Finiteness Theorem for the Ray Transform on a Riemannian manifold

Vladimir A. Sharafutdinov*
Institute of Mathematics, Novosibirsk, 630090, Russia

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Abstract

The ray transform \( I \) on a compact Riemannian manifold \( M \) with boundary is the operator sending a symmetric tensor field \( f \) to the set of integrals of \( f \) over all geodesics joining boundary points. A field \( f \) is called potential if it can be represented as the symmetric part of the covariant derivative of another tensor field vanishing on the boundary. The main result asserts that the space of potential tensor fields is a subspace of a finite codimension in \( \text{Ker} \, I \) if \( M \) is simple. A Riemannian manifold is called simple if every two points are joined by a unique geodesic.

1 Introduction

The ray transform \( I \) on a Riemannian manifold is the linear operator sending a symmetric tensor field \( f \) of degree \( m \) to the set of integrals of \( f \) over all maximal geodesics (the precise definition is given below). There are a few works [1, 2] devoted to the ray transform on closed manifolds (integration is performed over closed geodesics). In the current paper we will investigate this operator on compact Riemannian manifolds with boundary. In this situation the operator \( I \) is of great applied interest for tomography. Indeed, in the case \( m = 0 \), inversion of \( I \) is the main mathematical problem of X-ray tomography of inhomogeneous media. Some questions of tomography of anisotropic media relate to the operator \( I \) for \( m > 0 \). In these works the operator \( I \) is considered for homogeneous media, and integration is performed over straight lines. In the more general case of inhomogeneous media the integration must be made over geodesics of the Riemannian metric \( dt^2 = |dx|^2/c(x) \), where \( c(x) \) is the sound velocity. Another example of application of the ray transform to tomography problems arises in photoelasticity. As is shown in [5], the inverse problem of integrated photoelasticity can be reduced to inversion of \( I \) for \( m = 0 \) and \( m = 1 \). Finally, the problem of determining the elasticity tensor from results of registration of compressional waves in quasi-isotropic elastic media is equivalent to inversion of \( I \) for \( m = 4 \) [6].

In geophysics the so-called inverse kinematic problem of seismics is well known which is formulated as follows. In a bounded domain \( D \subset \mathbb{R}^3 \) there is a Riemannian metric of the type \( dx^2 = n(x)|dx|^2 \); One has to determine the function \( n(x) \) from the known distances in this metric between boundary points. In geophysics the metrics of such type are called isotropic. Linearization of the problem in the class of isotropic metrics leads to the question of inverting the ray transform \( I \) for \( m = 0 \). A discussion of these problems is exposed in [7] as well as an extensive bibliography.

We will now discuss in more detail some version of the inverse kinematic problem for Riemannian metrics of general type. We will show that the linearization of the problem leads to the ray transform \( I \) for \( m = 2 \). The principal difference between the cases \( m = 0 \) and \( m > 0 \) is that in the last case the operator \( I \) has a nontrivial kernel. It is essential that in the process of linearization there arises a conjecture on the kernel of the ray transform.

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Let \((M, g)\) be a compact Riemannian manifold with boundary \(\partial M\). For a point \(x \in \partial M\), the second quadratic form of the boundary
\[
\Pi(\xi, \xi) = (\nabla_\xi \nu(x), \xi) \quad (\xi \in T_x(\partial M))
\]
is defined on \(T_x(\partial M)\). Here \(\nu(x)\) is the unit vector of the outer normal to the boundary, \(\nabla\) is the covariant derivative in the metric \(g\), and \((\cdot, \cdot)\) is the scalar product in the metric \(g\). The boundary is called strictly convex if the form is positively determined for every \(x \in \partial M\).

A Riemannian manifold \((M, g)\) (or the Riemannian metric) is called simple if its boundary is strictly convex and, for every points \(p, q \in M\), there exists a unique geodesic \(\gamma_{pq}\) with endpoints \(p\) and \(q\) which depends smoothly on \(p, q\). The latter means that the mapping \(\exp'_p : T_pM \to T_pM\) is smooth for every \(p \in M\). For \(p, q \in \partial M\) we denote by \(\Gamma_g(p, q)\) the length of \(\gamma_{pq}\) in the metric \(g\). The function \(\Gamma_g : \partial M \times \partial M \to \mathbb{R}\) is called the hodograph of the metric \(g\) (the term is taken from geophysics). The problem of determining a metric by its hodograph is formulated as follows: for a given function \(\Gamma : \partial M \times \partial M \to \mathbb{R}\), one has to determine whether it is the hodograph of a simple metric and find all such metrics.

The following nonuniqueness of a solution is evident. Let \(\varphi\) be a diffeomorphism of \(M\) onto itself which is identical on \(\partial M\). It transforms a simple metric \(g^0\) to a simple metric \(g^1\) (the last equality means that, for a point \(x \in M\) and \(\xi, \eta \in T_xM\), the equality \(\langle d_x\varphi \xi, (d_x\varphi)\eta \rangle^0 = \langle \xi, \eta \rangle^1\) is valid where \(d_x\varphi : T_xM \to T_xM\) is the differential of \(\varphi\) and \(\langle \cdot, \cdot \rangle^0\) is the scalar product on \(T_xM\) in the metric \(g^0\)). These two metrics have different families of geodesics and the same hodograph. The question arises: is the nonuniqueness of the posed problem settled by the above mentioned construction? Let us formulate the precise statement.

**Problem 1 (the problem of determining a metric by its hodograph)** Let \(g^0, g^1\) be two simple metrics on a compact manifold \(M\) with boundary. Does the equality \(\Gamma_{g^0} = \Gamma_{g^1}\) imply existence of a diffeomorphism \(\varphi : M \to M\) such that \(\varphi|_{\partial M} = \text{Id}\) and \(\varphi^* g^0 = g^1\)?

This problem was formulated in [8] under another name: boundary rigidity problem.

