Invariant geometry on nilmanifolds and narrow Lie algebras

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The plan of my talk

The purpose of my talk is just to answer the question
"Are positively graded Lie algebras interesting to geometers?"

- nilmanifolds
- positively graded Lie algebras
- filiform Lie algebras and affine structures.Benoist's list.
- filiform Lie algebras and symplectic forms on nilmanifolds
- Fialowski's classification
- Carnot algebras of width $3/2$ and complex structures

Nilmanifolds

Definition

Nilmanifold M = a compact homogeneous space $M = G/\Gamma$ where G is a simply connected nilpotent Lie group. $\Gamma \subset G$ – is a cocompact lattice.

Example (Heisenberg nilmanifold)

$$H_3 = \left\{ \begin{pmatrix} 1 & x & t \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, x, y, t \in \mathbb{R}, \;\; \Gamma_3 = \{(\dots), x, y, t \in \mathbb{Z}\} \right\}$$

$M^3 = H_3/\Gamma_3, \; KT = M^3 \times S^1$ – Kodaira-Thurston manifold.

Anatoly Maltsev's construction 1949

Let $\mathfrak{g}$ be a nilpotent Lie algebra with a basis $e_1, \dots, e_n$ and

$$[e_i, e_j] = \sum_k c_{ij}^k e_k, \quad c_{ij}^k \in \mathbb{Q},$$

Remark

The vector space $\mathfrak{g}$ has a Lie group structure $\star$ (the Baker-Campbell-Hausdorff formula):

$$x \star y = x + y + \frac{1}{2}[x, y] + \dots$$

so G is a nilpotent Lie group
and $\Gamma \subset G$ is a lattice generated by $e_1, e_2, \dots, e_n$.
It follows that G/Γ is a compact nilmanifold

Nil-index

A Lie algebra $\mathfrak{g}$ is called nilpotent, if there exists a positive integer s such that

$$\mathfrak{g}^1 = \mathfrak{g} \supset \mathfrak{g}^2 = [\mathfrak{g}, \mathfrak{g}] \supset \cdots \supset \mathfrak{g}^s \supset \mathfrak{g}^{s+1} = [\mathfrak{g}, \mathfrak{g}^s] = 0, \ \mathfrak{g}^s \neq 0.$$

We call s nil-index of $\mathfrak{g}$. $\mathfrak{g}$. There is an upper bound

$$s(\mathfrak{g}) \leq \dim \mathfrak{g} - 1.$$

Definition (Vergne)

A finite dimensional Lie algebra $\mathfrak{g}$ is called filiform, if $s(\mathfrak{g}) = \dim \mathfrak{g} - 1$.

Positively graded Lie algebras

Consider a finite dimensional positively graded Lie algebra

$$\mathfrak{g} = \oplus_{\alpha \in S} \mathfrak{g}_\alpha, \ [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}, \alpha, \beta \in S.$$

where $S \subset \mathbb{R}_{>0}$ is a subset of positive real numbers. Obviously $\mathfrak{g}$ is nilpotent Lie algebra. On another hand **all nilpotent Lie algebras of dimensions ≤ 6 admit positive grading**.

Example (generic 6-dimensional nilpotent Lie algebra)

$$S = 1, 2, 3, 4, 5, \quad , 7, \qquad e_1, \ldots, e_5, \quad e_7,$$
$$[e_i, e_j] = \begin{cases} e_{i+j}, \ i < j, i+j \in S, \\ 0, i < j, i+j \notin S. \end{cases}$$

One can verify that it is a **filiform Lie algebra**.

Consider be a positively graded Lie algebra

$$\mathfrak{g} = \oplus_{\alpha \in S} \mathfrak{g}_\alpha, S \subset R_{>0}$$

Then a semisimple operator $\tau : \mathfrak{g} \to \mathfrak{g}$ defined by

$$\tau(x) = \alpha x, x \in \mathfrak{g}_\alpha,\ \alpha \in S.$$

is a **derivation** of $\mathfrak{g}$

$$\underbrace{\tau([x, y])}_{=(\alpha+\beta)[x,y]} = \underbrace{[\tau(x), y]}_{=[\alpha x,y]} + \underbrace{[x, \tau(y)]}_{=[x,\beta y]}, \quad x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_\beta.$$

In the opposite direction: an arbitrary **semisimple derivation** τ **with positive eigenvalues** determines a **positive grading** of $\mathfrak{g}$. One can define the solvable Lie algebra $\hat{\mathfrak{g}}$ = "left extension" of $\mathfrak{g}$

$$\hat{\mathfrak{g}} = \langle h \rangle \oplus \mathfrak{g},\ [h, x] = \tau(x), x \in \mathfrak{g}.$$

Yu. Neretin, "A remark on nilpotent Lie algebras that do not admit gradings"// Zapiski Nauchnykh Seminarov POMI, **481** (2019).

