Infinite excitation limit: horocyclic chaos

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Geodesic flow
(Anosov Th.
on ergodicity)

homotopy of classical systems

Horocycle flow
(Furstenberg Th
on unique ergodicity)

quantization

quantization

Free quantum particle
(Rudnick-Sarnak
QUE conjecture
etc.)

Quantum homotopy
(excitation)

Quantum horocyclic chaos

Bohr correspondence:

Quantum mech. $\xrightarrow[\hbar \to 0]{}$ Classical mech.

$D = \frac{\partial}{i}, \hbar > 0$

Semiclassical analysis

$\mathbb{R}^d \quad a(x,\xi) \in C_0^\infty(\mathbb{R}^d \times \mathbb{R}^d) \rightsquigarrow A = a(x, \hbar D) =$

$\hbar \in \mathbb{R}^d$

$= Op_\hbar a : C_0^\infty(\mathbb{R}^d)$

$\downarrow$

$C^\infty(\mathbb{R}^d)$

$(Op_\hbar a)\, u(x) = \int_{\mathbb{R}^d} a(x, \hbar\xi)\, e^{ix\cdot\xi}\, \hat{u}(\xi)\, d\xi$

$u : \mathbb{R}^d \to \mathbb{C}$

$Op_\hbar [a,b] = \frac{i}{\hbar} [Op_\hbar a, Op_\hbar b] + O_{L^2 \to L^2}(\hbar)$

Question: suppose that classical system is chaotic.

? Is the quantization of such system chaotic?

Rudnick-Sarnak conjecture (Quantum Unique Ergodicity (QUE))

$\mathbb{H} = \mathbb{C}^+ = \{x+iy : x \in \mathbb{R}, y>0\}$ $\quad ds = \frac{dx^2+dy^2}{y^2}$

X - hyperbolic surface $= \Gamma \backslash \mathbb{H}$

$\dim_{\mathbb{R}} X = 2$ $\quad \partial X = \emptyset$ $\quad X$ - compact

$K \equiv -1$

$\Gamma < \mathrm{Isom}^+ \mathbb{H}$

$\gamma \in \Gamma : \gamma z \mapsto \frac{az+b}{cz+d}$

$ad-bc=1$

$-\Delta$ - Laplace-Beltrami op. on X (on $\mathbb{H}$)

X comp. $\Rightarrow \sigma(-\Delta)$ is discr.

$-\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$

$-\Delta u^{(s)} = s^2 u^{(s)}$

$s \in \sqrt{\sigma(-\Delta)}$ $\quad u = u^{(s)}$

$\int_X u^2 d\mathrm{vol}_X = 1$

$h = 1/s$

$h^2 \Delta u = u$

Conjecture: $(u^{(s)})^2 \cdot \mathrm{vol}_X \xrightarrow[s\to\infty]{} \frac{\mathrm{vol}_X}{\mathrm{vol}_X(X)}$?

Semiclassical measures

$\exists \mathbb{N}_1 \subset \sqrt{\sigma(-\Delta)}$ $\exists\, y < y_1$ $\exists \mu \in \mathrm{Meas}^+(T^*X)$:

1. $\forall a \in C_0^\infty(T^*X)$

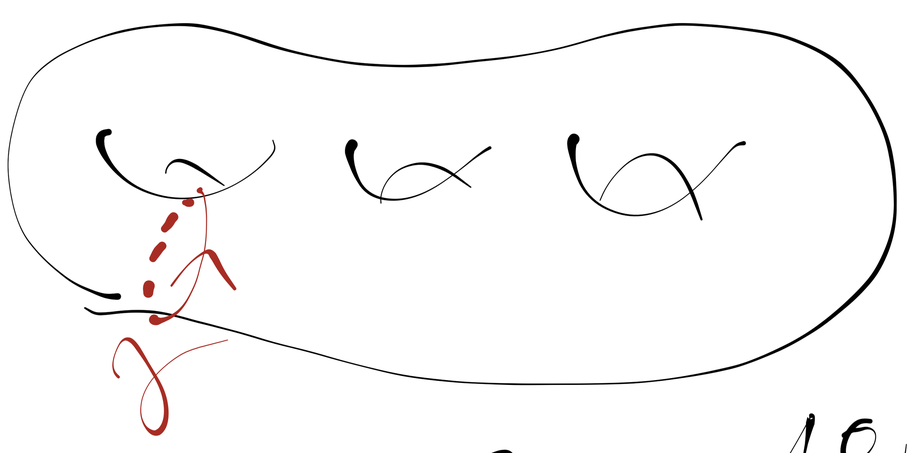
$\left\langle \left(\mathrm{Op}_{1/s} a\right) u^{(s)}, u^{(s)} \right\rangle_{L^2(X)} \xrightarrow[s\in\mathbb{N}]{s\to+\infty} \int_{T^*X} a \, d\mu$

