Isoperimetric inequalities for Laplace eigenvalues on the sphere and the real projective plane

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Geometry, Topology and their Applications

Outline

Laplace-Beltrami operator on manifolds

- Laplace-Beltrami operator on a Riemannian manifold

$$\Delta f = -\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^i}\left(\sqrt{|g|}g^{ij}\frac{\partial f}{\partial x^j}\right),$$

where g_{ij} is the metric tensor, g^{ij} are the component of the matrix inverse to g_{ij} and $g = \det g$.

Spectral problem for the Laplace-Beltrami operator

- Spectral problem for the Laplace-Beltrami operator on a Riemannian manifold M without boundary

$$\Delta f = \lambda f$$

- The spectrum consists only of eigenvalues

$$0 = \lambda_0(M,g) < \lambda_1(M,g) \leqslant \lambda_2(M,g) \leqslant \dots$$

Geometric optimization of eigenvalues

- Let us fix M. Then $\lambda_k(M, g)$ is a functional on the space of Riemannian metrics on M

$$g \longmapsto \lambda_k(M, g)$$

- Natural geometric optimization problem: find

$$\Lambda_k(M) = \sup_g \lambda_k(M, g),$$

  where g belongs to the the space of Riemannian metrics on M such that $\mathrm{Vol}(M, g) = 1$
- This is a good question only for surfaces

Rescaling of a metric

- Let us remark that $\bar{\lambda}_k(M,g) = \lambda_k(M,g)\,\mathrm{Vol}(M,g)$ is invariant under rescaling $g \mapsto tg$.
- This means that instead looking for

$$\sup_g \lambda_k(M,g),$$

where g belongs to the the space of Riemannian metrics on M such that $\mathrm{Vol}(M) = 1$ one can look for

$$\sup_g \bar{\lambda}_k(M,g),$$

where g belongs to the the space of *all* Riemannian metrics on M.

Geometric optimization vs isoperimetric inequality

- Geometric optimization problem: find

$$\Lambda_k(M) = \sup_g \bar{\lambda}_k(M, g),$$

where g belongs to the the space of Riemannian metrics on M.

- Isoperimetric inequality: for any metric g on M such that $\mathrm{Vol}(M, g) = 1$ the inequality

$$\lambda_k(M, g) \leqslant \Lambda_k(M)$$

holds

Upper bounds

- Yang and Yau (1980): for an orientable surface M of genus γ we have
$$\bar{\lambda}_1(M, g) \leqslant 8\pi(\gamma + 1).$$
- In fact, Yang and Yau argument implies
$$\bar{\lambda}_1(M, g) \leqslant 8\pi \left[\frac{\gamma + 3}{2}\right].$$
- Karpukhin (2016): for a non-orientable surface M of genus γ we have
$$\bar{\lambda}_1(M, g) \leqslant 16\pi \left[\frac{\gamma + 3}{2}\right].$$

Upper bounds

- Korevaar (1993): there exists a constant C such that for any $k > 0$ and any compact surface M of genus γ the functional $\bar{\lambda}_k(M,g)$ is bounded,

$$\bar{\lambda}_k(M,g) \leqslant C(\gamma+1)k.$$

- As a result,

$$\Lambda_k(M) < +\infty.$$

Maximal metric

- Definition. Let M be a closed surface. A metric g_0 on M is called *maximal* for the functional $\bar\lambda_k(M,g)$ if

$$\Lambda_k(M) = \bar\lambda_k(M,g_0)$$

Eigenvalues as functions of a metric

- The functional $\bar{\lambda}_k(M, g)$ depends continuously on the metric g, but this functional is not differentiable.
- However, it was shown by Berger, Bando & Urakawa, El Soufi & Ilias that for analytic deformations g_t the left and right derivatives of the functional $\bar{\lambda}_k(M, g_t)$ with respect to t exist.

