty set $\Sigma$ (called the alphabet), we denote by $\Sigma^*$, $\Sigma^\omega$ and $\Sigma^\mathbb{Z}$, respectively, the set of finite words, the set of (right) infinite words, and the set of biinfinite words over the alphabet $\Sigma$. Given a finite word $u = u_1u_2 \ldots u_n$ with $n \geq 1$ and $u_i \in \Sigma$, we denote the length $n$ of $u$ by $|u|$. The empty word will be denoted by $\varepsilon$ and we set $|\varepsilon| = 0$. A factor of a word is any consecutive sequence of its symbols. We say that a biinfinite word $w$ is periodic, if there exists a finite word $v$ such that $w = v^\mathbb{Z}$. A right infinite word $w$ is ultimately periodic, if for some finite words $u$ and $v$ it holds $w = uv^\omega$; $w$ is purely periodic (or briefly periodic) if $u = \varepsilon$.

In the Critical Factorization theorem, given a word $a_1a_2 \ldots a_n$, for any integer $i$ ($1 \leq i \leq n - 1$) one looks at the shortest repetition (a square) centered in that position, i. e., one looks at the shortest (virtual) suffix of $a_1a_2 \ldots a_i$ which is also a (virtual) prefix of $a_i a_{i+2} \ldots a_{i+n}$. The local period at position $i$ is defined as the length of this shortest suffix. The Critical Factorization Theorem states that the global period of $a_1 \ldots a_n$ is the maximum of the local periods. We are interested in consequences of this theorem for infinite words (see, e. g., [7], p. 296–297):

**Theorem 1.** A biinfinite word $w$ is periodic if and only if there exists an integer $l$ such that $w$ has at every position a centered square with period at most $l$. 


**Theorem 2.** A right infinite word \( w \) is periodic if and only if there exists an integer \( l \) such that \( w \) has at every position a (virtual) centered square with period at most \( l \).

In [5, 8] by local period the authors mean powers immediately to the left from each position. Denote by \( \text{pref}_k(w) \) a prefix of \( w \) of length \( k \). For a rational number \( k \geq 1 \), we say that a word \( w \) is a \( k \)-th power if there exists a word \( v \) such that \( w = v^k \), where \( v^k \) denotes the word \( v'v'' \) with \( v' = v^{\lfloor k \rfloor} \) and \( v'' = \text{pref}_{|v|(|k|−|k|)}v \). Next we say that \( w \) contains a repetition of order \( \rho \geq 1 \) if it contains as a factor a \( k \)-th power with \( k \geq \rho \). Note that here \( \rho \) is allowed to be any real number \( \geq 1 \). We say that \( w \) has a \( \rho \)-suffix, if \( w \) contains a repetition of order \( \rho \) as its suffix.

A remarkable result of Mignosi, Restivo and Salemi [8] connects \( \rho \)-suffixes with periodicity:

**Theorem 3.** A right-infinite word \( w \) is ultimately periodic if and only if there exists \( n_0 \) such that for every \( n \geq n_0 \) the word \( \text{pref}_n(w) \) has a \( \varphi^2 \)-suffix, where \( \varphi = (1 + \sqrt{5})/2 \).

As also shown in [8], Theorem 3 is optimal:

**Theorem 4.** For any real number \( \varepsilon > 0 \) there exists a natural number \( n_0 > 0 \) such that for any \( n \geq n_0 \) the prefix of length \( n \) of the infinite Fibonacci word has a \( (\varphi^2 - \varepsilon) \)-suffix.