Let us linearize Problem 1. To this end we suppose \(g^\tau\) be a family, of simple metrics on \(M\), smoothly depending on \(\tau \in (\varepsilon, \varepsilon)\). We fix \(p, q \in \partial M\), \(p \neq q\), and put \(a = \Gamma_{g^\tau}(p, q)\). Let \(\gamma^\tau : [0, a] \to M\) be a geodesic, of the metric \(g^\tau\), for which \(\gamma^\tau(0) = p\), \(\gamma^\tau(a) = q\). Let \(\gamma^\tau = (\gamma^\tau(1), \tau), \ldots, \gamma^\tau(n, \tau)\) be the coordinate representation of \(\gamma^\tau\) in a local coordinate system, \(g^\tau = (g^\tau_{ij})\). Simplicity of \(\gamma^\tau\) implies smoothness for the functions \(\gamma_i^\tau(\tau)\). The equality
\[
\frac{1}{\alpha} |\Gamma_{g^\tau}(p, q)|^2 = \int_0^a g^\tau_{ij}(\gamma^\tau(t)) \dot{\gamma}_i^\tau(t, \tau) \dot{\gamma}_j^\tau(t, \tau) dt
\]
is valid where the dot denotes differentiation with respect to \(t\). In (1.1) and through what following the next rule is used: on repeating sub- and super-indices in a monomial the summation from 1 to \(n\) is assumed. Differentiating (1.1) with respect to \(\tau\) and putting \(\tau = 0\), we get
\[
\frac{1}{\alpha} \frac{\partial}{\partial \tau} \left| \Gamma_{g^\tau}(p, q) \right|^2 = \int_0^a f_{ij}(\gamma^0(\tau)) \dot{\gamma}_i^\tau(t, 0) \dot{\gamma}_j^\tau(t, 0) dt + \int_0^a \left[ \frac{\partial g^0_{ij}}{\partial x^l} (\gamma^0(\tau)) \dot{\gamma}_i^\tau(t, 0) \frac{\partial \gamma_j^\tau}{\partial \tau}(t, 0) + 2 g^0_{ij}(\gamma^0(\tau)) \ddot{\gamma}_i^\tau(t, 0) \frac{\partial \gamma_j^\tau}{\partial \tau}(t, 0) \right] dt
\]
where
\[
f_{ij} = \frac{\partial}{\partial \tau} \bigg|_{\tau = 0} g^\tau_{ij}.
\]
The second integral on the right-hand side of (1.2) is equal to zero since \(\frac{\partial \gamma_i^\tau}{\partial \tau}(0, 0) = \frac{\partial \gamma_i^\tau}{\partial \tau}(a, 0) = 0\) and the geodesic \(\gamma^0\) is an extremal of the functional
\[
E_0(\gamma) = \int_0^a g^0_{ij}(\gamma(t)) \dot{\gamma}_i^0(t) \dot{\gamma}_j^0(t) dt.
\]
(One can also verify the vanishing of this integral by transforming the second term in brackets with the help of differentiation by parts and use made of the equation of geodesics). We thus come to the equality

$$\frac{1}{a} \frac{\partial}{\partial \tau} \bigg|_{\tau=0} [\Gamma g^\tau(p, q)]^2 = If(\gamma_{pq}) \equiv \int_{\gamma_{pq}} f_{ij}(x)\dot{x}^i\dot{x}^j dt$$

(1.3)

in which $\gamma_{pq}$ is a geodesic of the metric $g^0$ and $t$ is the arc length of this geodesic in the metric $g^0$.

If the hodograph $\Gamma g^\tau$ does not depend on $\tau$ then the left-hand side of (1.3) is equal to zero. On the other hand, if Problem 1 has a positive answer for the family $g^\tau$ then there exists a one-parameter group of diffeomorphisms $\varphi^\tau : M \to M$ such that $\varphi^\tau|_{\partial M} = Id$ and $g^\tau = (\varphi^\tau)^*g^0$. Written in coordinate form, the last equality gives

$$g^\tau_{ij} = (g^0_{ik} \circ \varphi^\tau) \frac{\partial \varphi^k(x, \tau)}{\partial x^i} \frac{\partial \varphi^j(x, \tau)}{\partial x^j}$$

where $\varphi^\tau(x) = (\varphi^1(x, \tau), \ldots, \varphi^n(x, \tau))$. Differentiating this relation with respect to $\tau$ and putting $\tau = 0$, we get the equation

$$(d\tau)_{ij} \equiv \frac{1}{2}(v_{i, j} + v_{j, i}) = \frac{1}{2}f_{ij}$$

(1.4)

for the vector field $v$ generating the group $\varphi^\tau$, where $v_{i, j}$ are covariant derivatives of the field $v$ in the metric $g^0$. The condition $\varphi^\tau|_{\partial M} = Id$ implies that $v|_{\partial M} = 0$. We thus come to the following question which is a linearization of Problem 1: to what extent is a symmetric tensor field $f = (f_{ij})$ on a simple Riemannian manifold $(M, g^0)$ determined by the family of integrals (1.3) which are known for all $p, q \in \partial M$? In particular, is it true that the equality $If(\gamma_{pq}) = 0$ for all $p, q \in \partial M$ implies existence of a vector field $v$ such that $v|_{\partial M} = 0$ and $dv = f^0$?

By the ray transform of the field $f$ we will mean the function $If$ that is determined by formula (1.3) on the set of geodesics joining boundary points. In the next section this linear problem will be generalized to a wider class of metrics and to symmetric tensor fields of arbitrary order.

2 Posing the problem and formulation of the result

To formulate the precise definition of the ray transform we need some preliminary notions.

Given a compact Riemannian manifold $(M, g)$ with the boundary $\partial M$, let $\tau_M = (TM, p, M)$ and $\tau'_M = (T'M, p', M)$ be the tangent and cotangent bundles, and $S^m\tau'_M = (S^mT'M, p'^m, M)$ be the complexification of the $m$-th symmetric power of $\tau'_M$. The space $C^\infty(S^m\tau'_M)$ of its sections is called the space of (smooth covariant) symmetric tensor fields of degree $m$ on $M$.

