Action-Minimizing Methods in Hamiltonian Dynamics and Invariant Lagrangian Graphs

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Seminar: "Geometry, Topology and their Applications"

25th May 2020

GOAL OF THIS TALK: DESCRIBE WHAT KIND OF INFORMATION THE PRINCIPLE OF LEAST ACTION CONVEYS INTO THE STUDY OF THE EXISTENCE OR THE NON-EXISTENCE OF INVARIANT LAGRANGIAN GRAPHS.

IN PARTICULAR:

- WEAK INTEGRABILITY $\longrightarrow$ A NON-COMMUTATIVE VERSION OF LIOUVILLE-ARNOLD THEOREM FOR CERTAIN CLASSES OF HAMILTONIANS.
- REGULARITY OF THE MINIMAL AVERAGED ACTION

1. THE PRINCIPLE OF LEAST ACTION IN HAMILTONIAN DYNAMICS

$\longrightarrow$ AUBRY-MATHER THEORY (1980-90's)

"NATURE IS THRIFTY IN ALL ITS ACTIONS" - PIERRE-LOUIS MOREAU DE MAUPERTUIS (1744)

HISTORICAL REMARK:

KÖNIG PUBLISHED A NOTE CLAIMING PRIORITY FOR LEIBNIZ IN THE BERLIN ACADEMY CORRESPONDENCES OVERSEEN BY MAUPERTUIS.

PRIORITY DISPUTE BROUGHT IN EULER, VOLTAIRE AND ULTIMATELY A COMMITTEE CONVENED BY THE KING OF PRUSSIA.

IN 1913, THE BERLIN ACADEMY REVERSED ITS PREVIOUS DECISION AND FOUND LEIBNIZ HAD PRIORITY

1.1 CARTOON EXAMPLE — A (VERY SPECIAL) INTEGRABLE HAMILTONIAN SYSTEM

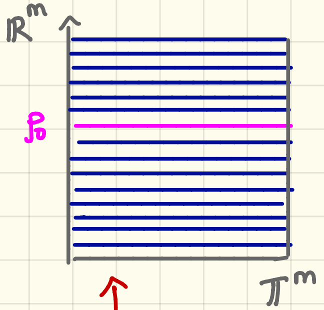
$$h : \mathbb{T}^m \times \mathbb{R}^m \longrightarrow \mathbb{R} \qquad \text{HAMILTONIAN}$$
$$(x,p) \longmapsto h(p)$$

ASSUMPTIONS

- h ONLY DEPENDS ON p
- h IS C^2
- h IS STRICTLY CONVEX IN p (I.E., POSITIVE DEFINITE HESSIAN)
- h SUPERLINEAR IN p (I.E., $\lim\limits_{\|p\|\to\infty} \frac{h(p)}{\|p\|} = +\infty$)

$$\begin{cases} \dot{x} = \partial_p h(p) \\ \dot{p} = 0 \end{cases} \qquad \longleftarrow \text{ EQUATIONS OF MOTION}$$

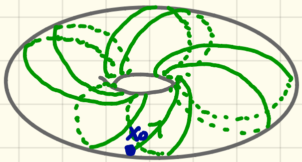
$$\phi_h^t(x_0,p_0) = \Big(x_0 + \underbrace{\partial_p h(p_0)}_{=\,\rho(p_0)}\, t \,,\; p_0\Big) \qquad \longleftarrow \text{ FLOW}$$



$$\longleftarrow \quad \mathcal{T}^*_{p_0} = \mathbb{T}^m \times \{p_0\}$$

INVARIANT TORUS

THE PHASE SPACE IS FOLIATED BY INVARIANT TORI

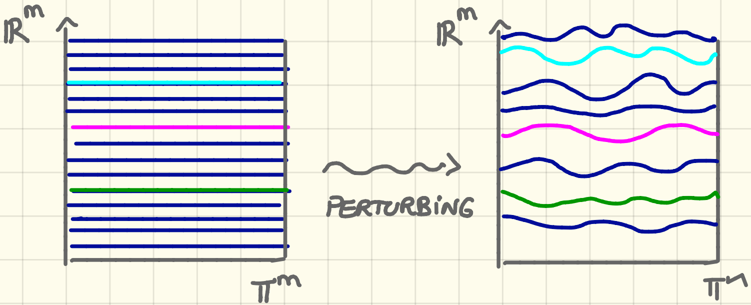


$\longleftarrow$ ROTATION WITH ROTATION VECTOR $\rho(p_0)$

QUESTION: WHAT HAPPENS TO THIS FOLIATION IF ONE PERTURBS THE SYSTEM?

- TORI FOLIATED BY PERIODIC ORBITS WILL IN GENERAL DISAPPEAR
- IN 1954 KOLMOGOROV (AND LATER ARNOL'D AND MOSER IN DIFFERENT SETTINGS) PROVED THAT THE MAJORITY OF THESE TORI WILL SURVIVE ← KAM THEORY

THOSE CORRESPONDING TO DIOPHANTINE ROTATION VECTORS



- WHAT HAPPENS TO THE DESTROYED TORI?
- WHAT HAPPENS IN THOSE GAPS?