In [5] powers with bounded periods are considered. Let \( \rho \geq 1 \) be a real number and \( p \geq 1 \) an integer. An infinite word \( w \) is \((\rho, p)\)-repetitive if there exists an integer \( n_0 \) such that each prefix of \( w \) of length at least \( n_0 \) ends with a repetition of order \( \rho \) of a word of length at most \( p \). It is clear that \((\rho, p)\)-repetitivity implies \((\rho, p')\)-repetitivity for any \( p' \geq p \) and \((\rho', p)\)-repetitivity for any \( \rho' \leq \rho \). The goal of the paper [5] is to establish connection between \((\rho, p)\)-repetitive and ultimately periodic words. In the paper it is proved, that there exist non-ultimately periodic \((2, 5)\)-repetitive words, and all such words are described. Moreover, the pair \((2, 5)\) is optimal in the sense that \((\rho, p)\)-repetitive word with \( \rho > 2 \) and \( p = 5 \) or \( \rho = 2 \) and \( p = 4 \) is ultimately periodic.

### 2.2 Definitions and notation

Here we define our basic notions for the abelian case. Given a finite word \( u = u_1u_2 \ldots u_n \) with \( n \geq 1 \) and \( u_i \in \Sigma \), for each \( a \in \Sigma \), we let \( |u|_a \) denote the number of occurrences of the letter \( a \) in \( u \). Two words \( u \) and \( v \) in \( \Sigma^* \) are **abelian equivalent** if and only if \( |u|_a = |v|_a \) for all \( a \in \Sigma \). It is easy to see that abelian equivalence is indeed an equivalence relation on \( \Sigma^* \). An **abelian \( k \)-power** is a non-empty word of
the form \( u = v_1 v_2 \ldots v_k \) where the words \( v_i \) are pairwise abelian equivalent. In this case we refer to \( |v_1| \) as length of abelian period of \( u \) (or briefly period).

Let \( u \) be a biinfinite word over an alphabet \( \Sigma \); \( k \), \( l \) integers. We say that \( u \) is \((k, l)\)-abelian central repetitive (or briefly \((k, l)\)-ACR), if it has at every position a centered abelian 2\( k \)-power with length of period at most \( l \) as its factor. Formally, it means that for every \( i \) there exists \( l' \leq l \) such that \( w_{i-l'k+1} \ldots w_{i+l'k} \) is an abelian 2\( k \)-power. We say that \( u \) is \((k, l)\)-abelian right (resp. left) repetitive, if it has an abelian \( k \)-power immediately to the right (resp. left) from every position with length of period at most \( l \), i. e., for every \( i \) the word \( w_{i+1} \ldots w_{i+l'k} \) (resp. \( w_{i-l'k+1} \ldots w_i \)) is an abelian \( k \)-power for some \( l' \leq l \). We will refer to \((k, l)\)-ACR (resp. \((k, l)\)-ARR) for brevity. It is easy to see that a periodic biinfinite word with the period \( l \) is \((k, l)\)-ACR, ALR and ARR for every \( k \).

Remark that a \((k, l)\)-ACR word is also \((k, l)\)-ARR and ALR. It is easy to see that a word having at every position a centered abelian square of fixed length is periodic, and the same holds for left and right squares.

3 Central abelian powers

In this section we study central abelian powers. In particular, contrary to usual Critical Factorization Theorem, we will prove that there exist infinite non-periodic words with bounded central abelian powers at every position. To build such words, we will need some auxiliary notation.

Let \( w \) be a biinfinite word. Let \( w \) contain at every position \( i \) an arithmetical progression of one letter of length at least \( 2k+1 \) with common difference at most \( l \) centered at this position, i. e., for every \( i \) there exists \( l' \leq l \) such that \( \omega_i = \omega_{i+l'} = \omega_{i+2l'} = \omega_{i+2l'} = \cdots = \omega_{i+kl'} = \omega_{i-kl'} \). Then we say that \( w \) is \((k, l)\)-arithmetically centered.

Let \( \Sigma = \{a_1, \ldots, a_n\} \) be an alphabet. Denote by \( \mu_\Sigma \) the generalized Thue-Morse morphism:

\[
\mu_\Sigma(a_1) = a_1 a_2 \ldots a_n,
\mu_\Sigma(a_2) = a_2 a_3 \ldots a_n a_1,
\vdots
\mu_\Sigma(a_n) = a_n a_1 \ldots a_{n-1}.
\]

Notice that \( \mu = \mu_{\{0,1\}} \) is the usual Thue-Morse morphism, and its fixed point is the Thue-Morse word.