Extremal metrics

- **Definition** (Nadirashvili, 1986, El Soufi and Ilias, 2000). A Riemannian metric g on a closed surface M is called *extremal metric* for the functional $\bar\lambda_k(M,g)$ if for any analytic deformation g_t such that $g_0 = g$ the following inequality holds,

$$\frac{d}{dt}\bar\lambda_k(M,g_t)\Big|_{t=0+}\cdot\frac{d}{dt}\bar\lambda_k(M,g_t)\Big|_{t=0-}\leqslant 0.$$

What can we say about particular surfaces?

- $\lambda_1(\mathbb{S}^2, g)$. Hersch proved in 1970 that $\Lambda_1(\mathbb{S}^2) = 8\pi$ and the maximum is reached on the canonical metric on $\mathbb{S}^2$. This metric is the unique extremal metric.
- $\lambda_1(\mathbb{R}P^2, g)$. Li and Yau proved in 1982 that $\Lambda_1(\mathbb{R}P^2) = 12\pi$ and the maximum is reached on the canonical metric on $\mathbb{R}P^2$. This metric is the unique extremal metric.
- $\lambda_1(\mathbb{T}^2, g)$. Nadirashvili proved in 1996 that $\Lambda_1(\mathbb{T}^2) = \frac{8\pi^2}{\sqrt{3}}$ and the maximum is reached on the flat equilateral torus. El Soufi and Ilias proved in 2000 that the only extremal metric for $\bar{\lambda}_1(\mathbb{T}^2, g)$ different from the maximal one is the metric on the Clifford torus.

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What can we say about particular surfaces?

- $\lambda_1(\mathbb{K}, g)$. Jakobson, Nadirashvili and I. Polterovich proved in 2006 that the metric on a Klein bottle realized as the Lawson bipolar surface $\tilde{\tau}_{3,1}$ is extremal. El Soufi, Giacomini and Jazar proved in the same year that this metric is the unique extremal metric and the maximal one. There is a common belief that $\Lambda_1(\mathbb{K}) = \bar{\lambda}_1(\mathbb{K}, g_{\tilde{\tau}_{3,1}}) = 12\pi E\left(\frac{2\sqrt{2}}{3}\right)$, where E is a complete elliptic integral of the second kind,

$$E(k) = \int_0^1 \frac{\sqrt{1 - k^2\alpha^2}}{\sqrt{1 - \alpha^2}} d\alpha.$$

What can we say about particular surfaces?

- $\lambda_2(\mathbb{S}^2, g)$. Nadirashvili proved in 2002 that $\Lambda_2(\mathbb{S}^2, g) = 16\pi$ and maximum is reached on a singular metric which can be obtained as the metric on the union of two spheres of equal radius with canonical metric glued together. The proof contained some gaps filled later by Petrides (2012).
- $\lambda_3(\mathbb{S}^2, g)$. Nadirashvili and Sire proved in 2015 that $\Lambda_3(\mathbb{S}^2, g) = 24\pi$ and maximum is reached on a singular metric which can be obtained as the metric on the union of three spheres of equal radius with canonical metric glued together.

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The most recent results

- $\lambda_1(\Sigma_2, g)$. It was shown by Jakobson, Levitin, Nadirashvili, Nigam, and I. Polterovich in 2005 using a combination of analytic and numerical tools that the maximal metric for the first eigenvalue on the surface of genus two Σ_2 is the metric on the Bolza surface $\mathcal{P}$ induced from the canonical metric on the sphere using the standard covering $\mathcal{P} \longrightarrow \mathbb{S}^2$. The result was stated as a conjecture, because the argument is partly based on a numerical calculation.
- The this conjecture was recently proved by Nayatani and Shoda (2019).

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The most recent results

- $\lambda_2(\mathbb{RP}^2, g)$. Nadirashvili, A.P., 2016: $\Lambda_2(\mathbb{R}P^2) = 20\pi$ and the supremum can not be attained on a smooth metric, it is realized in the limit on a sequence of metrics degenerating to a union of a projective plane with the canonical metric and a sphere with the canonical metric touching each other such that the ratio of volumes is 3 : 2.