2. $\mathrm{supp}\,\mu \subset S^*X$

3. μ is invariant w.r.t. geodesic flow on S^*X.

$\exp\left(\frac{-h^2\Delta}{ih}t\right) \circlearrowright L^2(X)$

Conjecture: ? μ = Liouville meas. on S^*X



μ_γ

? $\mu = \mu_\gamma$?

? $\mu = 1\% \cdot \mu_\gamma + 99\% \cdot$ (Prob. Liouville meas on S^*X)

$$H_B = -y^2(\xi_1^2+\xi_2^2) - 2By\xi_1 \quad \cdot 1/2$$

$$D_{B,\hbar} = -\hbar^2 y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) + 2iB\hbar y\frac{\partial}{\partial x}$$

$D_{B,\hbar} \curvearrowright C^\infty(H)$ $\not\curvearrowright C^\infty(X)$ $\quad X = \Gamma\backslash H$

curv. $B/\sqrt{B^2+1}$, speed $= \sqrt{B^2+1}$

$D_{B,\hbar} \curvearrowright \{\tau\text{-forms}\}$ $\quad \tau \in 2\mathbb{Z}$

Def $u: H \to \mathbb{C}$ is called τ-form, w.r.t. $\Gamma < I_{som}^+(HH)$

$\forall z \in \mathbb{C}\ \forall \gamma \in \Gamma$, $\gamma z = \frac{az+b}{cz+d}$ $\quad \gamma' z = \frac{1}{(cz+d)^2}$

$u(\gamma z) = u(z)\cdot\left(\frac{cz+d}{c\bar z+d}\right)^\tau$ $\quad \left(\frac{dz}{d\bar z}\right)^\tau u(z)$

$F^\tau(\Gamma) = \{\tau\text{-forms w.r.t. } \Gamma\}$

$D_{B,\hbar} = \hbar^2 D^\tau$, $D^\tau = -y^2(\partial_x^2+\partial_y^2)+2i\tau y\partial_x$

$\tau = B/\hbar$

τ-Laplacian

$D^\tau \curvearrowright F^\tau(\Gamma)$ $\quad \exp\left(\frac{D_{B,\hbar}}{i\hbar}t\right) \curvearrowright F^{B/\hbar}(\Gamma)$

Maass derivatives

Def $\tau \in \mathbb{R}$ $K_\tau u := 2iy\frac{\partial u}{\partial z} + \tau u$

$u: \mathbb{C}^+ \to \mathbb{C}$

Properties: 1. $u \in F^\tau(\Gamma) \Rightarrow K_\tau u \in F^{\tau+1}(\Gamma)$

2. $K_\tau D^\tau = D^{\tau+1} K_\tau$

Rem. $u \in F^\tau \Rightarrow |u| \in F^0 \Rightarrow \|u\|_{L^2(X)}$ is defined