The abstract of his article **"We explain why nilpotent Lie algebras usually are characteristically nilpotent, i.e., do not admit $\mathbb{Z}$-gradings".**

Definition

A Lie algebra $\mathfrak{g}$ is called **characteristically nilpotent** if and only if all derivations of $\mathfrak{g}$ are nilpotent.

The first example of **characteristically nilpotent** Lie algebra was constructed by Dixmier and Lister in 1957 in dimension 8.

Metric Lie algebras $\quad \mathfrak{g}_\alpha \perp \mathfrak{g}_\beta, \ \alpha \neq \beta$

Let G be a simply connected **Lie group** endowed with a **left-invariant metric** g. We may identify the Riemannian manifold (G, g) with the metric Lie algebra $(\mathfrak{g}, (\cdot,\cdot))$=the tangent space T_eG at the identity $e \in G$ with the inner product $(\cdot,\cdot)$ determined by the left-invariant metric g.

We have for the Levi-Civita connection ∇, sectional curvature K the following formulas

$$\nabla_X Y = \frac{1}{2}\left(ad_X Y - ad^*_X Y - ad^*_Y X\right),$$
$$K(X \wedge Y) = \|\nabla_X Y\|^2 - (\nabla_X X, \nabla_Y Y) - ([Y,[Y,X]],X) - \|[X,Y]\|^2,$$

For a positively graded Lie algebra $\mathfrak{g} = \oplus_{\alpha \in S}\mathfrak{g}_\alpha$ usually require that the homogeneous subspaces $\mathfrak{g}_\alpha$ be orthogonal to each other.

A positively graded Lie algebra $\mathfrak{g} = \oplus_{i=1}^{+\infty} \mathfrak{g}_i$ is called a Lie algebra **of finite width**, if

$$\exists C \geq 0, \;\; \dim \mathfrak{g}_i \leq C, \;\; \forall i \in \mathbb{N}.$$

Definition (Shalev, Zelmanov 97)

The **width** $d(\mathfrak{g})$ of a positively graded Lie algebra $\mathfrak{g} = \oplus_{i=1}^{+\infty} \mathfrak{g}_i$ of finite width is called

$$d(\mathfrak{g}) = \max_{i \in \mathbb{N}} \dim \mathfrak{g}_i.$$

Narrow Lie algebras = Lie algebras **of width one or two**.

Truncated Lie algebras

Consider an infinite dimensional $\mathbb{N}$-graded Lie algebra $\mathfrak{g} = \oplus_{i=1}^{+\infty} \mathfrak{g}_i$.
One can consider its finite dimensional quotient

$$\hat{\mathfrak{g}}_n = \oplus_{i=1}^{+\infty} \mathfrak{g}_i / \oplus_{i=n+1}^{+\infty} \mathfrak{g}_i$$

Obviously $\hat{\mathfrak{g}}_n$ is the finite dimensional positively graded Lie algebra.

The narrowest Lie algebras

Example (Vergne, 1970)

The Lie algebra $\mathfrak{m}_0$ is defined by its infinite basis $e_1, e_2, e_3, \dots$ and relations

$$[e_1, e_i] = e_{i+1}, i \geq 2, \quad [e_i, e_k] = 0, \ i, k \neq 1.$$

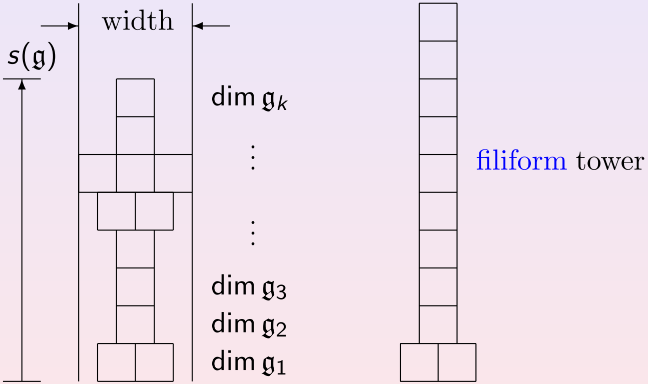
One-dimensional homogeneous subspaces $(\mathfrak{m}_0)_k = \langle e_k \rangle$.