**Lemma 1.** Let \( w \) be a \((k, l)\)-arithmetically centered word over an alphabet \( \Sigma \). Then \( \mu_\Sigma(w) \) is \((k, |\Sigma|l)\)-ACR word.
Proof. Consider a position \( i \) in \( \mu_\Sigma(w) \). If \( i = |\Sigma|m \) for an integer \( m \), then we have a central abelian \( 2k \)-power with length of period \( |\Sigma| \). The period coincides with a full image of some letter, so in the period all letters from the alphabet are used once. For \( i = |\Sigma|m + r \), \( 0 < r < |\Sigma| \), if \( w \) contains at the position \( m \) a central arithmetical progression of length \( 2k + 1 \) with common difference \( l' \leq l \), then \( \mu_\Sigma(w) \) contains at the position \( i \) a central abelian \( 2k \)-power with length of period \( |\Sigma|l' \). The period includes \( l' - 1 \) full images of letters and two pieces of image of a letter \( w_m \) in the beginning and in the end of period, these two pieces constitute full image of the letter. So in the abelian period every letter is used \( l' \) times. \( \square \)

**Theorem 5.** For every integer \( k \), there exists a biinfinite non-periodic \((k, l)\)-ACR word with \( l = 2(k + 1)^2 \).

Proof. Lemma 1 implies that it is enough to build a \((k, l/2)\)-arithmetically centered word. We build it as follows:

\[
w_i = \begin{cases} 
1, & \text{if } i \text{ is not divisible by } (k + 1), \\
0, & \text{if } i = j(k + 1), j \text{ is not divisible by } (k + 1), \\
?, & \text{if } i \text{ is divisible by } (k + 1)^2.
\end{cases} \tag{1}
\]

Here \( ? \) means that we can use either 0 or 1.

To prove that this word is \((k, l/2)\)-arithmetically centered, we consider separately letters from each group from definition (1). Every \( w_i = 1 \) with \( i \) not divisible by \( k + 1 \) is a center of an arithmetical progression with common difference \( l/2 \). Every \( w_i = 0 \) with \( i = j(k + 1), j \) not divisible by \( k \), is a center of an arithmetical progression with common difference \( l/2 \). Every \( ? = 0 \) is a center of an arithmetical progression with common difference \( (k + 1) \), every \( ? = 1 \) is a center of an arithmetical progression with common difference 1.

Applying the Thue-Morse morphism to this sequence, we obtain \((k, l)\)-ACR word. \( \square \)

There are some remarks to be made about the construction from Theorem 5.

**Remark 1.** The construction gives uncountably many non-periodic \((k, 2(k + 1)^2)\)-ACR words.

**Remark 2.** Note that instead of Thue-Morse morphism we can also apply general Thue-Morse morphism for an alphabet \( \Sigma, |\Sigma| > 2 \). In this case we obtain a \((k, |\Sigma|(k + 1)^2)\)-ACR word over \( \Sigma \).

**Remark 3.** Let \( w \) be a \((k, l)\)-ACR word over an alphabet \( \Sigma \). Consider an alphabet \( \Sigma' \) satisfying \( |\Sigma'| < |\Sigma| \), and a surjective map \( \varphi : \Sigma \to \Sigma' \). Then the word \( \varphi(w) \) is a \((k, l)\)-ACR word over \( \Sigma' \). This construction allows to construct non-periodic \((k, l)\)-ACR words from non-periodic \((k, l)\)-ACR words over a larger alphabet.
Theorem 6. There exists a biinfinite non-periodic word \( w \), such that for every integer \( k \) there exists \( l \) such that \( w \) is \((k,l)\)-ACR word.