CHANGE OF POINT OF VIEW: LAGRANGIAN POINT OF VIEW

$H : \mathbb{T}^m \times \mathbb{R}^m \to \mathbb{R}$ HAMILTONIAN $\rightsquigarrow$ $L : \mathbb{T}^m \times \mathbb{R}^m \to \mathbb{R}$ LAGRANGIAN

$$L(x,v) = \sup_{p \in \mathbb{R}^m} \left(\langle p, v \rangle - H(x,p) \right)$$

FENCHEL-LEGENDRE INEQUALITY

$$\boxed{\begin{aligned} L(x,v) + H(x,p) &\geq \langle p, v \rangle \qquad \forall\, v, p \in \mathbb{R}^m \\ = \quad &\iff \quad p = \partial_v L(x,v) \\ &\iff \quad v = \partial_p H(x,p) \end{aligned}}$$

$$\begin{tikzcd} \mathbb{T}^m \times \mathbb{R}^m \arrow[r, "\phi_H^t"] \arrow[d, "{(x, \partial_p H)}"'] & \mathbb{T}^m \times \mathbb{R}^m \arrow[d, "{(x, \partial_p H)}"] \\ \mathbb{T}^m \times \mathbb{R}^m \arrow[r, "\phi_L^t"'] & \mathbb{T}^m \times \mathbb{R}^m \end{tikzcd}$$

$$\frac{d}{dt} \frac{\partial L}{\partial v} = \frac{\partial L}{\partial x} \quad \Leftarrow \quad \text{EULER-LAGRANGE EQUATION (E-L)}$$

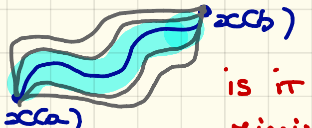
ACTION FUNCTIONAL

$(x(t), v(t))\ t \in [a,b]$ (E-L)

$\Updownarrow$

$x(t)$ is a "CRITICAL POINT" FOR THE FUNCTIONAL

$$A_L(x(t)) = \int_a^b L(x(t), v(t))\, dt$$



IS IT A MINIMUM?

(BACK TO THE CARTOON EXAMPLE) $\quad \ell(v) = \sup_{p\in\mathbb{R}^m} \left(\langle p, v\rangle - h(p) \right)$

$$\mathcal{T}^*_{p_0} = \mathbb{T}^m_x \{p_0\} \quad \rightsquigarrow \quad \tilde{\mathcal{T}}_{p_0} = \mathbb{T}^m_x \{v_0\} \quad \text{INVARIANT FOR } \phi^t_\ell$$

INVARIANT FOR ϕ^t_h $\qquad\qquad v_0 = \partial_p h(p_0)$

GOAL: CHARACTERIZE $\tilde{\mathcal{T}}_{p_0}$ IN TERMS OF ITS ACTION PROPERTIES

IDEA 1 (J. MATHER): LOOK AT INVARIANT PROBABILITY MEASURES + THEIR ACTION PROPERTIES

- μ_0 INVARIANT PROB. MEASURE SUPPORTED ON $\tilde{\mathcal{T}}_{p_0} = \mathbb{T}^m_x \{v_0\}$

$$A_\ell(\mu_0) = \int_{\mathbb{T}^m_x \mathbb{R}^m} \ell(v)\, d\mu_0 = \ell(v_0) = \langle p_0, v_0\rangle - h(p_0)$$

HOW TO COMPARE THEM?

- ν ANY INVARIANT PROB. MEASURE

$$A_\ell(\nu) = \int_{\mathbb{T}^m_x \mathbb{R}^m} \ell(v)\, d\nu \geq \int_{\mathbb{T}^m_x \mathbb{R}^m} \left(\langle p_0, v\rangle - h(p_0) \right) d\nu = \langle p_0, \int v\, d\nu \rangle - h(p_0)$$

IDEA 2 (J. MATHER): MODIFY THE LAGRANGIAN : $\ell(v) - \langle p_0, v\rangle$ ← SAME E-L FLOW BUT DIFFERENT ACTION

$$A_{\ell - \langle p_0, v\rangle}(\mu_0) = \int_{\mathbb{T}^m \times \mathbb{R}^m} (\ell(v) - \langle p_0, v\rangle)\, d\mu_0 = -h(p_0)$$

$$A_{\ell - \langle p_0, v\rangle}(\nu) \geq \int_{\mathbb{T}^m \times \mathbb{R}^m} (\ell(v) - \langle p_0, v\rangle)\, d\nu = -h(p_0)$$

$$\Longrightarrow \boxed{A_{\ell - \langle p_0, v\rangle}(\mu_0) \leq A_{\ell - \langle p_0, v\rangle}(\nu)}$$

CONCLUSION 1:

- EVERY INV. PROB. MEASURE SUPPORTED ON $\tilde{\mathcal{T}}_{p_0}$ MINIMIZES $A_{\ell - \langle p_0, v\rangle}(\cdot)$
- $$\boxed{\tilde{\mathcal{T}}_{p_0} = \bigcup \{ \operatorname{supp} \mu_0 : \mu_0 \text{ MINIMIZES } A_{\ell - \langle p_0, v\rangle}(\cdot) \}}$$
- $$\boxed{\min_{\mu \text{ INV. PB MEAS}} A_{\ell - \langle p_0, v\rangle}(\mu) = -h(p_0)}$$

MATHER SET

VARIATIONAL CHARACTERIZ OF THE TORUS

VARIATIONAL CHARACT. OF THE HAMILTONIAN

MINIMAL AVERAGE ACTION

QUESTION: CAN WE AVOID TO CHANGE THE LAGRANGIAN?