Example

The positive part W^+ of the Witt algebra. The basis $\{e_i, i \in \mathbb{N}\}$ and relations

$$[e_i, e_j] = (j - i)e_{i+j}, \ \forall i, j \in \mathbb{N}.$$

Positively graded Lie algebras and lego towers



Benoist's classification 1992

Yves Benoist classified Lie algebras $\mathfrak{a}_r$, $r \in \mathbb{R}$, defined by

1) two generators f_1 and f_2;
2) two relations

$$\overbrace{[f_2,[f_1,f_2]]}^{\sum i=5} = \overbrace{[f_1,[f_1,[f_1,f_2]]]}^{\sum i=5},$$
$$\underbrace{[f_2,[f_1,[f_1,[f_1,f_2]]]]}_{\sum i=7} = r\underbrace{[f_1,[f_1,[f_1,[f_1,[f_1,f_2]]]]]}_{\sum i=7},$$

Theorem (Benoist, 1992)

If $r \neq \frac{9}{10}, 1$ then $\mathfrak{a}_r$ is a finite-dimensional Lie algebra
1) Let $r = \frac{9}{10}$ then $\mathfrak{a}_r \cong W^+$,
2) Let $r = 1$ then $\mathfrak{a}_r \cong \mathfrak{m}_2$.
3) Let $r \neq 0, \frac{9}{10}, 1, 2, 3$ then $\mathfrak{a}_r$ is a 11-dimensional graded filiform

Why did Benoit classify graded Lie algebras?

In 1992 **Benoist gave a negative answer to the question (conjecture) of Milnor** of 1977, who asked, whether a left-invariant affine structure always exists on a simply connected nilpotent group?

Benoist found an example of 11-dimensional nilmanifold, **which does not admit any complete invariant affine structure.**

Affine structure on a manifold M = **flat torsionfree affine connection** ∇ on M, i.e.

- torsionfree $\quad \nabla_X Y - \nabla_Y X - [X, Y] = 0;$
- flat $\quad \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} = 0.$

There are two operators:

- left $L(X) : \mathfrak{g} \to \mathfrak{g}, L(X)Y = \nabla_X Y$
- right $R(Y) : \mathfrak{g} \to \mathfrak{g}, R(Y)X = \nabla_X Y$

An affine structure ∇ is complete $\Leftrightarrow$ $R(Y)$ is nilpotent $\forall Y \in \mathfrak{g}$.
There is a faithfull representation

$$\alpha : \mathfrak{g} \to \mathfrak{aff}(\mathbb{R}^n) = \mathfrak{gl}_n(\mathbb{R}) \oplus \mathbb{R}^n,$$
$$\alpha(X) = \begin{pmatrix} L(X) & X \\ 0\dots0 & 0 \end{pmatrix}.$$

Operator $\alpha(X)$ acts in $(n+1)$-dimensional vector space.

Benoist considered 11-dimensional **filiform** graded Lie algebra $\mathfrak{a}_{-2}$ and then its **generic deformation** $\mathfrak{a}_{-2,1,t}$.

The hard part of his proof: he shown that the Lie algebra $\mathfrak{a}_{-2,1,t}$ **does not admit faithfull linear representations of dimension** 12.

Almost simultaneously with the work of Benoist, **Burde and Grunewald** obtained a very close result. Moreover they added counterexamples in dimensions 10 and 12 and proved that there are **no counterexemples in dimensions** ≤ 9.

Conjecture

Generic deformation $[,] + \Psi$ *of the truncated positive Witt algebra* W_n^+ *does not admit any affine structure for* $n \geq 13$.

Symplectic nilmanifolds are compact non-Kähler manifolds

Example

Kodaira (Kodaira-Thurston) surface $K = M_3 \times S^1$ with a symplectic form $\omega = \frac{1}{2\pi} dy \wedge d\phi + dx \wedge dz$, where M_3 is the Heisenberg manifold and $d\phi$ is the standard volume form on S^1.

In 1998 a special family of nilmanifolds was used by **Babenko and Taimanov** for their construction **examples of non-formal simply connected compact symplectic manifolds** $M^{2k}, k \geq 5$.

A natural question arose "How many other filiform graded Lie algebras admit a symplectic structure?"

