Closed geodesics on non-simply-connected manifolds

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We consider the Hilbert manifold $\Lambda M$ consisting of the $H^1$-maps of the parametrized circle $S^1$ into a closed Riemannian manifold $M^n$ [1]. Let $h \in \pi_1(M^n, x_0)$, $[h]$ be the corresponding free homotopy class of closed paths, $\Lambda M([h])$ the connected component of $\Lambda M$ containing the curves in the class $[h]$, and $\delta_1$ the automorphism of $\pi_1(M^n, x_0)$ corresponding to the element $h$ under the standard action of $\pi_1$ on $\pi_1$.

Any point $c \in \Lambda M$ can be realized by a map $\gamma : [0, 1] \rightarrow M^n$, where $\gamma(0) = \gamma(1)$. We define a map $p : \Lambda M \rightarrow M^n$ by $p(c) = \gamma(0)$, that is, we assign distinguished points to closed parameterized curves. This is the Serre fibration, where a fibre is a space of closed curves with a fixed distinguished point. Let $p(c) = x_0$ and let $c$ realize the element $h \in \pi_1(M^n, x_0)$. We restrict the fibration to $\Lambda M([h])$ and consider an exact homotopy sequence. Since $\pi_1(p^{-1}(x_0), e) = \pi_{1+p}(M^n, x_0)$ this takes the form

$$\pi_1(\Lambda M([h]), e) \rightarrow \pi_1(M^n, x_0) \rightarrow \pi_1(M^n, x_0) \rightarrow \pi_1\left(M^n, x_0\right) \rightarrow.$$ 

**Theorem 1.** 1) $p(\pi_1(\Lambda M([h]), e)) = St(h)$, where $St(h)$ is the subgroup of $\pi_1(M^n, x_0)$ consisting of the elements fixed under $h_\lambda$. 2) $f_i = I - h_i$, $i > 1$, where $I$ is the identity automorphism.

(The original study of $\pi_1(\Lambda M([h]))$ was carried out by the author by direct geometrical constructions; D.V. Anosov drew his attention to the inclusion of these in the fibration structure and the consequent form of $f_i$.)

**Theorem 2.** Suppose that all closed geodesics on a closed manifold are non-singular and that there is an $h \in \pi_1(M^n, x_0)$ such that for an infinite sequence of natural numbers $k_j (j = 1, 2, ...)$ all the classes $[h_{j\lambda}]$ are distinct. If at least one of the following conditions holds: 1) $(I - h_{j\lambda})$ are not automorphisms for all $l$, for some $N \geq 2$; 2) $St(h_{1\lambda}) = Z$, then $M^n$ has infinitely many geometrically distinct closed geodesics.

Proof. By $\Lambda A_j$ and $\Lambda A_j^+$ denote $\Lambda M([h_{j\lambda}])$ and its subspace consisting of curves of length at most $a$. Each $\Lambda A_j$ has a minimal extremal of index 0 and length $r_j$. We remark that $\Lambda A_j^+$ is homeomorphic to a circle. If $St(h_{1\lambda}) = Z$ or $p_\pi(M^n)/(I - h_{1\lambda}^\pi \pi_1(M^n) = 0$, then $\pi_1(\Lambda A_j)$ has at least two generators. If $\ker(I - h_{1\lambda}^\pi \pi_1(M^n) = \pi_1(M^n)/(I - h_{1\lambda}^\pi \pi_1(M^n) = 0$ for $i \geq 2$, then $\pi_1(\Lambda A_j) = 0$.

Considering by means of Morse theory the rearrangements of the $\Lambda A_j$ and their homotopy groups as a passes through a critical level of the length functional, we find that under the conditions of the theorem each $\Lambda A_j$ contains an extremal of index $\lambda$, where $1 < \lambda < N$. The following fact is standard: let $g$ be a closed geodesic and $g_0^m > 0$ for some $m > 1$ (ind is the Morse index of an extremal). Then there are an $a > 0$ and a $0 > 0$ such that $\text{ind} g^{l+1} = \text{ind} g - a l + b$ for all integers $l \geq 1$ ([11], [2]).

It follows that the resulting infinite set of closed geodesics of positive bounded index cannot be formed by multiples of finitely many simple ones, which establishes the theorem.

Using the fact that if $A$ is an automorphism of a finitely generated Abelian group, then for infinitely many natural numbers $k_j (j = 1, 2, ...)$ the homomorphisms $I - A_j^+$ (I is the identity automorphism) are not automorphisms, when all the $[h_{j\lambda}]$, $n > 1$, are distinct, one can simplify the statement of Theorem 2. In particular, we have the following theorem.

**Theorem 3.** Suppose that all closed geodesics on a closed manifold $M^n$ are non-singular and that there is an $h \in \pi_1(M^n, x_0)$ for which all the $[h_{j\lambda}]$, $m > 1$, are distinct. If $\pi_k(M^n)$ is finitely generated for some $k > 2$, and $\pi_k(M^n) = 0$, then $M^n$ has infinitely many geometrically distinct closed geodesics.

When $\pi_1(M^n) = Z$, Theorem 3 gives an answer to the question, posed in [3] and [4], about an estimate of the number of closed geodesics on such manifolds when there are non-trivial finitely generated higher homotopy groups.

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After this note was sent to the printers D.V. Anosov informed the author in December 1984 of the appearance of a paper by V. Bangert and N. Hingston in the Journal of Differential Geometry (1984), 277–282, whose results overlap those of this note.

References

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