DEFINITION: μ INV. PROB. MEASURE $\Longrightarrow$ $\rho(\mu) := \int_{\mathbb{T}^n \times \mathbb{R}^n} v \, d\mu \in \mathbb{R}^n$ ROTATION VECTOR OF μ

IDEA: COMPARE ONLY INV. PROB. MEASURES WITH THE SAME ROTATION VECTOR

- μ_0 INVARIANT PROB. MEASURE SUPPORTED ON $\tilde{\mathcal{G}}_{p_0} = \mathbb{T}^n \times \{v_0\}$ $\leftarrow$ ROTATION VECTOR $\rho(\mu_0) = v_0$

$$A_\ell(\mu_0) = \int_{\mathbb{T}^n \times \mathbb{R}^n} \ell(v)\, d\mu_0 = \ell(v_0) = \langle p_0, v_0 \rangle - h(p_0)$$

- ν ANY INVARIANT PROB. MEASURE WITH $\rho(\nu) = v_0$

$$A_\ell(\nu) = \int_{\mathbb{T}^n \times \mathbb{R}^n} \ell(v)\, d\nu \geq \int_{\mathbb{T}^n \times \mathbb{R}^n} \left(\langle p_0, v \rangle - h(p_0) \right) d\nu = \langle p_0, \underbrace{\int v \, d\nu}_{= v_0} \rangle - h(p_0)$$

$$\Longrightarrow \quad \boxed{A_\ell(\mu_0) \leq A_\ell(\nu)}$$

CONCLUSION 2: • EVERY INV. PROB. MEASURE SUPPORTED ON $\tilde{\mathcal{T}}_{\rho_0}$ MINIMIZES $A_\ell(\cdot)$

AMONG ALL INV. PB MEASURES WITH THE SAME ROTATION VECTOR

MATHER SET

• $\boxed{\tilde{\mathcal{T}}_{\rho_0} = \bigcup\{\operatorname{supp}\mu_0 : \mu_0 \text{ MINIMIZES } A_\ell \text{ AMONG MEAS. WITH SAME ROT. VECT}\}}$

VARIATIONAL CHARACTERIZ OF THE TORUS

MINIMAL AVERAGE ACTION

• $\boxed{\min_{\substack{\mu \text{ inv.} \\ \rho(\mu)=\nu_0}} A_\ell(\mu) = \ell(\nu_0)}$

VARIATIONAL CHARACT. OF THE LAGRANGIAN

SUMMARY OF THIS CARTOON EXAMPLE

$$\boxed{\bigcup \left\{ \mathrm{supp}\,\mu : \ \mu \text{ minimizes } A_{\ell - \langle p_0, v\rangle} \right\}} \longleftarrow \tilde{\mathcal{T}}_{p_0} = \mathbb{T}^m \times \{v_0\} \longrightarrow \boxed{\bigcup \left\{ \mathrm{supp}\,\mu : \ \mu \text{ minimizes } A_\ell \text{ WITH CONSTRAINT } \rho(\mu) = v_0 \right\}}$$

INVARIANT TORUS

$$\tilde{\mathcal{M}}_{p_0} = \tilde{\mathcal{M}}^{v_0}$$

$$p_0 = \partial_v \ell(v_0)$$

$$v_0 = \partial_p h(p_0)$$

$$\boxed{h(p_0) = -\min_{\substack{\mu \text{ inv.} \\ \text{PB meas.}}} A_{\ell - \langle p_0, v\rangle}(\mu)} \longleftrightarrow \boxed{\ell(v_0) = \min_{\substack{\mu \text{ inv} \\ \text{PB meas} \\ \rho(\mu) = v_0}} A_\ell(\mu)}$$

FENCHEL-LEGENDRE DUALITY

$$h(p) = \sup_v \left(\langle p, v\rangle - \ell(v) \right) \qquad \ell(v) = \sup_p \left(\langle p, v\rangle - h(p) \right)$$

1.2 SETTING: TONELLI LAGRANGIAN AND HAMILTONIAN

- M COMPACT CONNECTED MANIFOLD W/OUT BOUNDARY $\dim M = m$
- g RIEMANNIAN METRIC

$L: \underset{(x,v)}{TM} \to \mathbb{R}$ IS A TONELLI LAGRANGIAN IF

- $L \in C^2(TM)$
- L FIBERWISE STRICTLY CONVEX: $\partial^2_{vv} L > 0$
- L SUPERLINEAR IN THE FIBER

$$\lim_{\|v\| \to \infty} \frac{L(x,v)}{\|v\|} = +\infty \quad \text{UNIF. IN } x$$

$H: T^*M \to \mathbb{R}$ IS A TONELLI HAMILTONIAN IF

- $H \in C^2(T^*M)$
- H FIBERWISE STRICTLY CONVEX: $\partial^2_{pp} H > 0$
- H SUPERLINEAR IN THE FIBER

$$\lim_{\|p\| \to \infty} \frac{H(x,p)}{\|p\|} = +\infty \quad \text{UNIF. IN } x$$

FENCHEL LEGENDRE DUALITY

$$L(x,v) = \sup_{p \in T^*_x M} \left(\langle p, v \rangle_x - H(x,p) \right)$$

$$H(x,p) = \sup_{v \in T_x M} \left(\langle p, v \rangle_x - L(x,v) \right)$$

EXAMPLES:

- GEODESIC FLOW $\qquad L(x,v)=\frac{1}{2}\|v\|_x^2$

- MECHANICAL SYSTEM $\qquad L(x,v)=\frac{1}{2}\|v\|_x^2 - V(x) \qquad V\in C^2(M)$

