Systems of isometries: dynamics and topology

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Interval Exchange Transformations: Definition

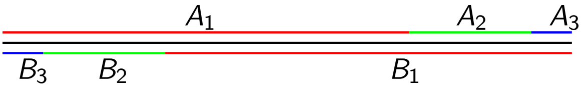
Definition

Let $I \subset \mathbb{R}$ be an interval and $\{A_i : i = 1 \cdots k\}$ be a partition of I. And *interval exchange transformation* is a bijective map $\phi : I \to I$ which is a translation of each A_i.

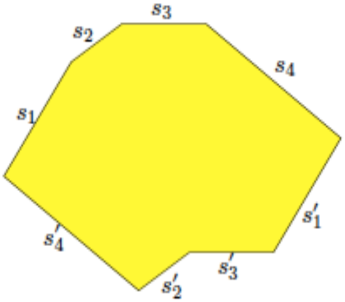
Restriction of map on each A_i is ϕ_i and $\phi_i(A_i) = B_i$. So, $\{B_i : i = 1 \cdots k\}$ is also a partition of I.

Motivation: measured foliations on oriented surface; rational billiards.

Two points $x, y \in I$ belong to the same *orbit* of IET if there exists a word consisting of ϕ_j and ϕ_j^{-1} that sends x to y.



Suspension surface



Interval Exchange Transformations: Main Results

Definition

IET is *minimal* if every orbit is everywhere dense.

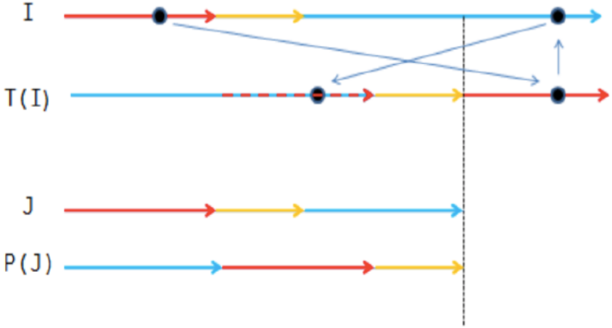
Generic IET is minimal (M. Keane, 1975)

Definition

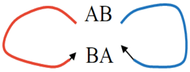
IET is *uniquely ergodic* if it admits exactly one invariant probability measure.

Almost every IET is uniquely ergodic (H. Masur - W. Veech, 1982). Every IET admits at most g invariant measures, where g is the genus of the suspension surface (A. Katok, 1973)

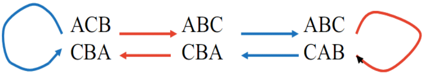
The Rauzy induction



Rauzy diagrams-genus 1

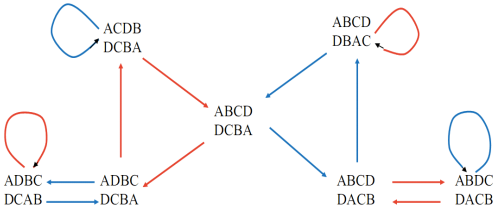


g=1, s=1,d=2



g=1,s=2,d=3

Rauzy diagrams-genus 2



g=2,s=1,d=4

IET with flips

- Motivation: measured foliations on non-orientable surface;
- Minimality: almost all IET with flips have periodic orbit (A. Nogueira, 1989);

Theorem (S. - S. Troubetzkoy, 2015)

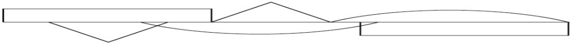
The Hausdorff dimension of the set X_n of minimal n-fIET satisfies:

$$n - 2 \leq Hdim(X_n) < n - 1.$$

Open questions: invariant measure, ergodic properties.

Interval Translation Mappings: definitions

ITM were introduced by M. Boshernitzan and I. Kornfeld in 1994.



With each ITM $T : I \to I$ we associate the set
$\Omega_i = I \cap TI \cdots \cap T^i I$.