We denote points of $TM$ by the pairs $(x, \xi)$ where $x \in M, \xi \in T_xM$. Let $\Omega M = \{(x, \xi) \in TM \mid |\xi|^2 = \langle \xi, \xi \rangle = 1\}$ be the manifold of unit tangent vectors. Its boundary $\partial \Omega M$ can be represented as the union of two submanifolds

$$\partial_{\pm} \Omega M = \{(x, \xi) \in \Omega M \mid x \in \partial M, \pm \langle \xi, \nu(x) \rangle \geq 0\}$$

of outer and inner vectors.

$(M, g)$ is called a compact dissipative Riemannian manifold (CDRM briefly) if the boundary $\partial M$ is strictly convex and, for every $(x, \xi) \in \Omega M$, the maximal geodesic $\gamma_{x, \xi}(t)$ satisfying the initial conditions $\gamma_{x, \xi}(0) = x$ and $\gamma_{x, \xi}'(0) = \xi$ is defined on a finite segment $[\tau_-(x, \xi), \tau_+(x, \xi)]$. A simple compact Riemannian manifold is a CDRM.

We recall simultaneously that a geodesic $\gamma : [a, b] \to M$ is called maximal if it can not be extended to a geodesic $\gamma' : [a - \varepsilon_1, b + \varepsilon_2] \to M$ such that $\varepsilon_1 \geq 0$ and $\varepsilon_1 + \varepsilon_2 > 0$. Together with the definition of a CDRM, we have defined two functions $\tau_{\pm} : \Omega M \to R$. As can be easily shown, strict convexity of the boundary implies smoothness of the restriction $\tau_{\pm}|_{\partial_{\pm} M}$.

By the ray transform on a CDRM $(M, g)$ we mean the linear operator

$$I : C^\infty(S^m\tau'_M) \to C^\infty(\partial_{\pm} \Omega M)$$

(2.1)
For what classes of CDRMs and for what values of \( k \) and \( m \) can the inclusion in (2.5) be replaced with equality?

As can be easily shown, if the answer is positive for \( k = k_0 \), then it is positive for \( k \geq k_0 \). In [6] the positive answer is obtained for \( k = 1 \) and for all \( m \) under some assumption (depending on \( m \)) on the curvature of the metric.

We can now formulate the main result of the paper.

**Theorem 1** Given a simple compact Riemannian manifold \((M, g)\), inclusion (2.5) is of a finite codimension for all \( m \) and \( k \geq 1 \).

Together with the proof of Theorem 1, we shall establish the next

**Theorem 2** If \((M, g)\) is a simple compact Riemannian manifold, then inclusion (2.5) is the equality for \( m = 0 \) or \( m = 1 \) and for all \( k \geq 1 \).

The last claim is not new; for \( m = 0 \) it was proved in [9, 10]; and for \( m = 1 \) it was proved in [11].

In conclusion of the section we formulate some problems.

**Problem 3** Does there exist a simple compact Riemannian manifold for which inclusion (2.5) is not equality?

To author’s opinion, such manifolds exist; but the author had no success in constructing an example.

**Problem 4** Given a simple Riemannian manifold, is the codimension \( c_{k,m}(M, g) \) of inclusion (2.5) independent of \( k \)? In other words, does there exist a complement of \( P^k(S^m \tau'_{M}) \), in \( \ker^{k,m} I \), consisting of smooth tensor fields?

**Problem 5** Does there exist a CDRM for which inclusion (2.5) is of infinite codimension?
3 Proof of Theorem 1

We recall that the Hilbert space \( H^k(\Omega M) \) was defined for a compact Riemannian manifold \((M, g)\). In particular, \( H^0(\Omega M) = L^2(\Omega M) \). The next claim is of some interest by itself.

**Theorem 3** Given a CDRM \((M, g)\), the operator

\[
L : C(\Omega M) \to C(\Omega M)
\]

defined by the equality

\[
(LF)(x, \xi) = u(x, \xi) = \int_{\tau_{\cdot}(x, \xi)}^0 F(\gamma_{x, \xi}(t), \gamma_{x, \xi}(t)) \, dt
\]

is extendible to the bounded operator

\[
L : L^2(\Omega M) \to L^2(\Omega M).
\]

The difference between formulas (2.2) and (3.2) is that the first formula is considered for \((x, \xi) \in \partial_\iota \Omega M\) while the latter, for \((x, \xi) \in \Omega M\). The function \(\tau_{\cdot}(x, \xi)\) has singularities on \(\Omega M\) while its restriction \(\tau_{\cdot}|_{\partial_\iota \Omega M}\) is a smooth function; so Theorem 3 is not trivial.

For a Riemannian manifold \((M, g)\), the geodesic flow \(G^t : TM \to TM\) is the local one-parameter group of diffeomorphisms defined by the equality \(G^t(x, \xi) = (\gamma_{x, \xi}(t), \gamma_{x, \xi}(t))\) where \(\gamma_{x, \xi}(t)\) is the geodesic satisfying the initial conditions \(\gamma_{x, \xi}(0) = x\) and \(\gamma_{\iota, \xi}(0) = \xi\). The vector field \(H\) on \(TM\) generating the geodesic flow is called the geodesic vector field. In local coordinates it is given by the equality

\[
H = \xi^i \frac{\partial}{\partial x^i} - \Gamma_{jk}^i(x, \xi) \xi^j \xi^k \frac{\partial}{\partial \xi^l}
\]

where \(\Gamma_{jk}^i\) are the Christoffel symbols. The submanifold \(\Omega M\) is invariant with respect to the geodesic flow, so \(H\) can be considered as a first-order differential operator

\[
H : C^\infty(\Omega M) \to C^\infty(\Omega M).
\]

To prove Theorem 3 we need the next

**Lemma 1** Let \((M, g)\) be a CDRM, \(d\Sigma\) and \(d\sigma\) be smooth volume forms (i.e., differential forms of highest degree not vanishing at any point) on \(\Omega M\) and \(\partial_\iota \Omega M\) respectively. By \(D\) we mean the closed domain in \(\partial_\iota \Omega M \times \mathbb{R}\) defined by the equality

\[
D = \{(x, \xi; t) \mid \tau_{\cdot}(x, \xi) \leq t \leq 0\}.
\]

Define the mapping \(G : D \to \Omega M\) by putting \(G(x, \xi; t) = G^t(x, \xi)\) where \(G^t\) is the geodesic flow. Then the equality

\[
(G^* d\Sigma)(x, \xi; t) = a(x, \xi; t) (\xi, \nu(x)) \, d\sigma(x, \xi; t) \wedge dt
\]

holds with some \(a \in C^\infty(D)\) not vanishing at any point.