Proof. We build the word based on the construction from Theorem 5 and Toeplitz construction [4, 11]. Remind the definition of Toeplitz words. Let ? be a letter not in \( \Sigma \). For a word \( w \in \Sigma(\Sigma \cup ?)^* \), let

\[ T_0(w) = ?w, T_{i+1}(w) = F_w(T_i(w)), \]

where \( F_w(u) \), defined for any \( u \in (\Sigma \cup ?)^\omega \), is the word obtained from \( w^\omega \) by replacing the sequence of all occurrences of ? by \( u \); in particular, \( F_w(u) = w^\omega \) if \( w \) contains no ?.

Clearly,

\[ T(w) = \lim_{i \to \infty} T_i(w) \in \Sigma^\omega \]

is well-defined, and it is referred to as the Toeplitz word determined by the pattern \( w \). Let \( p = |w| \) and \( q = |w|? \) be the length of \( w \) and the number of ?s in \( w \), respectively. Then \( T(w) \) is called a \((p,q)\)-Toeplitz word.

We modify this construction as follows. First we build a \((1,l_1)\)-ACR word using the construction from Theorem 5. Notice that all elements in it except ? are centres of infinite arithmetical progressions. Now we build a \((2,l_2)\)-ACR word, using this construction, and put this word on the places of ? of previously built \((1,l_1)\)-ACR word. Continuing this line of reasoning to infinity, we obtain a non-periodic binary word, such that for every integer \( k \) there exists \( l \) such that \( w \) is \((k,l)\)-ACR word.

Theorems 5 and 6 actually show differences between usual powers and abelian powers. Contrary to Theorem 1, they give the existence of non-periodic words having central abelian powers. To obtain conditions implying periodicity, we should put quite strong constraints for the powers.

If in Theorem 6 we replace quantifiers “there exists \( l \)” and “for every \( k \)”, then we obtain an opposite result:

Theorem 7. Let \( w \) be a biinfinite word. If there exists an integer \( l \) such that for every \( k \) the word \( w \) is \((k,l)\)-ACR, then \( w \) is periodic.

Proof. Since for every \( k \) there exists an abelian centered \( k \)-power of length not greater than \( l \), we take \( k = ll \). Then for every position \( i \) there exists \( l_i \leq l \) such that at this position we have \( 2i! \)-power of length \( l_i \). So, multiplying period by \( l!/l_{i} \), we get \( 2l_i \)-power (and so \( 2 \)-power since \( 1 \leq l_i \)) with period \( l! \). Therefore, \( w \) has a central \( 2 \)-power of length \( l! \) at every position. A word having at every position a central abelian square with a fixed period is periodic.

The proof of Theorem 7 implies that it can be formulated in the following form:
**Theorem 7'.** Let \( w \) be a biinfinite non-periodic \((k, l)\)-ACR word. Then \( l \geq k! \).

**Example 1.** Here we consider a family of non-periodic \((1, 8)\)-ACR words obtained by the construction described in Theorem 5.

First we build a \((1, 4)\)-arithmetically centered word:

\[
w_i = \begin{cases} 
1, & \text{if } i \text{ is not divisible by } 2, \\
?, & \text{if } i = 4j, j \in \mathbb{Z}, \\
0, & \text{otherwise.} 
\end{cases}
\]

Instead of each ? one can use either 0 or 1. It is easy to see that each letter is a centre of an arithmetical progression of length 1, 2 or 4.

Applying the Thue-Morse morphism

\[
\varphi(0) = 01, \varphi(1) = 10
\]

to this sequence, we obtain \((1, 8)\)-ACR word.

Notice that the bound is optimal, i.e., if \( w \) is a biinfinite non-periodic \((1, l)\)-ACR word, then \( l \geq 8 \). The conjecture about nonexistence of non-periodic \((1, l)\)-ACR words for \( l < 8 \) was checked by hand for \( l < 6 \) (with a technical case study) and with a computer for \( l < 8 \) (and we are grateful for computer experiments to A. Saarela and K. Saari). Note that this fact is similar to results about bounded left squares implying periodicity from [5], which we mentioned in Subsection 2.1.