Graded filiform Lie algebras

Theorem (M., Russian Math. Surveys 2002, AMS Transl. 2004)

The classification list of $\mathbb{N}$-graded filiform Lie algebras

$$\begin{gathered} \mathfrak{g} = \bigoplus_{i=1}^{n} \mathfrak{g}_i, \\ \dim \mathfrak{g}_i = 1,\ 1 \le i \le n; \quad [\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1},\ i \ge 2. \end{gathered} \tag{1}$$

consists of

- 6 *infinite sequences:* $\mathfrak{m}_0(n), \mathfrak{m}_2(n), W_n^+, \mathfrak{m}_{0,1}(2k+1), \mathfrak{m}_{0,2}(2k+2), \mathfrak{m}_{0,3}(2k+3).$
- 5 *one-parametric families* $\mathfrak{g}_{n,\alpha}$ *in dimensions* $n = 7 - 11.$

Provided with tables with bases $e_1, \dots, e_n$ *and structure constants of the form*

$$[e_i, e_j] = c_{ij} e_{i+j}.$$

A few historical remarks

We recall that it was Y. Khakimdjanov who discovered in 1991 (Geom. Dedicata) that there exists only a finite number of non-isomorphic $\mathbb{N}$-graded filiform Lie algebras over $\mathbb{C}$ in dimensions ≥ 12.
In his first paper he missed for ≥ 12 three sequences $\mathfrak{m}_{0,1}(2k+1)$, $\mathfrak{m}_{0,2}(2k+2)$, $\mathfrak{m}_{0,3}(2k+3)$. They were missed also in the book M. Goze, Y. Khakimdjanov "Nilpotent Lie algebras" as well as explicit formulae for one-parameter families in dimensions ≤ 11 (for an arbitrary field $\mathbb{K}$ of zero characteristic).
The reason for these errors and inaccuracies was the lack of full evidence without using the Mathematics program

In our proof we use an inductive procedure of construction of filiform Lie algebras by means of one-dimensional central extensions.

Theorem (M., Russian Math Surveys 2002, AMS Translations 2004)

Symplectic positively graded filiform *Lie algebras:*

- $\mathfrak{m}_0(2k)$, $\Omega_{2k+1} = e^1 \wedge e^{2k} + \sum_{i=2}^{k} (-1)^i e^i \wedge e^{2k-i+1}$;
- $W^+(2k), k \geq 6$, $\Omega_{2k+1} = (2k-1) e^1 \wedge e^{2k} + \ldots + e^k \wedge e^{k+1}$;
- $\mathfrak{g}_{8,\alpha}$, $\Omega_9(\alpha)$,
  $\Omega_9(\alpha) = e^1 \wedge e^8 + \frac{2\alpha^2+\alpha-1}{2\alpha+5} e^2 \wedge e^7 + \frac{2\alpha-1}{2\alpha+5} e^3 \wedge e^6 + \frac{3}{2\alpha+5} e^4 \wedge e^5$;
- $\mathfrak{g}_{10,\alpha}$, $\Omega_{11}(\alpha) =$
  $e^1 \wedge e^{10} + \frac{2\alpha^3+2\alpha^2+3}{2(\alpha^2+4\alpha+3)} e^2 \wedge e^9 + \frac{4\alpha^3+8\alpha^2-8\alpha-21}{2(\alpha^2+4\alpha+3)(2\alpha+5)} e^3 \wedge e^8 +$
  $\frac{3(2\alpha^2+4\alpha+5)}{2(\alpha^2+4\alpha+3)(2\alpha+5)} e^4 \wedge e^7 + \frac{3(4\alpha+1)}{2(\alpha^2+4\alpha+3)(2\alpha+5)} e^5 \wedge e^6$.

The previous classification was used by several geometers. In particular I would like to mention here several papers by **Yuri Nikolayevsky** (2008) on construction of **Einstein solvmanifolds** and by **Tracy Payne** (2010, 2015) on **geometry of filiform Lie groups** and parametrization of some components of the variety

Narrow Lie algebras $\mathfrak{n}_1$ and $\mathfrak{n}_2$

- $\mathfrak{n}_1$ (the positive part of the Kac-Moody algebra $A_1^{(1)}$), two generators e_1, e_2 and two relations

$$ad^3 e_1(e_2) = 0, \; ad^3 e_2(e_1) = 0; \; \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

- $\mathfrak{n}_2$ (the positive part of the Kac-Moody algebra $A_2^{(2)}$), two generators e_1, e_2 and two relations

$$ad^5 e_1(e_2){=}0, \; ad^2 e_2(e_1){=}0, \; \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}.$$

Narrowest infinite dimensional Lie algebras

Theorem (A. Fialowski, 1983)