Definition

ITM T is of *finite type* if for some n $\Omega_n = \Omega_{n+1}$. If all inclusions $\Omega_n \subset \Omega_{n+1}$ are proper, ITM is of *infinite type*.

Conjecture (Boshernitzan - Kornfeld)

Almost all ITM are of finite type.

Interval Translation Mappings: results

The result is only known for *double rotations* (H. Bruin - G. Clack, 2011) based on the induction described by Suzuki, Ito, Aihara.

Theorem (Bruin - Clack)

For any measure invariant with respect to SIA induction almost all double rotations of infinite type are uniquely ergodic.

Work in progress with M. Artigiani, Ch. Fougeron and P. Hubert):

- where to find such a measure?
- what about mixing properties?

Theorem (Buzzi-Hubert, 2004)

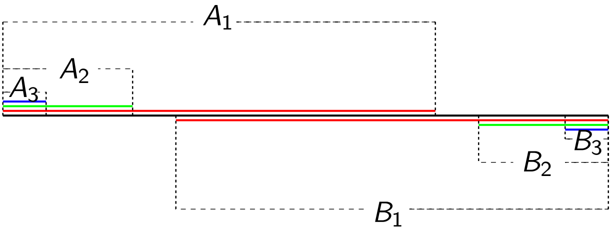
ITMs without flips admit at most n invariant measures where n is the number of monotone intervals; in case of flipped intervals this number doubles.

Systems of isometries

Systems of isometries were introduced by D. Gaboriau, G. Levitt and F. Paulin in 1994.

Definition

A *system of isometries* is a pair $S = (D, \{\phi_j\}_{j=1,\cdots,k})$ where D is a multi-interval and each $\phi_j : A_j \to B_j$ is an isometry between closed subintervals of D.



Systems of isometries of thin type

Definition

System of isometries S is called *balanced* if the following hold:

- all ends of D are covered by some subintervals A_i or B_i;
- $\Sigma_{i=1}^{n}|A_i| = |D|$, where $|A|$ means length of subinterval A;

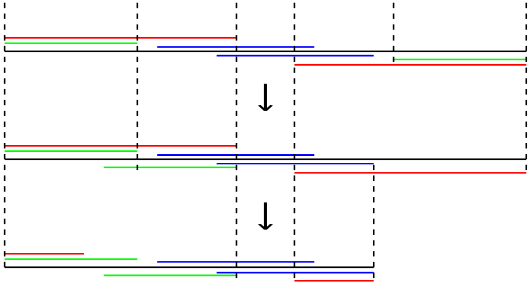
Definition

Balanced system of isometires is of *thin type* if all orbits are everywhere dense.

Thin case was discovered by G. Levitt in terms of $\mathbb{R}$-trees in 1993.
Main conjecture:
(S.P. Novikov, 2003; I. Dynnikov, 2008)
Thin systems of isometries form a zero measure set in parameter space.

Rauzy induction: translation + reduction

We say that an system of isometry has a *hole* if there are some points in the support interval that are not covered by an interval from S. The Rauzy induction stops when we obtain a hole.

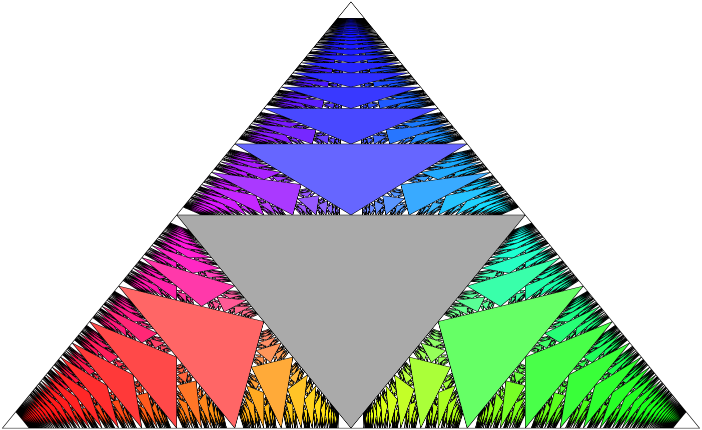


Baby Example: the Rauzy Gasket

Let us start from the easiest, two-dimensional case: $D = [0,1]$, all A_i start in 0, all B_i end in 1.