- LET X BE A C^2 VECTOR FIELD ON M $\qquad$ LET φ_t^X BE THE FLOW OF X

LET $L(x,v)=\frac{1}{2}\|v - X(x)\|_x^2$ $\qquad \longleftarrow$ MAÑE LAGRANGIAN

$\text{GRAPH}(X)=\{(x,X(x)) : x\in M\}$ IS INVARIANT UNDER Φ_L^t

$$\begin{tikzcd}
\text{GRAPH}(X) \arrow[r, "\Phi_L^t"] \arrow[d, "\pi"'] & \text{GRAPH}(X) \arrow[d, "\pi"] \\
M \arrow[r, "\varphi_t^X"] & M
\end{tikzcd}$$

THE FLOW OF X IS EMBEDDED IN THE FLOW OF A TONELLI LAGRANGIAN

1.3 MATHER THEORY FOR TONELLI LAGRANGIAN

$L : TM \longrightarrow \mathbb{R}$ TONELLI LAGRANGIAN

$\Phi_L : TM \longrightarrow TM$ E-L FLOW

$\mathfrak{M}_L = \{$ INVARIANT PROBABILITY MEASURES OF $\Phi_L \} \neq \emptyset$

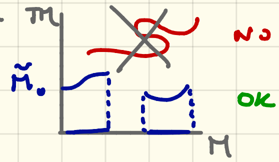
$A_L : \mathfrak{M}_L \longrightarrow \mathbb{R}$ ACTION FUNCTIONAL

- $\exists\, \mu \in \mathfrak{M}_L$ MINIMIZING A_L
- $\alpha(0) := -\min\limits_{\mu \in \mathfrak{M}_L} A_L(\mu)$
- $\boxed{\tilde{\mathcal{M}}_0 := \bigcup \{ \operatorname{supp} \mu : A_L(\mu) = -\alpha(0) \}}$
  - IT IS NON-EMPTY, INVARIANT, COMPACT
  - IT LIES IN THE ENERGY VALUE $= \alpha(0)$ (CARNEIRO, 1995)
  - IT IS SUPPORTED ON A LIPSCHITZ GRAPH (MATHER GRAPH THEOREM 1991)

MATHER GRAPH THEOREM

$$\begin{array}{c} \tilde{\mathcal{M}}_0 \subset TM \\ \big\downarrow \pi \\ \pi(\tilde{\mathcal{M}}_0) \subset M \end{array}$$

BI-LIPSCHITZ HOMEOM



HOW TO MODIFY THE LAGRANGIAN ?

- IF η IS A CLOSED ONE FORM ON M $\Rightarrow$ $L_\eta(x,v) = L(x,v) - \langle \eta, v\rangle$ HAS THE SAME E-L FLOW AS L (EXERCISE: JUST WRITE E-L EQUATIONS)
- IF $\eta = df$ $f \in C^1(M)$ $\Rightarrow$ $\int_{TM} \langle df, v\rangle \, d\mu = 0 \quad \forall \mu \in \mathfrak{M}_L$

STRATEGY:

- LET $c \in H^1(M,\mathbb{R})$ AND η_c A REPRESENTATIVE
- $L_{\eta_c}(x,v) = L(x,v) - \langle \eta_c, v\rangle$ IS A TONELLI LAGRANGIAN
- FIND MINIMIZING MEASURES FOR $A_{L_{\eta_c}}$
- DEFINE $\alpha(c) := -\min_{\mathfrak{M}_L} A_{L_{\eta_c}}$
- CONSTRUCT MATHER SETS $\boxed{\widetilde{M}_c := \bigcup \{\operatorname{supp} \mu : A_{L_{\eta_c}}(\mu) = -\alpha(c)\}}$

THESE DEFINITIONS DEPEND ONLY ON c

↳ COMPACT, INVARIANT, LIES IN ENERGY LEVEL $\alpha(c)$, SUPPORTED ON A LIPSCHITZ GRAPH

ROTATION VECTOR OF AN INVARIANT MEASURE

(SCHWARTZMAN ASYMPTOTIC CYCLE)

$$\rho : \mathfrak{M}_L \longrightarrow H_1(M, \mathbb{R})$$

LET $\mu \in \mathfrak{M}_L$ DEFINE

$$I_\mu : H^1(M, \mathbb{R}) \longrightarrow \mathbb{R}$$
$$c \longmapsto \int_{TM} \langle \eta_c, v \rangle \, d\mu$$

IT IS INDEPENDENT OF THE CHOICE OF THE REPRESENTATIVE η_c

I_μ LINEAR FUNCTIONAL $\xrightarrow{\text{RIESZ THEOREM}}$ $\exists !\ \rho(\mu) \in (H^1(M, \mathbb{R}))^* \simeq H_1(M, \mathbb{R})$

S.T. $I_\mu(c) = \langle \rho(\mu), c \rangle \quad \forall c \in H^1(M, \mathbb{R})$

STRATEGY:

- $\forall h \in H_1(M, \mathbb{R})$, $\mathfrak{M}_L^h = \{\mu \in \mathfrak{M}_L : \rho(\mu) = h\} \neq \emptyset$
- FIND MINIMIZING MEASURES FOR A_L ON $\mathfrak{M}_L^h$
- DEFINE $\beta(h) := \min\limits_{\mathfrak{M}_L^h} A_L$
- CONSTRUCT MATHER SETS $\boxed{\tilde{M}^h := \bigcup \{\operatorname{supp} \mu : \rho(\mu) = h,\ A_L(\mu) = \beta(h)\}}$