**Remark 4.** Notice that all statements from this paragraph can be reformulated for right infinite words as follows. In all the statements for right infinite words instead of periodicity we can write ultimate periodicity, and we suppose that all conditions for abelian repetitions hold starting from some position. For example, the word \( w \in \Sigma^\omega \) is \((k, l)\)-ACR, if there exists \( i_0 \) such that for every \( i \geq i_0 \) there is a central \( 2k \)-power with length of period at most \( l \) at position \( i \).

### 4 Right and left abelian powers

In this section we discuss abelian powers occurring immediately to the right and to the left from each position. Similarly to usual powers, right and left abelian powers give weaker conditions for periodicity than central abelian powers, but still can be used for establishing periodicity.

Note first, that for every \( k \) there exist non-periodic biinfinite \((k, 2(k + 1)^2)\)-ARR (ALR) words. Moreover, there exists a non-periodic biinfinite word, such that for every \( k \) there exists \( l \) for which \( w \) is a \((k, l)\)-ARR (ALR) word. This follows from Theorems 5, 6 and observation that every \((k, l)\)-ACR word is \((k, l)\)-ARR and ALR.
For central squares the minimal value of $l$ for the existence of non-periodic $(1, l)$-ACR word was established in previous section, and it is equal to 8. Here we will find the minimal value for right (resp. left) squares.

**Example 2.** Non-periodic biinfinite $(2, 2)$- and $(2, 3)$-ARR binary words.

Let $w$ be a $(2, 2)$-ARR word. If $w$ contains the subword $010$, then its suffix is $(10)\omega$. If $w$ contains subword $100$, then its suffix is of the form $(1100)^*(10)\omega$ or $(1100)^\omega$. Denote by $\Sigma^{-\omega}$ left infinite words. So, $w$ is contained in the following set of words: $0^Z$, $(01)^Z$, $(1100)^{-\omega}(10)^\omega$, $0^{-\omega}(1100)^\omega$, $(0)^{-\omega}(0011)^*(01)^\omega$, $(0)^{-\omega}(1100)^*(10)^\omega$ (up to replacing 0 and 1). Though some of these words are non-periodic in the sense of our definition, they have some periodicity structure similar to ultimately periodic right infinite words. So, we will say that a biinfinite word $w$ is ultimately periodic, if there exist finite words $u$, $v_1$, $v_2$, such that $w = v_1^{-\omega}uv_2^\omega$.

Words in $\{01, 011\}^Z$ give a family of non-periodic biinfinite $(2, 3)$-ARR words. Notice that this family gives uncountably many words.

So in the case of right abelian squares the bound distinguishing periodic and non-periodic words is the following: for $l = 1$ there exist only periodic words, namely, words of the form $a^Z$, where $a \in \Sigma$. For $l = 2$ there exist only periodic and ultimately periodic words, for $l \geq 3$ there exist non-periodic words.

## 5 Examples

In this section we consider the examples of Sturmian words and the Thue-Morse word.

**Example 3:** Sturmian words. Let $s$ be a Sturmian word. Then

- $s$ is not $(k, l)$-ACR for every pair $(k, l)$;
- for every $k$ there exists $l$ such that $s$ is $(k, l)$-ARR.

The nonexistence of bounded centered abelian squares follows from the fact that Sturmian word contains factors of the form $a01u$, where $u$ is a special palindromic factor, and the lengths of bispecial palindromic factors are unbounded (see, e. g., [7], Chapter 2).

The existence of bounded right abelian powers follows from the following theorem by G. Richomme, K. Saari and L. Zamboni:

**Theorem 8.** (Theorem 6.1. in [9].) For every Sturmian word $s$ and every positive integer $k$, there exist two integers $l_1$ and $l_2$ such that each position in $s$ begins in an abelian $k$-power $u_1u_2\ldots u_k$ with abelian period $l_1$ or $l_2$, that is $|u_i| \in \{l_1, l_2\}$. In
particular, every Sturmian word begins in an abelian \( k \)-power for all positive integers \( k \).

