PERIODIC EXTREMALS OF MANY-VALUED OR NOT-EVERYWHERE-POSITIVE FUNCTIONALS

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I. We consider (possibly many-valued) functionals of the type of length in Finsler metric, which in local coordinates \( z^i_\alpha \) have the form

\[
I^{(\alpha)}(\gamma) = \int F(z, \dot{z}) \, dt + \int \Omega^{(\alpha)}_j (z) \, dz^j. 
\]

Here it is assumed that \( F(z, \lambda \dot{z}) = \lambda F(z, \dot{z}) \), for \( \lambda > 0 \), \( F > 0 \), \( \partial^2 F/\partial \dot{z}^i \partial \dot{z}^j \) is a positive form, and the expression \( \Omega^{(\alpha)}_j = \partial_i A_j^{(\alpha)} - \partial_j A_i^{(\alpha)} \) (magnetic field) is uniquely defined on the whole manifold \( M^n \) as a closed but possibly nonexact 2-form. The first summand is defined as a positive Finsler metric, but the second may make the functional nonpositive or many-valued. This note is a continuation of [1]–[5].

Remark. This note contains, in particular, corrections of certain errors in [2]–[4]. It is necessary to remove the term “nonselfintersecting” from all statements except those in §II of this article. Moreover, in §5 of [3] there is an assertion in which the term “flat section” should be replaced by the term “T-invariant curve”. These curves, however, can be considered nonselfintersecting only for functionals invariant under reversal of direction.

II. Two-dimensional problems. Functionals of the form (1) occur, for example, as Maupertuis-Fermat functionals for a charged particle in an external magnetic field or (if \( M^n = SO_3 \)) for the motion of a solid body under the action of gyroscopic forces. These functionals, as was shown in [2], occur on the two-dimensional sphere \( S^2 \) as the result of a reduction of the Hamiltonian formalism of the Kirchhoff-Thompson equations for the motion of a solid in a fluid or of a body around a fixed point, with the total flux of the “effective magnetic field” \( \Omega_{12} \) through the sphere \( S^2 \) proportional to the area constant. Let \( \Omega^+ \) denote the space of directed nonselfintersecting (piecewise smooth) curves. On this space a functional of the form (1) is single-valued. We normalize it so that it takes the value zero on one-point curves. In the many-valued case the functional is normalized so that it takes a positive value on the second component of the set of one-point curves, which is “split into two” by the construction of this single-valued branch of the functional (see [3], the beginning of §2).

Theorem 1. If on \( S^2 \) there exists at least one nonselfintersecting closed curve \( \gamma \) such that \( I(\gamma) < 0 \), then there exists a nonselfintersecting periodic extremal \( \gamma_0 \) such that \( I(\gamma_0) < 0 \).

Remark. This extremal may not be a minimum of the functional among nonselfintersecting curves. This is the point of an error in §1 of [3].

Theorem 1 follows from the following lemmas.

Lemma 1. If a minimum \( \gamma \) of the functional \( I \) lying in the closure of the set of nonselfintersecting curves is not a periodic extremal, then there exists a closed curve \( \gamma \).
consisting of a single nonselfintersecting closed piece of an extremal containing one turning point (zero angle) with \(l(\gamma) < 0\).

**Lemma 2.** Given a functional of the form (1) on the plane and a convex extremal polygon \(\gamma\) with all the angles between the links not greater than \(\pi\) (the zero angle is allowed) and \(l(\gamma) < 0\), there exists within its interior a smooth nonselfintersecting periodic extremal \(\gamma^*\) such that \(l(\gamma) < 0\).

Theorem 2 follows from the fact that, due to the convexity, the shortening deformations lead into the interior of \(\gamma\).

**Theorem 2.** Suppose that on the torus \(T^2\) there is given a functional of the form (1), where the total flux of the field \(\Omega_{12}\) through \(T^2\) (or a chamber in the plane \(R^2\)) is nonzero. Then in the general case there exist at least 4 nondegenerate periodic extremals on \(R^2\) (up to a translation by a vector in the lattice \(Z + Z\)) with index \(1, 2, 2, 3\) for which the Whitney number is zero and \(l > 0\).

**Corollary 1.** All these extremals are geometrically distinct, since the Whitney number of an iterated curve is nonzero.

The proof follows [3], §4.1II (see the footnote), starting from the “principle of throwing out cycles”, in this case first formulated in [3]; for its generalization see [5]. However, in the process of deformation “downwards” there appear selfintersecting curves, and only the Whitney number remains invariant.

**III.** We turn now to arbitrary dimensions.

**Lemma 3.** If a functional of the form (1) is single-valued and positive on the space of curves nullhomotopic in \(\Omega^+_g(M^n)\), then it is semibounded and single-valued on all spaces of curves \(\Omega^+_g(M^n)\) of arbitrary homotopy classes \(g\) (with a fixed starting point).

If the functional is many-valued on some space \(\Omega^+_g(M^n)\), \(g \neq 1\), then in \(\Omega^+_g\) the group \(\pi_1\) is nontrivial. A map \(S^1 \subseteq \Omega^+_g\) defines a map \(T^2 \rightarrow M^n\), \(f \circ \pi_1(T^2) \neq 0\), such that the flux of the magnetic field \(\Omega_{ij}\) through the cycle \(f(T^2)\) is nonzero. We obtain a map of coverings \(R^2 \rightarrow M^n\). The image \(\gamma_R = f(\gamma_R)\) of a circle of radius \(R\) in \(R^2\) as \(R \rightarrow \infty\) is a curve for which \(l(\gamma_R) < 0\). From this Lemma 3 follows easily. Hence we obtain

**Theorem 3.** On an arbitrary non-simply-connected manifold (closed) the functional (1) possesses at least one periodic extremal.

For the simply-connected case the theorem was obtained in [1].

**Condition. a)** \(H_{2k+1}(M^n; R) \neq 0\) for some \(k\).

**b)** For some \(g\), \(g \neq 1\), the number of generators of \(\pi_1(\Omega^+_g(M^n))\) is not less than two and the class \(g^{2m+1}\) does not coincide with \(g^{2n}\) for any \(n, m \geq 0\).

**Theorem 4.** If the condition holds, then there exist at least two nondegenerate extremals of the functional (1) in general position on a closed manifold, which extremals are geometrically distinct and homotopic to each other.

The existence of at least two generators in \(\pi_1(\Omega^+_g)\) gives the existence of two extremals in \(\Omega^+_g\) of index 0 and 1, if the functional is single-valued and semibounded on \(\Omega^+_g\). Otherwise, the functional is nonpositive on \(\Omega^+_g\), where, therefore, there exist geometrically distinct extremals of index 1 and \(2k + 2\) [5]. The claim that the extremals in \(\Omega^+_g\) are geometrically distinct is proved using Bott’s theorem [6].

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BIBLIOGRAPHY


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