Let $\mathfrak{g} = \oplus_{i=1}^{+\infty} \mathfrak{g}_i$ *be a Lie algebra of width one, generated by two one-dimensional components* $\mathfrak{g}_1$ *and* $\mathfrak{g}_2$.

a) suppose that $[\mathfrak{g}_1, \mathfrak{g}_4] \neq 0$ *and* $[\mathfrak{g}_2, \mathfrak{g}_3] \neq 0$. *If it will be* $[\mathfrak{g}_3, \mathfrak{g}_4] \neq 0$ *then* $\mathfrak{g} \cong W^+$,
if $[\mathfrak{g}_3, \mathfrak{g}_4] = 0$ *then* $\mathfrak{g} \cong \mathfrak{m}_2$.

b) Let $[\mathfrak{g}_2, \mathfrak{g}_3] = 0$. *If* $[\mathfrak{g}_3, \mathfrak{g}_4] \neq 0$ *then* $\mathfrak{g} \cong \mathfrak{n}_2$,
if $[\mathfrak{g}_3, \mathfrak{g}_4] = 0$ *then* $\mathfrak{g} \cong \mathfrak{m}_0$.

c) Let $[\mathfrak{g}_1, \mathfrak{g}_4] = 0$. *If* $[\mathfrak{g}_3, \mathfrak{g}_4] \neq 0$ *then* $\mathfrak{g} \cong \mathfrak{n}_1$,
if $[\mathfrak{g}_3, \mathfrak{g}_4] = 0$ *then we've got the multiparametric family* $\mathfrak{g}(\lambda_8, \lambda_{12}, \dots, \lambda_{4k}, \dots)$ *of Lie algebras.*

A positively graded Lie algebra $\mathfrak{g} = \oplus_{i=1}^{+\infty} \mathfrak{g}_i$ can admit other (non equivalent) positive gradings.

Definition

A grading $\mathfrak{g} = \oplus_{i=1}^{+\infty} \mathfrak{g}_i$ is called **natural**, if

$$[\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1}, \ \forall i \in \mathbb{N}.$$

Example (natural grading $\mathfrak{m}_0$)

$$(\mathfrak{m}_0)_1 = \langle e_1, e_2 \rangle, (\mathfrak{m}_0)_2 = \langle e_3 \rangle, (\mathfrak{m}_0)_3 = \langle e_4 \rangle, (\mathfrak{m}_0)_4 = \langle e_5 \rangle, \dots$$

The natural grading of the free Lie algebra $\mathcal{L}(m)$: the degree of a homogeneous word $[a_{i_1}, [a_{i_2}, [\dots, \dots]], a_{i_k}]$ is equal to its length k.

The invariant meaning of natural grading

The ideals $\mathfrak{g}^m$ of the lower descending series of the Lie algebra $\mathfrak{g}$ determine a decreasing filtration of the Lie algebra $\mathfrak{g}$. For a positively graded Lie algebra $\mathfrak{g}$ the **associated graded Lie algebra** is defined

$$\mathrm{gr}\mathfrak{g} = \oplus_{i=1}^{+\infty} \mathfrak{g}^i / \mathfrak{g}^{i+1}.$$

Proposition

A Lie algebra $\mathfrak{g}$ is naturally graduable if and only if

$$\mathfrak{g} \cong \mathrm{gr}\mathfrak{g}.$$

Carnot algebras

A finite dimensional naturally graded Lie algebra is called in subRiemanniian geometry as *Carnot algebra*

$$\mathfrak{g} = \oplus_{i=1}^{n} \mathfrak{g}_i, \ [\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1}, i = 1, \dots, n-1.$$

Narrow Carnot algebras

Theorem (M. Vergne, 1970)

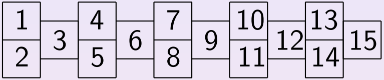
Let $\mathfrak{g} = \oplus_{i=1}^{+\infty} \mathfrak{g}_i$ *be a naturally graded Lie algebra such that*

$$\dim \mathfrak{g}_1 = 2, \dim \mathfrak{g}_i = 1, i \geq 2.$$

Then $\mathfrak{g} \cong \mathfrak{m}_0$

Natural gradings of $\mathfrak{n}_1$ and $\mathfrak{n}_2$

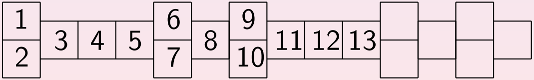
$\mathfrak{n}_1$:



$\dots$

$$
\begin{array}{cccc}
e_1 & & e_4=[e_1,[e_1,e_2]] & \\
 & e_3=[e_1,e_2] & & \dots \\
e_2 & & e_5=[e_2,[e_1,e_2]] & \\
1 & 2 & 3 & \dots
\end{array}
$$

$\mathfrak{n}_2$:



$\dots$

Central extensions $\mathfrak{m}_0^R$ of the Lie algebra $\mathfrak{m}_0$

Consider some odd number $r = 2k+1 \geq 3$. Define one-dimensional central extension $\mathfrak{m}_0^r$ of the Lie algebra $\mathfrak{m}_0$. We add a new basic element z_r that satisfies the relations

$$[e_i, e_{2k+3-i}] = (-1)^i z_{2k+1}, i = 2, \dots, k+1, \; [z_{2k+1}, e_i] = 0, i = 1, 2.$$

Using the same scheme, we can construct a central extension $\mathfrak{m}_0^R$ corresponding to an arbitrary subset $R \subset \{3, 5, 7, \dots\}$.

Naturally graded Lie algebras of width 3/2

Theorem (M., Doklady 2018, Math. Sbornik 2019)

Let $\mathfrak{g} = \oplus_{i=1}^{+\infty}\mathfrak{g}_i$ *be a real positively graded Lie algebra such that*

$$[\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1},\ \dim \mathfrak{g}_i + \dim \mathfrak{g}_{i+1} \leq 3, \forall i \in \mathbb{N}.$$

Then $\mathfrak{g} = \oplus_{i=1}^{+\infty}\mathfrak{g}_i$ *is isomorphic to one and only one Lie algebra from the list*

$$\mathfrak{m}_0, \mathfrak{n}_1^{\pm},\ \mathfrak{n}_2, \mathfrak{n}_2^3, \left\{\mathfrak{m}_0^R \mid R \subset \{3, 5, 7, 9, \dots\}\right\}.$$

Carnot algebras of width 3/2

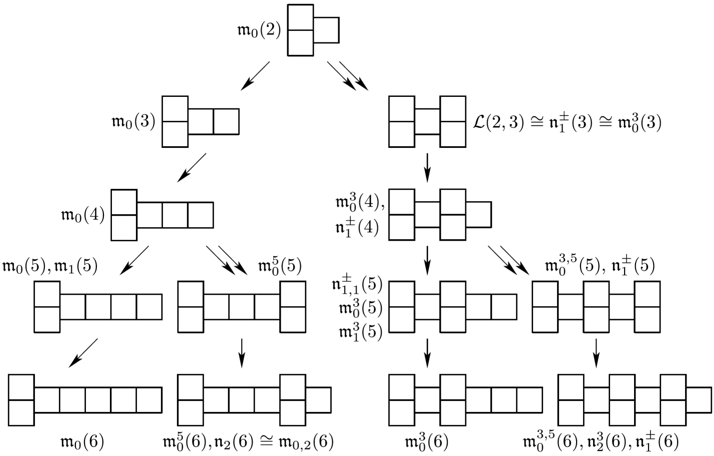
Theorem (M., 2019)

Let $\mathfrak{g} = \oplus_{i=1}^{+\infty} \mathfrak{g}_i$ *be a finite dimensional Carnot algebra such that*

$$\dim \mathfrak{g}_i + \dim \mathfrak{g}_{i+1} \leq 3, \quad i \geq 1.$$

Then $\mathfrak{g}$ *is isomorphic to one and only one Lie algebra from the list, consisting of 4 groups A, B, C, D – finite-dimensional quotients (and extensions) of infinite dimensional Lie algebras* $\mathfrak{m}_0, \mathfrak{n}_1^{\pm}, \ \mathfrak{n}_2, \mathfrak{n}_2^3$*, respectively.*

Genealogy table of Carnot algebras of width 3/2



A naturally graded Lie algebra $\mathfrak{g} = \oplus_{i=1}^{+\infty}\mathfrak{g}_i$ is generated by $\mathfrak{g}_1$

$$[\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_2, [\mathfrak{g}_1, \mathfrak{g}_2] = \mathfrak{g}_3, \dots$$

Denote by V_n the linear span of all commutators (words) $[[\dots[\dots],],\dots]$ of elements from $\mathfrak{g}_1$ of the length $\le n$ with an arbitrary arrangement of brackets.

$$\mathfrak{g}_1 = V_1 \subset V_2 \subset \dots \subset V_n \subset \dots, \ \cup_{i=1}^{+\infty} V_i = \mathfrak{g}.$$

Define **the natural growth function**

$$F_{\mathfrak{g}}^{gr}(n) = \dim V_n = \dim \mathfrak{g}_1 + \dots + \dim \mathfrak{g}_n.$$

- the free Lie algebra $\mathcal{L}(m)$ has the fastest growth

$$F_{\mathcal{L}(m)}^{gr}(n) \sim \frac{1}{n} m^n$$

- $\mathfrak{m}_0$ grows slowly than all algebras $F_{\mathfrak{m}_0}^{gr}(n) = n+1$.