**Example 4: Thue-Morse word.** For the Thue-Morse word \( t \) we will prove that

- \( t \) is (1,14)-ACR and not \((k,l)\)-ACR for every \( k > 1 \) or \( l < 14 \);
- \( t \) is (2,5)-ARR and not \((k,l)\)-ARR for every \( k > 2 \) or \( l < 5 \).

These facts are proved using the definition of the Thue-Morse word through binary expansion and Lemma 1. Recall that the Thue-Morse word can be defined in the following way: \( t_i = 1 \), if the number of ones in the binary expansion of \( i \) is odd, and \( t_i = 0 \) otherwise. Note that in this definition the starting index is 0.

First, we prove that the Thue-Morse word has bounded central abelian squares at every position. At every odd position (between blocks) \( t \) has a central abelian square with length of period 2. At even positions (inside blocks) central abelian squares with odd periods do not exist. Odd periods are impossible, since a word of odd length to the right from an even position contains several full blocks and a letter \( a \) immediately to the right from the position, and a word to the left from this even position contains several full blocks and a letter \( \bar{a} \) immediately to the left from the position. So these words cannot be abelian equivalent. It is easy to see that \( t \) contains an abelian square at a position \( 2i \) with period 2 iff \( t \) contains at the position \( i \) an arithmetical progression of length 3 with common difference \( l \). So, it is enough to prove that every position \( i \) is the centre of an arithmetical progression of length 3 with bounded common difference. To prove it, we will consider several cases. Denote the binary expansion of \( i \) by \( f(i) \). We will consider different forms of \( f(i) \) and in each case we will find a number \( m \), such that adding and subtracting \( f(m) \) from \( f(i) \) does not change the parity of 1’s. So the elements \( t_{i-m}, t_i, t_{i+m} \) form an arithmetical progression of one letter.

If \( f(i) \) ends with \( 10^{2j}, j \geq 1 \), then we take \( f(m) = 11 \). Then \( f(i) + f(m) \) ends with \( 10^{2j-2}11, f(i) - f(m) \) ends with \( 01^{2j-2}01 \), the beginning is the same as \( f(i) \).

If \( f(i) \) ends with \( 10^{2j+1}, or 10^{2j}/10, or 01^{2j}/01, or 01^{2j+1}, j \geq 1 \), then we take \( f(m) = 110 \).

If \( f(i) \) ends with \( 01^{2j}, or 10^{2j+1}, or 01^{2j}/01, j \geq 1 \), then we take \( f(m) = 11 \).

If \( f(i) \) ends with \( 10^{2j+1}, or 01^{2j+1}, j \geq 1 \), then we take \( f(m) = 111 \).

If \( f(i) \) ends with \( 10^{2j+1}/10, or 01^{2j+1}/01, j \geq 0 \), then we take \( f(m) = 101 \).

It is not difficult to check that all possible endings of \( f(i) \) are considered.

The maximal \( m \) we used is 7, \( f(7) = 111 \). It is not difficult to see that smaller \( m \) is not possible in \( f(i) \) ending with \( 10^{2j+1}, j \geq 1 \). So the Thue-Morse word is (1,14)-ACR word, and it is not (1,l)-ACR for \( l < 14 \).
To prove the nonexistence of bounded abelian central 4-powers, we first notice that at positions inside blocks periods of central abelian 4-powers should be of even period (since it holds already for squares). So in $\mu^{-1}(t)$ there should be central arithmetic progressions of length 5 with bounded difference at every position, and hence right arithmetical progressions with bounded difference of length 3. We will prove that the existence of right bounded abelian progressions of length 3 is impossible. Namely, we will prove that the positions of the form $i = 2^j - 1$ start with arithmetical progressions of length 3 only with growing differences. Let $m$ be the difference of an arithmetical progression of length 3 starting in $i$. Notice that $f(i) = 1^j$, $f(2m) = f(m)0$. We can take $i$ as large as needed, while $m$ is bounded. So for $i$ large enough the number of 1’s in $1^j + f(m)$ and $1^j + f(m)0$ differs by 1 and hence $t_{i+m} \neq t_{i+2m}$.