3. $X = \Gamma\backslash H$ is compact

$D^\tau u = s^2 u$

$u \in F^\tau(\Gamma)$

$\Rightarrow K_\tau u \in F^{\tau+1}(\Gamma)$

$\Rightarrow D^{\tau+1}(K_\tau u) = s^2(K_\tau u)$

$\|K_\tau u\|^2_{L^2(X)} = (s^2+\tau(\tau+1))\cdot\|u\|_{L^2(X)}$

$$\frac{K_\tau}{\sqrt{s^2+\tau(\tau+1)}} =$$

$u_0 = u_0^{(s)} : X \to \mathbb{R}$ $\quad -\Delta u_0 = s^2 u_0$

$u_0 \in \mathcal{F}^0(\Gamma)$

$B \geq 0$ $\quad u_B = u_B^{(s)} =$ B-excitation of $u_0^{(s)} :=$

$$\xrightarrow{K_{\lfloor Bs \rfloor - 1}} \circ \cdots \circ \xrightarrow{K_1} \circ \xrightarrow{K_0} u_0^{(s)}$$

$$\sqrt{s^2 + \lfloor Bs \rfloor (\lfloor Bs \rfloor - 1)} \qquad \sqrt{s^2 + 1 \cdot (1+1)} \qquad \sqrt{s^2 + 0 \cdot (0+1)}$$

$u_B \in \mathcal{F}^{\lfloor Bs \rfloor}(\Gamma)$; $\quad D^{\lfloor Bs \rfloor} u_B = s^2 u_B$

$\|u_B\|_{L^2(X)} = 1$

Claim: u_B depends smoothly on B

$H_B = \frac{1}{2}\left((\xi_1 - B)^2 + (\xi_2 y)^2\right)$

$\phi_B : T^* \mathbb{H} \to T\mathbb{H}$, $\quad \phi_B(x, y, \xi_1, \xi_2) =$

$$= \left(x, y, \frac{\partial H_B}{\partial \xi_1}, \frac{\partial H}{\partial \xi_2}\right)$$

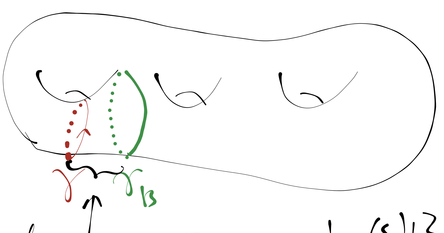
Th I $\quad \exists c \sqrt{2(-\Delta)}$: $\{u_0^{(s)}\}_{s \in \mathbb{Y}}$ has semiclassical measure μ_0

Then: $\{u_B^{(s)}\}_{s \in \mathbb{Y}}$ has semiclassical measure μ_B

$$(\phi_B)_\# \mu_B = (T_B)_\# (\phi_0)_\# \mu_0 .$$

$1 = r_0$ $\qquad$ $r_{\lfloor B \rfloor} := \sqrt{B^2 + 1}$

$\ln(B + \sqrt{B^2+1})$



$\ln(B + \sqrt{B^2+1})$

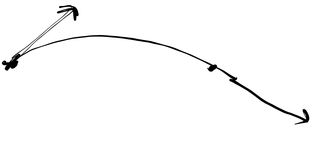
$|u_0^{(s)}|^2 \cdot \mathrm{Ft}_X \to^* 1\% \cdot \frac{H'|_{\gamma_1}}{H'(\gamma_1)} +$

$+ 99\% \cdot \frac{\mathrm{Ft}_X}{\mathrm{Ft}_X(X)}$

$|u_B^{(s)}|^2 \cdot \mathrm{Ft}_X \to^*$

$\to 1\% \frac{H'|_{\gamma_B}}{H'(\gamma_B)} + 99\% \cdot \frac{\mathrm{Ft}_X}{\mathrm{Ft}_X(X)}$