$\hookrightarrow$ COMPACT, INVARIANT, SUPPORTED ON LIPSCHITZ GRAPHS

SUMMARIZING:

$$\{\widetilde{\mathcal{M}}_c\}_{c\in H^1(M,\mathbb{R})} \overset{?}{\longleftrightarrow} \{\widetilde{\mathcal{M}}^h\}_{h\in H_1(M,\mathbb{R})}$$

MATHER SET OF COHOMOLOGY CLASS c

MATHER SET OF HOMOLOGY CLASS h

$$\alpha: H^1(M,\mathbb{R}) \longrightarrow \mathbb{R}$$
$$c \longmapsto -\min_{\mathfrak{M}_L} A_{L-\langle\eta_c, v\rangle}$$

MATHER'S α-FUNCTION (EFFECTIVE HAMILTONIAN)

FENCHEL-LEGENDRE DUALITY

$$\beta: H_1(M,\mathbb{R}) \longrightarrow \mathbb{R}$$
$$h \longmapsto \min_{\mathfrak{M}_L^h} A_L$$

MATHER'S β-FUNCTION (EFFECTIVE LAGRANGIAN)

$$\alpha(c) = \sup_{h\in H_1(M,\mathbb{R})} \left(\langle c,h\rangle - \beta(h)\right)$$

$$\beta(h) = \sup_{c\in H^1(M,\mathbb{R})} \left(\langle c,h\rangle - \alpha(c)\right)$$

FENCHEL-LEGENDRE INEQUALITY:

$$\alpha(c)+\beta(h) \geq \langle c,h\rangle \qquad \forall c\in H^1(M,\mathbb{R})\ \ \forall h\in H_1(M,\mathbb{R})$$

$$= \iff c = \partial\beta(h) \iff h\in\partial\alpha(c)$$

SUBDERIVATIVES

WARNING:
α, β ARE CONVEX BUT NOT NECESSARILY STRICTLY CONVEX OR DIFFERENTIABLE

$$\{\tilde{M}_c\}_{c\in H^1(M,\mathbb{R})} \overset{?}{\longleftrightarrow} \{\tilde{M}^h\}_{h\in H_1(M,\mathbb{R})}$$

RELATIONS:

- $$\bigcup_{c\in H^1(M,\mathbb{R})} \tilde{M}_c = \bigcup_{h\in H_1(M,\mathbb{R})} \tilde{M}^h$$
- $$\tilde{M}^h \subseteq \tilde{M}_c \iff h\in\partial\alpha(c) \iff c\in\partial\beta(h)$$
- IF $c, c' \in \partial\beta(h) \Rightarrow \tilde{M}^h \subseteq \tilde{M}_c \cap \tilde{M}_{c'}$ ($\tilde{M}_c$ AND $\tilde{M}_{c'}$ ARE NOT DISJOINT)
- IF $h, h' \in \partial\alpha(c) \Rightarrow \tilde{M}^h \cup \tilde{M}^{h'} \subseteq \tilde{M}_c$ ($\tilde{M}_c$ CONTAINS MOTIONS OF DIFFERENT ROT. VECTORS)

QUESTIONS

- IF β IS C^1, IS IT TRUE THAT THE SYSTEM IS INTEGRABLE?
- CAN ONE DETECT THE EXISTENCE OF AN INVARIANT LAGRANGIAN GRAPH FROM PROPERTIES OF β?

2. DIFFERENTIABILITY OF MATHER'S β-FUNCTION AND INTEGRABILITY

$\beta : H_1(M, \mathbb{R}) \longrightarrow \mathbb{R}$ IS A CONVEX FUNCTION

$\hookrightarrow$ NOT NECESSARILY STRICTLY CONVEX
NOR DIFFERENTIABLE

- IF $\exists$ AN INVARIANT TORUS, WHOSE MOTION IS CONJUGATED TO A ROTATION OF ROTATION VECTOR h $\Rightarrow$ β IS DIFFERENTIABLE AT h

- $\boxed{\text{IF THE SYSTEM IS INTEGRABLE} \Rightarrow \beta \text{ IS } C^1 \text{ ON SOME OPEN SET}}$

$\overset{?}{\Leftarrow}$

- IF $\exists$ AN INVARIANT LAGRANGIAN GRAPHS Λ, S.T. ALL ITS INVARIANT MEASURES HAVE ROTATION VECTOR h AND THE UNION OF THEIR SUPPORTS COVERS Λ

$\Rightarrow$ β IS DIFFERENTIABLE AT h.

- if $\dim H_1(M,\mathbb{R})=0 \Rightarrow \beta$ is defined only at a point
- if $\dim H_1(M,\mathbb{R})=1 \Rightarrow \beta$ is differentiable everywhere (with the possible exception of the origin)

  $\downarrow$

  (Carneiro '95: β always diff. in radial direction)

Question: With the exception of these trivial cases, does the differentiability of β imply anything on integrability / $\exists$ inv. Lagrangian graphs?

Definitions:

- $h \in H_1(M,\mathbb{R})$ is *k-irrational* if k is the dimension of the smallest subspace generated by elements of $H_1(M,\mathbb{Z})$, containing h.