The natural growth functions for $\mathfrak{n}_1$ and $\mathfrak{n}_2$.

- $\mathfrak{g} = \mathfrak{n}_1$

$$F^{gr}_{\mathfrak{n}_1}(n) = \left[\frac{3n+1}{2}\right] \sim \frac{3n}{2}$$

- $\mathfrak{g} = \mathfrak{n}_2$

$$F^{gr}_{\mathfrak{n}_2}(n) \sim \frac{4n}{3}$$

Left-invariant complex structures on Lie groups.

The Newlander-Nirenberg theorem implies that
a left-invariant complex structure on a real simply connected Lie group G can be defined as an almost-complex structure J on the tangent Lie algebra $\mathfrak{g}$ of G ($J^2 = -1$) satisfying the **integrability condition**=vanishing of the Nijenhuis tensor:

$$[JX, JY] = [X, Y] + J[JX, Y] + J[X, JY], \ \forall X, Y \in \mathfrak{g}.$$

Eigen-spaces $\mathfrak{g}^{\mathbb{C}}_{\pm i}$

Extending an almost complex structure J on the complexification $\mathfrak{g}^{\mathbb{C}}$ we have a splitting

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}^{\mathbb{C}}_{-i} \oplus \mathfrak{g}^{\mathbb{C}}_{i},$$

where $\mathfrak{g}^{\mathbb{C}}_{\pm i} = \{x - \pm iJx : x \in \mathfrak{g}\}$ are the eigen-space of the complexification of J corresponding to the eigen-values $\pm i$. J **is integrable if** and only if

both $\mathfrak{g}^{\mathbb{C}}_{\pm i}$ are (complex) subalgebras of $\mathfrak{g}^{\mathbb{C}}$.

There are two special cases:

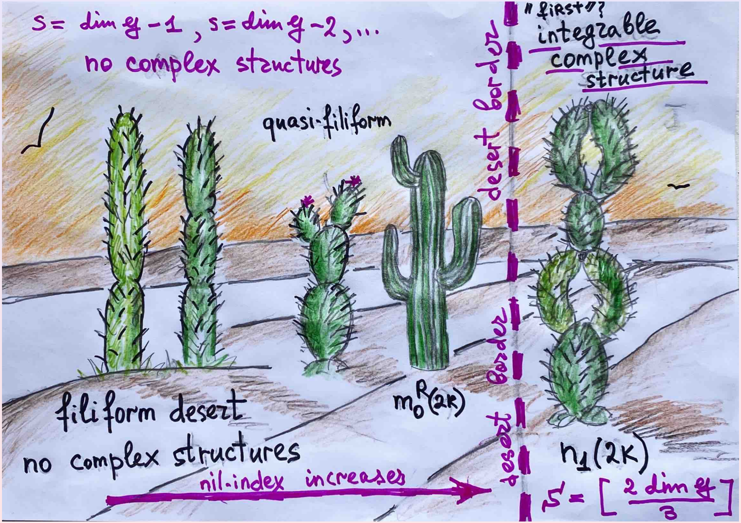
1) $\mathfrak{g}^{\mathbb{C}}_{\pm i}$ are **abelian subalgebras** of $\mathfrak{g}^{\mathbb{C}}$.
2) $\mathfrak{g}^{\mathbb{C}}_{\pm i}$ are **ideals** of $\mathfrak{g}^{\mathbb{C}}$.

Complex structure $\Rightarrow$ algebraic restrictions

The **existence of an integrable complex structure** impose some **algebraic restrictions on the Lie algebra $\mathfrak{g}$**

- S. Salamon (2001) classified all 6-dimensional nilpotent Lie algebras that admit integrable complex structure;
- $\dim \mathfrak{g} \geq 6$, a real **filiform** i.e. $s(\mathfrak{g}) = \dim \mathfrak{g} - 1$ Lie algebra $\mathfrak{g}$ **does not admit** any integrable complex structure (Goze and Remm 2002);
- $\dim \mathfrak{g} \geq 8$, a nilpotent Lie algebra $\mathfrak{g}$ with $s(\mathfrak{g}) = \dim \mathfrak{g} - 2$, i.e. **quasi-filiform, does not admit any integrable complex structure** (L. Garcia-Vergnolle, Remm 2008);