The most recent results

- $\lambda_k(\mathbb{S}^2, g)$. Karpukhin, Nadirashvili, A.P., I. Polterovich, 2017:
  The equality $\Lambda_k(\mathbb{S}^2) = 8\pi k$ holds for any $k \geqslant 1$. For $k = 1$ the supremum is attained if and only if g is the standard round metric on a sphere. For $k \geqslant 2$ the supremum can not be attained on a smooth metric, and is realized in the limit if and only if the corresponding sequence of metrics degenerates to a union of k touching identical round spheres.

The most recent results

- $\lambda_k(\mathbb{RP}^2, g)$. Karpukhin, 2019:
  The equality $\Lambda_k(\mathbb{RP}^2) = 4\pi(2k+1)$ holds for any $k \geqslant 1$. For $k = 1$ the supremum is attained if and only if g is the standard metric on $\mathbb{RP}^2$. For $k \geqslant 2$ the supremum can not be attained on a smooth metric, and is realized in the limit if and only if the corresponding sequence of metrics degenerates to a union of $\mathbb{RP}^2$ with the standard metric and $k-1$ touching identical round spheres such that the ratio of volumes is $3:2$.

Extremal metrics

- $\lambda_i(\mathbb{T}^2, g), \lambda_i(\mathbb{K}, g)$. Several series of extremal metrics on tori and Klein bottles:
- Bipolar Lawson τ-surfaces $\tilde{\tau}_{r,k}$ (Lapointe, 2008),
- Lawson tau-surfaces $\tau_{r,k}$ (A.P., 2012),
- Otsuki tori $O_{\frac{p}{q}}$ (A.P., 2013),
- Bipolar Otsuki tori $\tilde{O}_{\frac{p}{q}}$ (Karpuhin, 2014)
- Generalized Lawson τ-surfaces (A.P., 2015)

A classical theorem

- Let N be a submanifold of $\mathbb{R}^n$. Let Δ be the Laplace-Beltrami operator on N equipped with the induced metric.
- **Theorem.** The restrictions $x^1|_N, \ldots, x^n|_N$ on N of the standard coordinate functions of $\mathbb{R}^{n+1}$ are harmonic iff N is a minimal submanifold of $\mathbb{R}^n$.

Takahashi theorem (1966)

- Let N be a d-dimensional submanifold of $\mathbb{R}^{n+1}$. Let Δ be the Laplace-Beltrami operator on N equipped with the induced metric.
- **Theorem.** The functions $x^1|_N, \ldots, x^{n+1}|_N$ are eigenfunctions of Δ with eigenvalue $\frac{d}{R^2}$ iff N is a minimal submanifold of the sphere S^n_R of radius R.

Theorem by Nadirashvili (1996), El Soufi & Ilias (2008)

- Let us introduce the eigenvalue counting function
$$N(\lambda) = \#\{\lambda_i | \lambda_i < \lambda\}.$$
- **Theorem.** The metric g_0 induced on N by minimal immersion $N \subset \mathbb{S}^n$ is an extremal metric for the functional $\bar{\lambda}_{N\left(\frac{d}{R^2}\right)}(N, g)$.

How to find extremal metrics?

- Find a minimally immersed surface Σ in a unit sphere $\mathbb{S}^n$
- Find $N(2)$
- Then the induced metric on Σ is extremal for $\bar{\lambda}_{N(2)}(\Sigma, g)$.

Minimal maps and harmonic maps

- Let (M, g) and (N, h) be Riemannian manifolds. A smooth map $f : M \longrightarrow N$ is called *harmonic* if f is an extremal for the energy functional

$$E[f] = \int_M |df(x)|^2 \, dVol_g.$$

- Theorem. Let M, N be Riemannian manifolds. If $f : M \looparrowright N$ is an isometric immersion, then f is harmonic if and only if f is minimal.