$\frac{B}{\sqrt{B^2+1}} \xrightarrow[B \to \infty]{} 1$



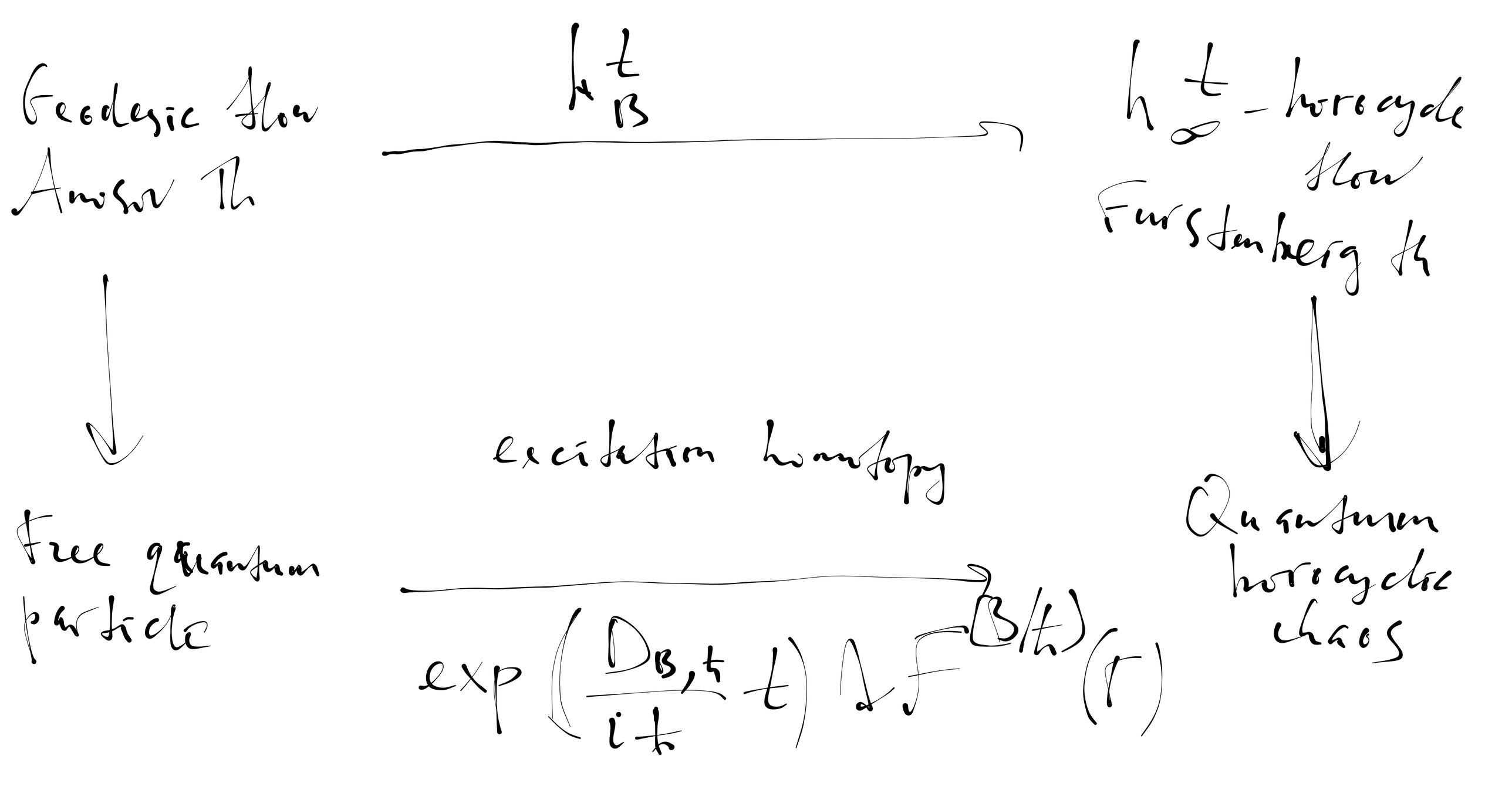
$h_\infty^t \curvearrowright S_1 \mathbb{H}$

$\curvearrowright S_1 X$

Th II $\quad \forall \{s_n\}_{n=1}^\infty \subset \sqrt{2(-\Delta)}$, $\quad \forall \{B_n\}_{n=1}^\infty$, $B_n \to +\infty$

$s_n \neq 0$

$$|u_{B_n}^{s_n}|^2 \cdot \mathrm{Ft}_X \xrightarrow{n \to \infty} * \frac{\mathrm{Ft}_X}{\mathrm{Ft}_X(X)}$$


Geodesic flow
Anosov Th
h_B^t
h_∞^t - horocycle flow
Furstenberg th
Free quantum particle
excitation homotopy
Quantum horocycle chaos
$\exp\left(\frac{D_{B,t}}{i\hbar}t\right)\Delta_F^{(B/\hbar)}(\Gamma)$