Now we proceed to the proof of the nonexistence of bounded abelian right cubes. Suppose that there exists $l_0$ such that $t$ is $(3, l_0)$-ARR. Similarly to evenness of periods of central squares inside blocks, we get for right cubes that in the positions inside blocks the periods cannot be odd. So, if at a position $2i$ the Thue-Morse word contains an abelian cube of length $2l \leq l_0$, then $\mu^{-1}(t) = t$ at the position $i$ starts with an arithmetical progression of length 4 with difference $l \leq l_0/2$, which as we have just seen is impossible at some positions.

It remains to prove the existence of bounded right abelian squares. Between the blocks we have right abelian powers of any length with period 2. We will prove, that at positions inside blocks we have bounded squares with odd periods. Considering $\mu^{-1}(t) = t$, one can see that the existence of abelian square at position $2i$ with odd period $l$ is equivalent to the condition $t_i = 1 - t_{i+l}$. We will prove that in every position we can find such $l \leq 5$. Suppose the converse, i. e., that there exists a position $i$ in which for all odd $l' \leq 5$ we do not have $t_i = 1 - t_{i+l'}$, which means $t_i = t_{i+1} = t_{i+3} = t_{i+5}$. Then a block begins in $i + 1$, $i$ is odd, and hence in the preimage we have $t_{i+1} = t_{i+1+1} = t_{i+1+2}$. This is impossible since the Thue-Morse word is cube-free (see, e. g., [7], p. 113). The optimality of $l = 5$ is trivially achieved, for example, at position 2.

Remark that K. Saari independently and simultaneously studied the right abelian powers for the example of Thue-Morse word.

We would like to emphasize that Sturmian and Thue-Morse words show very different properties concerning abelian powers. While Sturmian words have bounded right $k$-powers for every $k$, they do not have bounded central powers (even squares!). The Thue-Morse word has bounded central abelian squares and right cubes, and does not have bounded central and right abelian 4-powers.
6 Open questions

There remain many open questions about the abelian versions of the Critical Factorization Theorem. Below we summarize some of them.

**Open problem 1.** For any $k$, find $l$ such that there exists a non-periodic $(k, l)$-ACR (ARR) word, and all $(k, l - 1)$-ACR (ARR) words are periodic.

For abelian squares the answer is given by Examples 1 and 2. We have that for any $k$ there exists $l$ such that a non-periodic $(k, l)$-ACR (ARR) word exists (see Theorem 5). The general straightforward way to obtain such $l$ is as follows: for every $i = 2k, 4k, 6k, \ldots$ we build the directed graph whose vertices are all words of length $i$ having an abelian $2k$-power in the middle. There is an arc from a vertex $u$ to a vertex $v$, if $v$ is obtained from $u$ by deleting first letter and adding one letter in the end of the word. Actually, this graph is a subgraph of De Bruijn graph for length $i$. Each path in this graph corresponds to a word. If the graph does not contain two intersecting loops, then there exist only periodic words for this length and we increase $i$ by $2k$. If it does, then there exists a non-periodic $(k, i/2k)$-centered word and for the smallest such $i$ we have $l = i/2k$. But of course this way is ineffective, since the number of all words of length $i$ is exponential.

**Open problem 2.** Let $k$ be an integer. Does there exist a non-periodic word $w$ and integers $l_1, l_2$, such that $w$ contains a central abelian $2k$-power of length $l_1$ or $l_2$ at every position?

Notice that the construction from Theorem 5 gives an example of such a word using three periods. One period trivially implies periodicity. So two periods is an open question. Note also that for right (left) powers the answer to the question is given by Theorem 8, which implies that for every $k$ Sturmian words have abelian $k$-powers with one of two periods at every position.

**Open problem 3.** Find other natural conditions for abelian powers enforcing periodicity.

We found some conditions for abelian powers to imply periodicity, such as bound for the length of period and the number of possible periods. It would be interesting to find some other conditions giving abelian versions of Critical Factorization Theorem. In particular, some other equivalence relations on words intermediate between abelian equivalence and usual equality can be considered.
References