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Harmonic maps of surfaces

- It is well-known that if $\dim M = 2$, then the property of f to be harmonic depends only on the conformal class of the metric g.
- A harmonic map $f : M \longrightarrow N$ is called *conformal* if the metric induced by f belongs to the same conformal class on M for which the map f is extremal for the energy functional.

Harmonic maps and extremal metrics

- Proposition. Let M be a compact surface. Let a metric g on M be extremal for a functional $\bar\lambda_k(M,\cdot)$. Then there exists a conformal harmonic immersion $f : M \looparrowright \mathbb{S}^n$ from M (endowed with the conformal class of the metric g) to $\mathbb{S}^n$ (endowed with the canonical metric $g_{\mathbb{S}^n}$ of radius 1), such that $g = f^* g_{\mathbb{S}^n}$, i.e. g is induced by f.
- Conversely, let M be a compact surface with a fixed conformal class and $f : M \looparrowright \mathbb{S}^n$ be a conformal harmonic immersion from M to $\mathbb{S}^n$ endowed with the canonical metric $g_{\mathbb{S}^n}$ of radius 1. Then the metric $g = f^* g_{\mathbb{S}^n}$ induced by f is extremal for the functional $\bar\lambda_k(M,\cdot)$ for $k = N(2)$.

Harmonic maps and extremal metrics

- It is a well-known fact that there is only one conformal class of Riemannian metrics on $\mathbb{S}^2$ and $\mathbb{RP}^2$. Hence, for $\mathbb{S}^2$ and $\mathbb{RP}^2$ any harmonic immersion is conformal.
- Corollary. The extremal metrics for the eigenvalues of the Laplace-Beltrami operator on the sphere $\mathbb{S}^2$ and projective plane $\mathbb{RP}^2$ are exactly the metrics induced on $\mathbb{S}^2$ or $\mathbb{RP}^2$ by harmonic immersions $f : \mathbb{S}^2 \looparrowright \mathbb{S}^n$ or $f : \mathbb{RP}^2 \looparrowright \mathbb{S}^n$, respectively.

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Calabi-Barbosa theorem

- A harmonic map $f : \Sigma \longrightarrow \mathbb{S}^n \subset \mathbb{R}^{n+1}$ of a surface Σ to the standard unit sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ is called *linearly full* if the image $f(\Sigma)$ does not lie in a hyperplane of $\mathbb{R}^{n+1}$.
- Theorem (Calabi, Barbosa). Let $f : \mathbb{S}^2 \longrightarrow \mathbb{S}^n$ be a linearly full harmonic immersion (possibly, with branch points). Then
  (i) the area of $\mathbb{S}^2$ with respect to the induced metric $\mathrm{Area}(\mathbb{S}^2, f^* g_{\mathbb{S}^n})$ is an integer multiple of 4π;
  (ii) n is even, $n = 2m$, and

$$\mathrm{Area}(\mathbb{S}^2, f^* g_{\mathbb{S}^n}) \geqslant 2\pi m(m+1).$$

Harmonic degree

- If $\mathrm{Area}(\mathbb{S}^2, f^*g_{\mathbb{S}^n}) = 4\pi d$, then we say that f is of *harmonic degree d*.
- Let $f : \mathbb{S}^2 \longrightarrow \mathbb{S}^{2m}$ be a linearly full harmonic immersion with branch points. Then $d \geqslant \frac{m(m+1)}{2}$.
- Proposition. Extremal metrics (possibly with conical singularities) on $\mathbb{S}^2$ are induced by linearly full harmonic immersions with branch points $f : \mathbb{S}^2 \longrightarrow \mathbb{S}^{2m}$. The harmonic degree d of such an immersion satisfies the inequality $d \geqslant \frac{m(m+1)}{2}$.