A meeting at the Filiform desert border



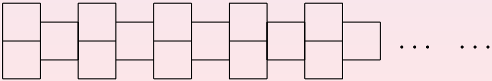
Two real forms of $\mathfrak{n}_1$

The real forms $\mathfrak{so}(3,\mathbb{R})$ and $\mathfrak{so}(1,2)$ of $\mathfrak{sl}(2,\mathbb{C})$ can be defined by the basis u, v, w and relations

$$[u,v]{=}w,\ [v,w]{=}\pm u,\ [w,u]{=}v.$$

Consider two subalgebras $\mathfrak{n}_1^{\pm}$ in the loop algebras $\mathfrak{so}(3,\mathbb{R})\otimes\mathbb{R}[t]$ and $\mathfrak{so}(1,2)\otimes\mathbb{R}[t]$ respectively. They are defined by bases

$$\begin{array}{c} u\otimes t^1 \\ v\otimes t^1 \end{array}, w\otimes t^2,\ \begin{array}{c} u\otimes t^3 \\ v\otimes t^3 \end{array}, w\otimes t^4,\ \begin{array}{c} u\otimes t^5 \\ v\otimes t^5 \end{array}, w\otimes t^6, \dots$$

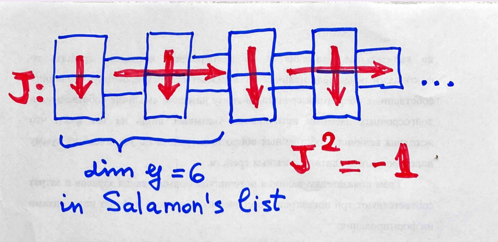


The integrable complex structure J on $\mathfrak{n}^+$

Define an almost complex structure J of infinite dimensional $\mathfrak{n}^+$

$$J(v \otimes t^{2l+1}) = u \otimes t^{2l+1}, \; J(w \otimes t^{4k+2}) = w \otimes t^{4k+4}.$$

One can directly verify $N(J) = 0$ for basic elements.



Define integrable complex structures on even dimensional quotients of $\mathfrak{n}^+$. The 6-dimensional example corresponds to quasi-filifrom Lie algebra in Salamon's classification list (2001).

Theorem (M., Rend. Sem. Mat. Univ. Pol. Torino, 2016)

Let a real nilpotent Lie algebra $\mathfrak{g}$ *of dimension* $\dim \mathfrak{g} \geq 6$*, admits an integrable complex structure* J*. Then for growth function of* $\mathrm{gr}\mathfrak{g}$ *the following estimate is valid*

$$F_{\mathrm{gr}\mathfrak{g}}(3) = \dim(\mathrm{gr}\mathfrak{g})_1 + \dim(\mathrm{gr}\mathfrak{g})_2 + \dim(\mathrm{gr}\mathfrak{g})_3 \geq 5.$$

Now we can add a new estimate for sufficiently large $\dim \mathfrak{g}$

$$F_{\mathrm{gr}\mathfrak{g}}(5) \geq 8.$$

Conjecture: Let a real nilpotent Lie algebra $\mathfrak{g}$ admits an integrable complex structure. Then its associated graded Lie algebra $\mathrm{gr}\mathfrak{g}$ should grow no slower than $\mathfrak{n}_1$."For instance

$$s(\mathfrak{g}) \leq \left[\frac{2\dim \mathfrak{g}}{3}\right].$$

Hypercomplex structures

Definition

A hypercomplex structure on a Lie algebra $\mathfrak{g}$ is a triple J_1, J_2, J_3 of integrable complex structures satisfying the quaternion identities

$$J_i^2 = -1, i = 1, 2, 3; \; J_1 J_2 = J_3 = -J_2 J_1.$$

Theorem (Dotti, Fino, Contemp. Math. 2001)

Let $\mathfrak{g}$ be a 8-dimensional Lie group endowed with a hypercomplex structure. Then $\mathfrak{g}$ is at most 2-step nilpotent and $b_1(\mathfrak{g}) \geq 4$.

Conjecture: Let a real nilpotent Lie algebra $\mathfrak{g}$ admits an integrable hypercomplex structure. Then

$$s(\mathfrak{g}) \leq \left[\frac{\dim \mathfrak{g}}{3}\right], b_1(\mathfrak{g}) \geq 4.$$

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