$\longrightarrow$ *1-irrational:* "on a line of rational slope"

$\longrightarrow$ *completely irrational:* $k = \dim H_1(M,\mathbb{R})$

RESULTS ON CLOSED SURFACES (WITH DANIEL MASSART, NONLINEARITY 2011)

THEOREM: LET M BE A CLOSED ORIENTED SURFACE OF GENUS g

LET $h_0 \in H_1(M,\mathbb{R})$ 1-IRRATIONAL AND NON-SINGULAR

(i) $\Rightarrow \dim \partial\beta(h_0) \geq g$

(ii) IF $M=\mathbb{T}^2$ AND β IS DIFFERENTIABLE AT h_0

$\Rightarrow \exists$ AN INVARIANT LAGRANGIAN GRAPH FOLIATED BY PERIODIC ORBITS, WHOSE HOMOLOGY IS A MULTIPLE OF h_0

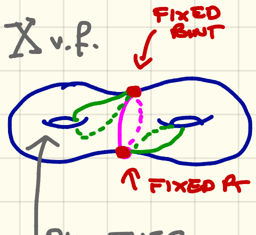
NON-SINGULAR MEANS THAT

$$\bigcup_{c\in\partial\beta(h_0)} \tilde{M}_c$$

DOES NOT CONTAIN FIXED POINTS OF THE FLOW

REMARKS:

- THE ABOVE RESULTS ARE FALSE IF h_0 IS SINGULAR



- (ii) GENERALIZES A RESULT BY MATHER FOR TWIST MAPS OF THE ANNULUS, AND A RESULT BY BANGERT FOR GEODESIC FLOWS ON $\mathbb{T}^2$

DEFINITION (M.-C. ARNAUD) A TONELLI HAMILTONIAN $H: T^*M$ IS SAID TO BE C^0-INTEGRABLE IF

$\exists$ A FOLIATION OF T^*M BY INVARIANT LAGRANGIAN GRAPHS, ONE FOR EACH COHOMOLOGY CLASS

REMARKS: LIOUVILLE-ARNOLD INTEGRABILITY ON T^*M $\overset{\Longrightarrow}{\underset{?}{\not\Longleftarrow}}$ C^0 INTEGRABILITY

$$\boxed{\text{THEOREM 2} \quad \beta \in C^1(H_1(M,\mathbb{R})) \iff \text{THE SYSTEM IS } C^0 \text{ INTEGRABLE}}$$

$$\Downarrow$$

$$(M \cong \mathbb{T}^2)$$

IN PARTICULAR:

- h 1-IRRATIONAL: $\exists$ INV. LAGRANGIAN GRAPH FOLIATED BY PER. ORBITS WITH HOMOLOGY h AND THE SAME MINIMAL PERIOD
- h COMPLETELY IRRATIONAL: $\exists$ INV. LAG. GRAPH WHOSE MOTION IS CONJUGATED TO AN IRRATIONAL ROTATION OR A DENJOY TYPE HOMEOMORPHISM.
  - $\hookrightarrow$ $\exists$ G_δ SET OF (CO)HOMOLOGIES FOR WHICH THE MOTION IS CONJUGATED TO A ROTATION
- $h = 0$ $\exists$ C^1 INVARIANT TORUS CONSISTING OF FIXED POINTS

SOME COMMENTS

- THIS ANALYSIS CAN BE EXTENDED TO NON ORIENTABLE SURFACES

IF $M \neq \mathbb{RP}^2$, KLEIN BOTTLE $\Rightarrow$ $\exists\, h \in H_1(M,\mathbb{R})$ S.T. β IS NOT DIFFERENTIABLE AT h

- ONE OF THE KEY PROPERTY IN DIMENSION 2 IS THAT IF h IS 1-IRRATIONAL AND NON-SINGULAR $\Rightarrow$ $\tilde{M}^h$ CONSISTS OF PERIODIC ORBITS.

- IN HIGHER DIMENSION NOT MANY RESULTS ARE KNOWN:

$\longrightarrow$ FOR NEARLY INTEGRABLE SYSTEMS

USING KAM THEORY ONE CAN DEDUCE SOME REGULARITY, IN THE WHITNEY SENSE, ON SOME SET OF DIOPHANTINE HOMOLOGIES.

3. WEAK INTEGRABILITY

RECALL LET (V,ω) BE A SYMPLECTIC MANIFOLD $H: V \longrightarrow \mathbb{R}$ HAMILTONIAN

(ω 2-FORM CLOSED AND NON DEGENERATE)

- HAMILTONIAN VECTOR FIELD: $i_{X_H}\omega = dH$
- $F: V \to \mathbb{R}$ IS AN INTEGRAL OF MOTION FOR X_H IF $\omega(X_F, X_H) = 0$

$\{F, H\}$ POISSON BRACKETS

$\rightarrow$ F REMAINS CONSTANT ALONG THE ORBITS OF X_H $\longleftrightarrow$ ORBITS LIE IN $\{F = c\}$

- IF $F_1, \dots, F_m$ ARE INTEGRALS OF MOTION $\Rightarrow$ $\{F_1 = c_1, \dots, F_m = c_m\}$ IS AN INVARIANT SET

THE MORE INTEGRALS OF MOTION EXIST $\Rightarrow$ THE MORE ORBITS ARE CONSTRAINED ON "SMALLER" SUBMANIFOLDS

THEY MUST HAVE SOME "TRANSVERSALITY"

$\Rightarrow$ FUNCTIONAL INDEPENDENCE

THEOREM (LIOUVILLE-ARNOLD)

LET $m = \frac{1}{2}\dim V$, $F_1, \dots, F_m : V \to \mathbb{R}$ INTEGRALS OF MOTION.