Proof. Under changing the form \(d\Sigma\) (or \(d\sigma\)), only the coefficient \(a\) changes in (3.6). Therefore it suffices to prove the claim for some concrete forms \(d\Sigma\) and \(d\sigma\).

We define the function \(r : M \to \mathbb{R}\) by putting \(r(x) = -\rho(x, \partial M)\) where \(\rho\) is the distance in the metric \(g\). The function \(r\) is smooth in some neighbourhood of \(\partial M\) and \(\nabla r(x) = \nu(x)\) for \(x \in \partial M\). We extend the form \(d\sigma\) onto some neighbourhood of the set \(\partial_\iota \Omega M\) in the manifold \(\Omega M\). By arguments of the previous paragraph, we can assume that \(d\Sigma(x, \xi) = d\sigma(x, \xi) \land d\nu(x)\) near \(\partial_\iota \Omega M\).

First we verify validity of (3.6) for \(t = 0\). The differential of the mapping \(G\) at a point \((x, \xi; 0)\) is identical on \(T_{(x, \xi)}(\partial_\iota \Omega M)\) and sends the vector \(\partial / \partial t\) to \(H\); therefore

\[
(G^* d\Sigma)(x, \xi; 0) = G^* (d\sigma \land d\nu)(x, \xi; 0) = Hr \cdot d\sigma(x, \xi) \land dt.
\]

(3.7)
By (3.4), \( Hr = \xi^i \frac{\partial r}{\partial x^i} = (\xi, \nu(x)) \). Inserting this value into (3.7), we obtain

\[
(G^*d\Sigma)(x, \xi; 0) = (\xi, \nu(x)) \, d\sigma(x, \xi) \wedge dt. \tag{3.8}
\]

The mapping \( G \) satisfies the equation \( G(x, \xi; t + s) = G^i(G(x, \xi; s)) \) that implies the relation

\[
(G^*d\Sigma)(x, \xi; t) = a(x, \xi, t)(G^*d\Sigma)(x, \xi; 0) \tag{3.9}
\]

with some positive function \( a \in C^\infty(D) \). Formulas (3.8) and (3.9) imply (3.6). The lemma is proved.

**Proof of Theorem 3.** First we consider the case of \( F \in C(\Omega M) \). Given by formula (3.2), the function \( u(x, \xi) \) belongs to \( C(\Omega M) \). We will obtain an estimate of the norm \( \|u\|_{L_2(\Omega M)} \). To this end, in the integral

\[
\|u\|_{L_2(\Omega M)}^2 = \int_{\Omega M} |u(y, \eta)|^2 \, d\Sigma(y, \eta)
\]

we change the integration variable with the help of the mapping \( G : D \to \Omega M \), defined in Lemma 1, that maps diffeomorphically the interior of \( D \) onto \( \Omega M \setminus \partial \Omega M \). As a result we obtain

\[
\|u\|_{L_2(\Omega M)}^2 = \int_{\partial_+ \Omega M} \int_0^\infty |u(G(x, \xi; t))|^2 \, \frac{d\Sigma(G(x, \xi, t))}{d\sigma(x, \xi)} \, dt \, d\sigma(x, \xi).
\]

By Lemma 1, the last equality implies the estimate

\[
\|u\|_{L_2(\Omega M)}^2 \leq C \int_{\partial_+ \Omega M} \int_0^\infty |u(G(x, \xi; t))|^2 \, \frac{dt \, d\sigma(x, \xi)}{(\xi, \nu(x))}. \tag{3.10}
\]

Definition (3.2) of the function \( u(y, \eta) \) can be rewritten as follows:

\[
u(y, \eta) = \int_0^\tau_{-(y, \eta)} F(G(y, \eta; s)) \, ds.
\]

Putting \( (y, \eta) = G(x, \xi; t) \) here, we obtain

\[
u(G(x, \xi; t)) = \int_0^\tau_{-(G(x, \xi; t))} F(G(G(x, \xi; t); s)) \, ds.
\]

Using the relations \( G(G(x, \xi; t); s) = G(x, \xi; t + s) \) and \( \tau_{-(G(x, \xi; t))} = \tau_{-(x, \xi)} - t \), we transform the previous formula to the form

\[
u(G(x, \xi; t)) = \int_0^{\tau_{-(x, \xi)} - t} F(G(x, \xi; t + s)) \, ds = \int_{\tau_{-(x, \xi)}}^t F(G(x, \xi; s)) \, ds.
\]

With the help of the Cauchy-Bunjakovskii inequality we obtain

\[
|u(G(x, \xi; t))|^2 \leq (t - \tau_{-(x, \xi)}) \int_{\tau_{-(x, \xi)}}^t |F(G(x, \xi; s))|^2 \, ds.
\]

The last inequality and (3.10) imply

\[
\|u\|_{L_2(\Omega M)}^2 \leq C \int_{\partial_+ \Omega M} \left[ \int_0^{t - \tau_{-(x, \xi)}} dt \int_{\tau_{-(x, \xi)}}^t |F(G(x, \xi; s))|^2 \, ds \right] \frac{d\sigma(x, \xi)}{(\xi, \nu(x))}.
\]
After changing the integration limits $t$ and $s$, this inequality takes the form

$$
\|u\|_{L^2(\Omega M)}^2 \leq C \int_{\partial \Omega M} \left[ \int_0^s (s\tau_-(x,\xi) - s^2/2)|F(G(x,\xi; s))|^2 \, ds \right] d\sigma(x,\xi).
$$

We return to the integration variable $(y,\eta) = G(x,\xi; s)$ in the last integral. Taking the relations $s = -\tau_-(y,\eta)$ and $\tau_+(x,\xi) = -\tau_+(y,\eta)$ into account, we obtain the inequality

$$
\|u\|_{L^2(\Omega M)}^2 \leq C \int_{\Omega M} \tau_+(x,\xi) \left( \frac{1}{2} \tau_+(x,\xi) - \tau_-(x,\xi) \right) |F(x,\xi)|^2 \, d\Sigma(x,\xi)
$$

which implies the estimate

$$
\|u\|_{L^2(\Omega M)} \leq C\|F\|_{L^2(\Omega M)}.
$$

Being proved for $u \in C(\Omega M)$, the last estimate allows us to finish the proof of the theorem by standard arguments.