Ejiri bound

- Theorem (Ejiri, 1998)
  Let $f:\mathbb{S}^2 \longrightarrow \mathbb{S}^{2m}$ be a linearly full harmonic map of harmonic degree $d>1$ of $\mathbb{S}^2$ to the standard unitary sphere $\mathbb{S}^{2m}$. Then
  $$N(2) \geqslant d+1.$$

Scheme of proof for $\mathbb{S}^2$

- 1) For a sequence converging to k touching spheres one has
$$\lim_{n\to\infty} \bar{\lambda}_k(\mathbb{S}^2, g_n) = 8\pi k \Longrightarrow \Lambda_k(\mathbb{S}^2) \geqslant 8\pi k.$$
- 2) Suppose there exists a sequence of metrics g_n such that
$$\lim_{n\to\infty} \bar{\lambda}_k(\mathbb{S}^2, g_n) > 8\pi k.$$
  Then by regularity results (Nadirashvili, Sire, 2015, Petrides, 2017) there exists the limit metric $g_{\lim}$, smooth except for a finite number of conical singularities.
- 3) Then $g_{\lim}$ is induced by a harmonic immersion $f : \mathbb{S}^2 \longrightarrow \mathbb{S}^{2m}$ of harmonic degree $d \geqslant \frac{m(m+1)}{2}$.

Scheme of the proof for $\mathbb{S}^2$

- 4) Ejiri theorem implies that for the metric $g_{\lim} = f^* g_{round}$ on $\mathbb{S}^2$ induced by f we have $N(2) \geqslant d+1$. On the other hand, we have $\mathrm{Area}(\mathbb{S}^2, f^* g_{round}) = 4\pi d$. Since the value of $\lambda_{N(2)} = 2$ by Takahashi theorem, we have

$$\bar\lambda_{N(2)}(\mathbb{S}^2, f^* g_{round}) = 2\,\mathrm{Area}(\mathbb{S}^2, f^* g_{round}) = 8\pi d.$$

If we denote $N(2) = k$, we have

$$\bar\lambda_k(\mathbb{S}^2, f^* g_{round}) = 8\pi d < 8\pi k,$$

i.e. any smooth metric (with possibly conical singularities) extremal for $\bar\lambda_k$ has the value of $\bar\lambda_k$ strictly less then limit of $\bar\lambda_k$ on a sequence converging to k touching spheres. Hence, any smooth extremal metric induced by a harmonic immersion of harmonic degree $d > 1$ is not a maximal metric.

Minimal surfaces and multiplicity of eigenvalues

- A minimal immersion by eigenfunctions $M \longrightarrow \mathbb{S}^n \subset \mathbb{R}^{n+1}$ requires $n+1$ linearly independent eigenfunctions of given eigenvalue λ_i.
- It follows that if there is an upper bound on multiplicity $m(\lambda_i) \leqslant M$ then one should study only minimal surfaces in $\mathbb{S}^{M-1}$ in order to study extremal metrics for the eigenvalue λ_i.

Scheme of the proof for $\lambda_2(\mathbb{RP}^2)$

- Prove that $m(\lambda_2, \mathbb{RP}^2) \leqslant 6$, hence it is sufficient to consider harmonic maps of $\mathbb{RP}^2$ to $\mathbb{S}^2$ (they are, in fact, trivial) and $\mathbb{S}^4$.
- Use Bryant theory of harmonic maps $\mathbb{S}^2 \longrightarrow \mathbb{S}^4$ in order to prove that either $d = 3$ or there are singularities.
- prove that if there is a singularity, then this map induces a metric extremal for λ_i with $i > 2$.
- the remaining part as for $\mathbb{S}^2$.

New idea in the proof for $\lambda_k(\mathbb{RP}^2)$

- Karpukhin's new idea: use the nullity and index of the second variation of the Area functional on a minimal map $\mathbb{RP}^2 \longrightarrow \mathbb{S}^n$.