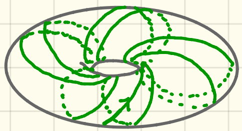
ASSUME:
- $F_1, \dots, F_m$ ARE FUNCTIONALLY INDEPENDENT ($dF_1, \dots, dF_m$ LINEAR INDEP.)
- $\{F_i, F_j\} = 0 \quad \forall i, j = 1 \dots m$ $\longleftrightarrow$ PAIRWISE IN INVOLUTION (POISSON COMMUTE)

CONSIDER $M_c := \{F_1 = c_1, \dots, F_m = c_m\} \neq \emptyset \qquad c = (c_1, \dots c_m) \in \mathbb{R}^m$

THEN:
- M_c IS AN INVARIANT LAGRANGIAN SUBMANIFOLD

  $\uparrow$ INTEGRALS OF MOTION $\qquad \uparrow$ INVOLUTION $\qquad \uparrow$ INDEPENDENCE

- IF M_c IS COMPACT (AND CONNECTED), THEN $M_c \cong \mathbb{T}^m$ ON WHICH THE MOTION IS CONJUGATED TO A ROTATION



- $\exists$ A NEIGHBOURHOOD OF M_c AND A CHANGE OF COORDINATES (ACTION-ANGLE) TRANSFORMING THE HAMILTONIAN:

$$\hat{H} : \mathcal{U} \subseteq T^*\mathbb{T}^m \to \mathbb{R}, \quad (x, p) \to \hat{H}(p) \qquad \{p = 0\} = M_c$$

CAN ONE WEAKEN THE HYPOTHESES IN LIOUVILLE-ARNOL'D THEOREM?

REMARK: IN LIOUVILLE-ARNOL'D THEOREM, INVOLUTION IS FUNDAMENTAL TO CONCLUDE THAT THE INVARIANT MANIFOLD IS LAGRANGIAN AND ITS DYNAMICAL PROPERTIES.

DEFINITION (WEAK-INTEGRABILITY) ← [SORRENTINO, TRANS. AMS 2011]

(V,ω) SYMPLECTIC MANIFOLD, $\dim V = 2n$, $H: V \to \mathbb{R}$ HAMILTONIAN

H IS SAID WEAKLY-INTEGRABLE IF IT HAS n INDEPENDENT INTEGRALS OF MOTION.

$$\text{LIOUVILLE-ARNOL'D INTEGRABILITY} \;\underset{\not\Leftarrow ?}{\Longrightarrow}\; \text{WEAK INTEGRABILITY}$$

NO!

LET G A COMPACT SEMI-SIMPLE LIE GROUP OF RANK ≥ 2
IN ANY NBHD OF THE BI-INVARIANT METRIC THERE ARE LEFT INV. METRICS WITH POSITIVE TOP. ENTROPY (BUTLER-PATERNAIN)

THESE LEFT INV. METRICS ARE WEAKLY INTEGRABLE BUT NOT LIOUVILLE-ARNOLD

EXAMPLE OF WEAKLY-INTEGRABLE SYSTEMS (SOME IDEAS)

G COMPACT LIE GROUP, $\mathfrak{g}$ LIE ALGEBRA

CONSIDER A LEFT INVARIANT RIEMANNIAN METRIC ON G $\longrightarrow$ GEODESIC FLOW

↳ $A: \mathfrak{g} \to \mathfrak{g}^*$ EUCLIDEAN STRUCTURE ON G DEFINING THE METRIC (SYMMETRIC POSITIVE DEFINITE)

↳ $$\left.\begin{aligned} A_g: T_g G &\longrightarrow T_g^* G \\ \dot{g} &\longmapsto L^*_{g^{-1}} A L_{g^{-1}*} \dot{g} \end{aligned}\right\}$$ IT CAN BE EXTENDED TO THE WHOLE TG

MAPS INDUCED BY LEFT TRANSLATION ON COTANGENT AND TANGENT SPACE

$$\tilde{A}(g,\dot{g}) = (g, A_g(\dot{g}))$$ MOMENT OF INERTIA OPERATOR

$$H: T^*G \longrightarrow \mathbb{R}$$
$$(g,p) \longmapsto \frac{1}{2}\langle p, A_g^{-1} p\rangle$$

$p \rightsquigarrow$ $p_b = L_g^* p \longrightarrow$ ANGULAR MOMENTUM RELATIVE TO THE BODY

$p_s = R_g^* p \longrightarrow$ " " " TO THE SPACE

$$\left.\right\} \quad \boxed{\frac{d}{dt} p_s = 0} \qquad \frac{d}{dt} p_b = ad^*_{A^{-1}_{p_b}} p_b$$

INTEGRALS OF MOTION ONE FOR EVERY $g \in G$

CO-ADJOINT REPR OF THE GROUP

WEAKLY-INTEGRABLE TONELLI HAMILTONIANS

$H: T^*M \longrightarrow \mathbb{R}$

IDEA: STUDY HOW THE PRESENCE OF INTEGRAL OF MOTIONS, IS REFLECTED BY THE ACTION-MINIMIZING PROPERTIES OF THE SYSTEM (i.E, ITS MATHER SETS $\{M_c^*\}_{c\in H^1(M,\mathbb{R})}$)

MATHER SETS IN T^*M VIA THE LEGENDRE TRANSFORM

KEY REMARKS:

- $\{M_c^*\}_c$ ARE SYMPLECTIC INVARIANT. LET $F: T^*M \longrightarrow \mathbb{R}$ INTEGRAL OF MOTION AND LET $\phi_F^t : T^*M \longrightarrow T^*M$ BE ITS FLOW.