Given a Riemannian manifold $(M,g)$, the differential operator

$$
\delta : H^{k+1}(S^{m+1}\tau'_M) \to H^k(S^m\tau'_M)
$$

defined in coordinate form by the equality

$$
(\delta f)_{i_1...i_m} = g^{jk} \nabla_j f_{k_{i_1...i_m}}
$$
is evidently independent of the choice of coordinates; here $\nabla f$ is the covariant derivative of the field $f$ and $(g^{jk})$ is the inverse matrix of $(g_{jk})$. The operator $\delta$ is called the divergence. The operators $d$ and $-\delta$ are formally dual to one other (see Theorem 3.3.1 of [6]).

Let $S^m\tau'_M|_{\partial M}$ be the restriction of the bundle $S^m\tau'_M$ to $\partial M$. By

$$
j_{\nu} : S^m\tau'_M|_{\partial M} \to S^{m-1}\tau'_M|_{\partial M}
$$

we denote the operator of convolution with vector $\nu$. It is defined in coordinate form by the equality

$$
(j_{\nu}f)_{i_1...i_m} = f_{i_1...i_m} \nu^m.
$$

The main step in our proof of Theorem 1 is the next

**Lemma 2** Let $(M,g)$ be a simple compact Riemannian manifold. For every field $f \in C^\infty(S^m\tau'_M)$, the function $Lf = u \in C(\Omega M)$ defined by the equality

$$
(Lf)(x,\xi) = u(x,\xi) = \int_{\tau_-(x,\xi)}^0 f_{i_1...i_m}(\gamma_{x,\xi}(t)) \gamma^{i_1}_{x,\xi}(t) \ldots \gamma^{i_m}_{x,\xi}(t) \, dt \quad (3.11)
$$

belongs to $H^1(\Omega M)$ and satisfies the estimate

$$
\|u\|_{H^1(\Omega M)}^2 \leq C \left[ m(m-1)\|u\|_{L^2(\Omega M)}^2 + m\|\delta f\|_{L^2(S^{m-1}\tau'_M)} \cdot \|u\|_{L^2(\Omega M)} + m\|f\|_{L^2(S^{m-1}\tau'_M|_{\partial M})} \cdot \|j_{\nu} f\|_{L^2(S^{m-1}\tau'_M|_{\partial M})} \cdot \|L\|_{H^1(\partial_+ \Omega M)} \right] \quad (3.12)
$$

with some constant $C$ independent of $f$.

The proof of the lemma will be given at the end of the paper, and now we will prove Theorem 1 with use made of the lemma. First of all, Lemma 2 implies the next

**Corollary 1** Given a simple compact Riemannian manifold $(M,g)$, the operator $L : f \mapsto u$ defined by formula (3.11) is extendible to the bounded operator

$$
L : H^1(S^m\tau'_M) \to H^1(\Omega M). \quad (3.13)
$$

For $f \in H^1(S^m\tau'_M)$ and $u = Lf$, estimate (3.12) is valid.
Proof. Given \( f \in H^1(S^m\tau'_M) \), let \( f_k \in C^\infty(S^m\tau'_M) \) \((k = 1, 2, \ldots)\) be a sequence converging to \( f \),
\[
f_k \to f \quad \text{in} \quad H^1(S^m\tau'_M) \quad \text{as} \quad k \to \infty.
\]
Then
\[
\delta f_k \to \delta f \quad \text{in} \quad L_2(S^{m-1}\tau'_M)
\]
and, by boundedness of the operator \( I \) (see Theorem 4.2.1 of [6]),
\[
I f_k \to I f \quad \text{in} \quad H^1(\partial_\Sigma \Omega M).
\]
Besides, by Lemma 1,
\[
L f_k = u_k \to u = L f \quad \text{in} \quad L_2(\Omega M). \tag{3.14}
\]
Applying estimate (3.12) to the difference \( u_k - u_1 \), we see that \( u_k \) is a Cauchy sequence in \( H^1(\Omega M) \) and, consequently, it converges in \( H^1(\Omega M) \). Therefore (3.14) implies that \( u \in H^1(\Omega M) \) and
\[
u_k \to u \quad \text{in} \quad H^1(\Omega M).
\]
Writing down estimate (3.12) for \( u_k \) and passing to the limit as \( k \to \infty \) in this inequality, we arrive at estimate (3.12) for \( u \).

Proof of Theorem 1. First of all we show that the claim of the theorem for \( k = 1 \) implies the same for arbitrary \( k \geq 1 \).

The kernel \( \text{Ker}^{k, m} I \) is the closed subspace in the Hilbert space \( H^k(S^m\tau'_M) \). Let
\[
A^{k, m} = \text{Ker}^{k, m} I \ominus P^k(S^m\tau'_M)
\]
be the orthogonal complement of the space of potential fields in \( \text{Ker}^{k, m} I \) with respect to the scalar product
\[
(u, v)_{L_2(\Omega M)} = \int_{\Omega M} \langle u(x, \xi), v(x, \xi) \rangle d\Sigma(x, \xi).
\]
Hereafter \( d\Sigma = d\Sigma^{2n-1} \) denotes the symplectic volume form on \( \Omega M \) defined in Section 3.6 of [6]. The claim of Theorem 1 is equivalent to finiteness of dimension of \( A^{k, m} \). It follows from the Green’s formula for \( d \) and \( \delta \) (formula (3.3.1) of [6]) that \( A^{k, m} \) consists of all fields \( f \in H^k(S^m\tau'_M) \) satisfying the relations
\[
\delta f = 0, \quad I f = 0. \tag{3.15}
\]
Consequently, \( A^{k, m} \subset A^{k', m} \) for \( k \geq k' \). Thus, in what follows we consider the case of \( k = 1 \).