- Triangle as a parameter space;
- Matrix of Rauzy induction has many in common with fully subtractive algorithm: $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$;
- Markov Partition (and therefore countable Markov shift);
- Contracting and Bounded Distortion Properties.

The Rauzy Gasket: Photo



The Rauzy Gasket: Bio

- **Date of Birth**: 2011 (2008, 1993)
- **Creators**: P. Arnoux and S. Starosta
- **Motivation**: episturmian words, multidimensional fraction algorithms
- **Alternative origin**:
  - I. Dynnikov and R. De Leo (3-dimensional topology: Novikov's problem);
  - G. Levitt (geometric group theory: pseudogroups of rotations);

The Rauzy Gasket: Metric Characteristics

Theorem

(G. Levitt and J.-C. Yoccoz, I. Dynnikov and R. De Leo, P. Arnoux and S. Starosta):
The Rauzy Gasket has zero Lebesgue measure.

Open question (P. Arnoux): estimate Hausdorff dimension?
Numerical estimations (I. Dynnikov and R. De Leo, 2008): between 1.7 and 1.8.

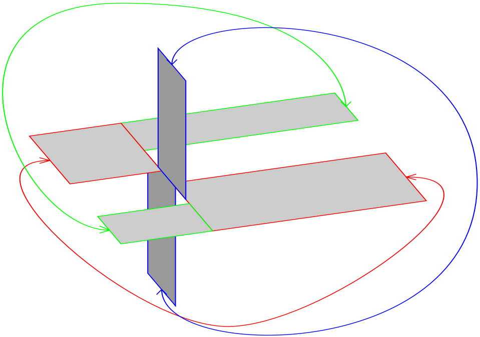
Theorem

(Artur Avila, Pascal Hubert and SS, 2013):
The Hausdorff dimension of the Rauzy gasket is strictly less than 2.

Lower bound (C. Matheus, R. Gutiérrez - Roma, 2019): lower bound is greater than 1.19.

2-dimensional foliated complex: band complex

There exists a kind of a suspension of systems of isometries that provides us with a foliated 2-complex (X, ω). Leaves of foliations correspond to orbits of systems.



The Cocycle and the Roof Function

- Accelerated matrix A:

$$\begin{pmatrix} n & 1 & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

- Matrix of the cocycle B:

$$\begin{pmatrix} 1 & 0 & 0 \\ n & 1 & 0 \\ n & 0 & 1 \end{pmatrix}$$

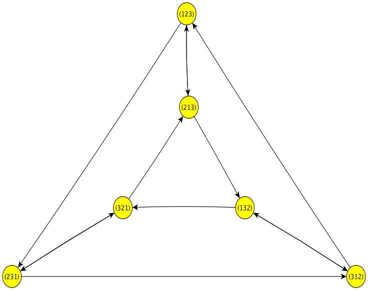
- the roof function: the first return time to the small simplex.

Theorem

The roof function r has exponential tails: there exists $\sigma > 0$ such that $\int_\Delta e^{\sigma r} dLeb < \infty$.

Combinatorics

The Rauzy graph illustrates that we work with a countable Markov shift:



Moreover, the shift is strongly recurrent (satisfies BIP property).

Key dynamical properties

Theorem (A. Avila - P. Hubert - SS, 2014)

There exists the measure of maximal entropy for the suspension flow of the Rauzy gasket, and this measure is unique.

The proof is based on thermodynamical formalism (O. Sarig' 2002).
Abramov's formula then gives us an invariant measure μ for the Rauzy gasket.

Thank you very much!