THEN: $\phi_F^t(M_c^*(H)) = M_c^*(H\circ\phi_F^{-t}) = M_c^*(H) \Longrightarrow M_c^*$ IS PRESERVED BY ϕ_F^t

- IF H HAS k INDEPENDENT INTEGRALS OF MOTION $\Longrightarrow$ RANK $T_{(x,p)}M_c^* \geq k$ $\quad \forall (x,p)\in M_c^* \quad \forall c\in H^1(M,\mathbb{R})$

M_c^* ARE NOT NEC. SUBMANIFOLDS

$T_{(x,p)}M_c^* = \{$ SET OF ALL VECTORS TANGENT TO M_c^* AT $(x,p)\}$

THE RANK OF $T^*_{(x,p)}M_c^*$ PROVIDES A CONSTRAINT TO HOW MANY INDEP. INTEGRALS OF MOTION, CAN EXIST IN A NEIGHBORHOOD OF (x,p)

- MATHER SETS FORCE INTEGRAL OF MOTIONS TO POISSON COMMUTE.

LET $F_1, F_2 : T^*M \to \mathbb{R}$ C^2 INTEGRALS OF MOTION. THEN $\forall c \in H^1(M, \mathbb{R})$

$$\{F_1, F_2\}(x, \pi_c^{-1}(x)) = 0 \qquad \forall x \in \overline{\mathrm{Int}(\pi_c(\tilde{M}_c^*))}^{(*)}$$

WHERE $\pi : TM \to M$ AND $\pi_c := \pi|\tilde{M}_c^*$

THIS SET MIGHT BE EMPTY

THIS FOLLOWS FROM THE FACT THAT $\tilde{M}_c^*$ IS A LIPSCHITZ GRAPH, WHICH IS A (SUB) SOLUTION ON HAMILTON-JACOBI EQUATION $H(x, \eta_c + du) = \alpha(c)$ $[\eta_c] = c$

$\Rightarrow$ IT INHERITS A ISOTROPIC TANGENT STRUCTURE ALMOST EVERYWHERE.

THEOREM [BUTLER - SORRENTINO] (S. TRANS. AMS 2011, BUTLER-S. CMP 2012)

LET $H: T^*M^m \to \mathbb{R}$ WEAKLY-INTEGRABLE C^r TONELLI HAMILTONIAN.

SUPPOSE $\exists\, c \in H^1(M,\mathbb{R})$: $\mathcal{M}_c^*$ INTERSECTS A REGULAR LEVEL SET OF $F = (F_1 \dots F_m)$

THEN:

i) • $\Lambda_c := \mathcal{M}_c^*$ IS A C^r INVARIANT LAGRANGIAN GRAPH;

• Λ_c IS STRICTLY SCHWARTZMAN ERGODIC (I.E., ALL INVARIANT PB MEASURES SUPPORTED ON IT, HAVE THE SAME ROTATION VECTOR, AND THE UNION OF THEIR SUPPORTS EQUALS Λ_c)

IN PARTICULAR, ALL ORBITS ARE CONJUGATE BY A SMOOTH DIFFEO ISOTOPIC TO THE IDENTITY

• Λ_c ADMITS THE STRUCTURE OF A SMOOTH $\mathbb{T}^d$-BUNDLE OVER A BASIS B^{m-d} THAT IS PARALLELISABLE, WITH $H_1(B) = 0$ $(d > 0)$

ii) • THE SAME IS TRUE FOR c' IN AN OPEN SET $\mathcal{O} \subseteq H^1(M,\mathbb{R})$

• α - FUNCTION IS DIFFERENTIABLE ON $\mathcal{O}$ & β-FUNCTION IS DIFFERENTIABLE ON $\partial\alpha(\mathcal{O})$

iii) • IF $\exists\ \Lambda_{c_0} \subseteq \mathrm{Int}\left(\bigcup_{c' \in \mathcal{O}} \Lambda_{c'}\right) \Longrightarrow \Lambda_{c_0}$ IS DIFFEOMORPHIC TO $\mathbb{T}^m$ AND THE

↑ (WE HAVE A FOLIATION)   MOTION ON IT IS CONJUGATE TO A ROTATION.

• IN PARTICULAR, $M \cong \mathbb{T}^m$ AND THE SYSTEM IS LIOUVILLE-ARNOL'D INTEGRABLE IN A NEIGHBORHOOD OF Λ_{c_0}

iv) IF $\dim H^1(M, \mathbb{R}) \geq \dim M$, THEN (iii) IS SATISFIED:

H IS LIOUVILLE-ARNOL'D INTEGRABLE IN A NEIGHBORHOOD OF Λ_c AND $M \cong \mathbb{T}^m$.

(v) • IF $\dim M \leq 3$, THEN $M \cong \mathbb{T}^m$

• IF $\dim M = 4$, $M \cong \mathbb{T}^4$ OR $\mathbb{T}^1 \times E$ WHERE E = ORIENTABLE 3-MANIFOLD FINITELY COVERED BY S^3

THANK YOU

FOR YOUR

ATTENTION!

ALL PUBLICATIONS CAN BE FOUND ON MY WEB-PAGE:

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