We have to prove that the space \( A^{1, m} \) has a finite dimension. To this end we consider the image \( L(A^{1, m}) \) of the space with respect to the operator \( L \) defined by (3.11). Note that the operator \( L \) is injective. Indeed, as is known (see Section 4.6 of [6]), the function \( u = L f \) satisfies the equation
\[
H u(x, \xi) = f_{i_1 \ldots i_m}(x)\xi^{i_1} \ldots \xi^{i_m} \tag{3.16}
\]
that recovers \( f \) from \( u \). Therefore to prove the theorem it suffices to show that the subspace \( L(A^{1, m}) \subset H^1(\Omega M) \) has a finite dimension.

For \( f \in A^{1, m} \) and \( u = L f \), estimate (3.12) is valid. By (3.15), the estimate takes the form
\[
\|u\|_{H^1(\Omega M)} \leq C m(m - 1)\|u\|_{L_2(\Omega M)}, \tag{3.17}
\]

Thus, estimate (3.17) holds for every \( u \in L(A^{1, m}) \). Since the imbedding \( H^1(\Omega M) \subset L_2(\Omega M) \) is compact, estimate (3.17) implies finiteness of the dimension of \( L(A^{1, m}) \). The theorem is proved.

Proof of Theorem 2. In the case of \( m = 0 \) or \( m = 1 \) estimate (3.17) implies that \( u = 0 \). Therefore \( A^{1, 0} = A^{1, 1} = 0 \), that is equivalent to the claim of Theorem 2.

Before proving Lemma 2 we will establish some auxiliary claims. In what follows we will use the machinery of semibasic tensor fields that is exposed in [6]. Given a Riemannian manifold \((M, g)\), by \( \beta^r M \) we mean the bundle of semibasic tensor fields of degree \( m \) and by \( \nabla, \nabla : C^\infty(\beta^r M) \to C^\infty(\beta^{r+1} M) \) we mean the vertical and horizontal covariant derivatives.
Lemma 3 Let \((M, g)\) be a CDRM, and \(\lambda \geq 0\) be a continuous function on \(\Omega M\). Assume a nonnegative function \(\varphi \in C(\Omega M)\) to be smooth on \(\Omega_{\varphi} = \{(x, \xi) \in \Omega M \mid \varphi(x, \xi) > 0\}\), satisfy the boundary condition

\[
\varphi|_{\partial \Omega M} = 0
\]

and the next condition

\[
\sup_{(x, \xi) \in \Omega_{\varphi}} |H \varphi(x, \xi)| < \infty.
\]

Then the estimate

\[
\int_{\Omega M} \lambda |\varphi|^2 \, d\Sigma \leq C \int_{\Omega_{\varphi}} |H \varphi|_+^2 \, d\Sigma
\]

holds with some constant \(C\) independent of \(\varphi\); here the notation

\[
[a]_+ = \begin{cases} a, & \text{if } a \geq 0 \\ 0, & \text{if } a < 0 \end{cases}
\]

is used.

The proof of this claim can be obtained by a slight modification of the proof of Lemma 4.5.2 of [6]. Namely, using nonnegativity of \(\varphi\), formula (4.5.7) of the book implies the inequality

\[
|\psi_y(t)| \leq t \int_0^t \left[ \frac{d\psi_y(\tau)}{d\tau} \right]_+ \, d\tau.
\]

The rest of arguments is not changed.

Lemma 4 Let \((M, g)\) be a CDRM, and \(a \in C^\infty(\beta^1_mM)\). By \(A : C^\infty(\beta^0_mM) \to C^\infty(\beta^0_mM)\) we denote the differential operator defined in coordinate form by the equality

\[
(Au)_{i_1 \ldots i_m} = (Hu)_{i_1 \ldots i_m} + a^j_{i_1} u_{ji_2 \ldots i_m}.
\]

If a field \(u \in C^\infty(\beta^0_mM)\) satisfies the boundary condition

\[
u|_{\partial \Omega M} = 0,
\]

then the estimate

\[
\|u\|_{L_2(\Omega M)} \leq C \|Au\|_{L_2(\Omega M)}
\]

holds with some constant \(C\) independent of \(u\).

Proof. The function \(\tilde{\varphi} = |u|\) is continuous on \(\Omega M\), smooth on \(\Omega_{\tilde{\varphi}} = \{(x, \xi) \in \Omega M \mid \tilde{\varphi}(x, \xi) > 0\}\) and satisfies the boundary condition \(\tilde{\varphi}|_{\partial \Omega M} = 0\). The equality \(H \tilde{\varphi} = \langle u, Hu \rangle / |u| \) holds on \(\Omega_{\tilde{\varphi}}\) and, consequently,

\[
|H \tilde{\varphi}| \leq |Hu|.
\]

From (3.20) we obtain the relation

\[
\frac{1}{2} H(|u|^2) = \langle Au, u \rangle - \langle au, u \rangle
\]

which implies the inequality

\[
|u| \cdot H(|u|) \leq |Au| \cdot |u| + |a| \cdot |u|^2.
\]

With the help of (3.23), it implies that the inequality

\[
H \tilde{\varphi} - |a| \tilde{\varphi} \leq |Au|
\]
holds on $\Omega_3$. It can be rewritten in the form

$$H(e^{-b}\varphi) \leq e^{-b}|Au|,$$

(3.24)

where $b$ is a function on $\Omega M$ satisfying the equation $Hb = |a|$. The function $\varphi = e^{-b}\varphi$ satisfies the conditions of Lemma 3. Applying this lemma with $\lambda = e^{2b}$ and using (3.24), we obtain

$$\|u\|_{L^2(\Omega M)}^2 = \int_{\Omega M} |\varphi|^2 d\Sigma = \int_{\Omega M} \lambda|\varphi|^2 d\Sigma \leq C \int_{\Omega M} |H\varphi|^2 d\Sigma \leq C \int_{\Omega M} |Au|^2 d\Sigma = C \|Au\|_{L^2(\Omega M)}^2.$$  

The lemma is proved.

Proof of Lemma 2. It suffices to prove the claim for a real field $f \in C^\infty(S^m\tau M)$. In what follows we agree to denote various constants independent of $f$ by the same letter $C$.

Let $T^0M = \{(x, \xi) \in TM | \xi \neq 0\}$ be the manifold of nonvanishing tangent vectors. Given $f$, we define the function $u \in C(T^0M)$ by formula (3.11). This function is smooth on $T^0M \setminus T(\partial M)$, satisfies equation (3.16), the boundary conditions (3.21) and the next one

$$u|_{\partial_0\Omega M} = If.$$  

(3.25)

This function is positively homogeneous in its second argument:

$$u(x, t\xi) = t^{-m-1}u(x, \xi) \quad (t > 0).$$  

(3.26)

By Lemma 8.3.1 of [6], we can find a semibasic field $a \in C^\infty(\beta^2M; T^0M)$ on $T^0M$ which is symmetric: $a^{ij} = a^{ji}$, orthogonal to the vector $\xi$: $a^{ij}\xi_j = 0$, positively homogeneous: $a^{ij}(x, t\xi) = ta^{ij}(x, \xi)$ $(t > 0)$ and such that the curvature tensor of the corresponding modified horizontal derivative $\overset{\sim}{\nabla}$ satisfies the equation

$$R_{ijkl}\xi^i\xi^j\xi^k = 0.$$  

(3.27)

Note that it is the unique point in our proof where simplicity of $(M, g)$ is used. The conditions of Lemma 8.3.1 of [6] require absence of conjugate points for the Jacobi equation; that is equivalent to simplicity of a CDRM.

We write down the Pestov identity (see Lemma 8.2.1 of [6]) for the function $u$:

$$2\langle \overset{\sim}{\nabla} u, \overset{\sim}{\nabla}(Hu) \rangle = |\overset{\sim}{\nabla} u|^2 + \overset{\sim}{\nabla} v_1 + \overset{\sim}{\nabla} w.$$  

(3.28)

The term containing the curvature vanishes because of (3.27). The semibasic fields $(v_1)$ and $(w)$ are defined by the formulas

$$v_1 = \xi_i \overset{\sim}{\nabla}^i u - \overset{\sim}{\nabla} \xi_i \overset{\sim}{\nabla} u, \quad (3.29)$$

$$w = \xi^i \overset{\sim}{\nabla}^i u.$$  

(3.30)

We transform the left-hand side of (3.28). By (3.16),

$$\overset{\sim}{\nabla}_i(Hu) = \overset{\sim}{\nabla}_i(f_{i_1...i_m}\xi^{i_1}...\xi^{i_m}) = m f_{i_1...i_m} \xi^{i_2}...\xi^{i_m}.$$  

(3.31)

Therefore

$$2\langle \overset{\sim}{\nabla} u, \overset{\sim}{\nabla}(Hu) \rangle = 2m \overset{\sim}{\nabla} u \cdot f_{i_2...i_m} \xi^{i_2}...\xi^{i_m} =$$

$$= \overset{\sim}{\nabla} (2mu f_{i_2...i_m} \xi^{i_2}...\xi^{i_m}) - 2mu \overset{\sim}{\nabla} (f_{i_2...i_m} \xi^{i_2}...\xi^{i_m}).$$

Introducing the notation

$$\overset{\sim}{v} = 2mu f_{i_2...i_m} \xi^{i_2}...\xi^{i_m},$$

we obtain

$$2\langle \overset{\sim}{\nabla} u, \overset{\sim}{\nabla}(Hu) \rangle = \overset{\sim}{\nabla} \overset{\sim}{v} - 2mu \overset{\sim}{\nabla} (f_{i_2...i_m} \xi^{i_2}...\xi^{i_m}).$$

Using the definition of the modified derivative (see Section 8.2 of [6]), the last formula is transformed as follows:

$$= \overset{\sim}{\nabla} \overset{\sim}{v} - 2mu \overset{\sim}{\nabla} (f_{i_2...i_m} \xi^{i_2}...\xi^{i_m}).$$
\[ 2\langle \nabla u, \nabla (Hu) \rangle = \nabla ^i \tilde{v}_i - 2m \mu \left[ \frac{h}{\rho} (f_{ij12...im} \xi^{i_2} \cdots \xi^{i_m}) + a^{ij} \nabla_j (f_{ij12...im} \xi^{i_2} \cdots \xi^{i_m}) \right] = \nabla ^i \tilde{v}_i - 2m \mu (\delta f)_{ij12...im} \xi^{i_2} \cdots \xi^{i_m} - 2m(m-1)a^{ij} u f_{ij12...im} \xi^{i_3} \cdots \xi^{i_m}. \]

Inserting this expression into (3.28), we obtain

\[ |\nabla u|^2 = -2m \mu (\delta f)_{ij12...im} \xi^{i_2} \cdots \xi^{i_m} - 2m(m-1)a^{ij} u f_{ij12...im} \xi^{i_3} \cdots \xi^{i_m} + \nabla ^i (\tilde{v}_i - v_i) - \nabla ^i w^i. \] (3.33)

We are going to integrate equality (3.33) over \( \Omega M \). In course of integration, some precautions are needed against singularities of the function \( u \) on the set \( T(\partial M) \). For this reason we will proceed as follows. Let \( r : M \to \mathbb{R} \) be the distance to \( \partial M \) in the metric \( g \). The manifold \( M = \{ x \in M \mid r(x) \geq \varepsilon \} \) has the smooth boundary for sufficiently small \( \varepsilon > 0 \), and \( u \) is smooth on \( T^0 M \). We multiply equality (3.33) by the volume form \( d\Sigma \) and integrate it over \( \Omega M_t \). Transforming the integrals of divergent terms by the Gauss-Ostrogradskiǐ formulas (Lemma 3.6.3 and formula (8.2.30) of [6]), we obtain

\[ \int_{\Omega M_t} |\nabla u|^2 \ d\Sigma = \int_{\partial \Omega M_t} \langle \tilde{v} - v, \nu \rangle \ d\Sigma^{2n-2} - (n+2m-2) \int_{\Omega M_t} \langle w, \xi \rangle \ d\Sigma - \int_{\Omega M_t} \left[ 2m \mu (\delta f)_{ij12...im} \xi^{i_2} \cdots \xi^{i_m} + 2m(m-1)a^{ij} u f_{ij12...im} \xi^{i_3} \cdots \xi^{i_m} \right] \ d\Sigma, \]

where \( n = \text{dim} \ M \). By (3.30), \( \langle w, \xi \rangle = |Hu|^2 \) and the previous formula takes the form

\[ \int_{\Omega M} \left( |\nabla u|^2 + (n+2m-2)|Hu|^2 \right) \ d\Sigma = \int_{\partial \Omega M} \langle \tilde{v} - v, \nu \rangle \ d\Sigma^{2n-2} - \int_{\Omega M_t} \left[ 2m \mu (\delta f)_{ij12...im} \xi^{i_2} \cdots \xi^{i_m} + 2m(m-1)a^{ij} u f_{ij12...im} \xi^{i_3} \cdots \xi^{i_m} \right] \ d\Sigma. \] (3.34)

Repeating the arguments of the end of Section 4.6 of [6], we insure that the first integral on the right-hand side of (3.34) can be estimated as follows:

\[ \lim_{\varepsilon \to 0} \int_{\partial \Omega M_t} \langle \tilde{v} - v, \nu \rangle \ d\Sigma^{2n-2} \leq CN^2(f) \equiv C(m\|j_0 f|_{\partial M}\|_0 \cdot \|f\|_0 + \|f\|_1^2). \] (3.35)

Hereafter we use the brief notations for norms:

\[ \|f\|_k = \|f\|_{H^k(\partial M)}, \quad \|f\| = \|f\|_{H^k(S^m \gamma^*_{\alpha})}, \quad \|u\|_k = \|u\|_{H^k(\Omega M)} \]

and so on.

By Theorem 3, \( u \in L_2(\Omega M) \). From this with the help of (3.34) and (3.35) we obtain that \( \nabla u \in L_2(\Gamma^0 M; \Omega M) \) and \( Hu \in L_2(\Omega M) \) as well as the inequality

\[ \|\nabla u\|_0^2 + \|Hu\|_0^2 \leq C \left( m\|u\|_0 \cdot \|\delta f\|_0 + m(m-1)|u|_0 \cdot \|f\|_0 + N^2(f) \right). \] (3.36)

Besides this, equation (3.16) implies the estimate

\[ \|f\|_0 \leq C \|Hu\|_0. \] (3.37)

It follows from (3.36) and (3.37) that

\[ \|Hu\|_0^2 \leq C \left( |u|_0 \cdot \|Hu\|_0 + |u|_0 \cdot \|\delta f\|_0 + N^2(f) \right). \] (3.38)
Considering (3.38) as a square inequality in \(\|Hu\|_0\), we obtain
\[
\|Hu\|_0 \leq C (\|u\|_0 + \|\delta f\|_0 + N(f)) .
\] (3.39)
The estimates (3.37) and (3.39) imply the inequality
\[
\|f\|_0 \leq C (\|u\|_0 + \|\delta f\|_0 + N(f))
\]
with help of which (3.36) gives
\[
\|\nabla u\|_0^2 \leq C \left( m(m - 1)\|u\|_0^2 + m\|u\|_0 \cdot \|\delta f\|_0 + N^2(f) \right) .
\] (3.40)

We now estimate \(\|\tilde{v}u\|_0\) by \(\|\nabla u\|_0\). From (3.31) with the help of the commutation formula \(\tilde{v}H - H\tilde{v} = \hat{\nabla} \hat{u}\), we obtain
\[
H\tilde{v}_i u = -\tilde{v}_{ij} u + m f_{ij2...im} \xi^{i2} \cdots \xi^{im} .
\] (3.41)
By the definition of the modified derivative
\[
\tilde{v}_i u = g_{ij} \tilde{v}^j u - a_i^{ij} \tilde{v}^j u .
\]
Inserting this expression into (3.41), we obtain
\[
(A\tilde{v}u)_i = (H\tilde{v}u)_i - a_i^{ij} \tilde{v}^j u = -g_{ij} \tilde{v}^j u + m f_{ij2...im} \xi^{i2} \cdots \xi^{im} .
\] (3.42)
By (3.21), the field \(\tilde{v} u\) satisfies the boundary condition \(\tilde{v} u|_{\partial M} = 0\). Applying Lemma 4 to the field \(\tilde{v} u\) and operator \(A\) defined by (3.42), we arrive at the estimate
\[
\|\tilde{v} u\|_0^2 \leq C (\|\tilde{v} u\|_0^2 + \|f\|_0^2) .
\] (3.43)
The equality \(Hu = \xi_i \tilde{v}^i u\) and estimate (3.37) imply the inequality \(\|f\|_0 \leq C \|\tilde{v} u\|\). With the help of the latter (3.43) gives
\[
\|\tilde{v} u\|_0 \leq C \|\tilde{v} u\|_0 .
\] (3.44)
Finally, the estimate
\[
\|u\|_0 \leq C\|Hu\|_0 \leq C_1 \|\tilde{v} u\|_0
\] (3.45)
is obtained by applying Lemma 4 with \(A = H\).

The next three norms
\[
\|u\|_{H^1_{\text{loc}}(M)}, \quad (\|\tilde{v} u\|_0^2 + \|v u\|_0^2 + \|u\|_0^2)^{1/2}, \quad (\|\tilde{v} u\|_0^2 + \|\tilde{v} u\|_0^2 + \|u\|_0^2)^{1/2}
\] (3.46)
are equivalent on the subspace, of \(H^1_{\text{loc}}(\mathcal{T}^0 M)\), consisting of functions possessing homogeneity (3.25). By (3.44) and (3.45), the last of these norms is equivalent to \(\|\tilde{v} u\|_0\). Therefore (3.40) implies the estimate
\[
\|u\|_{H^1_{\text{loc}}(M)}^2 \leq C \left( m(m - 1)\|u\|_0^2 + m\|u\|_0 \cdot \|\delta f\|_0 + N^2(f) \right)
\]
that coincides with (3.12). The lemma is proved.

References


