Braided Descriptions of Homotopy Groups of the 2-Sphere

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The Views from Configuration Spaces and Quillen's Plus Construction

The Views from Simplicial Groups

The Views from Braid Groups

Relative Lie Algebra of Brunnian Braids and the Unstable Adams Spectral Sequence

Configuration Spaces

Let M be a space. The n-th **ordered configuration space**[1]

$$F(M, n) = \{(x_1, \dots, x_n) \in M^{\times n} \mid x_i \neq x_j \text{ for } i \neq j\}.$$

The **configuartion space with labels in a pointed space** X

$$C(M, M_0; X) = \coprod_n F(M, n) \times_{\Sigma_n} X^n / \approx,$$

where $(m_1, \dots, m_n; x_1, \dots, x_n) \approx (m_1, \dots, m_{n-1}; x_1, \dots, x_{n-1})$ if $x_n = *$ or $m_n \in M_0$.

[1] Fadell, Edward; Neuwirth, Lee Configuration spaces. Math. Scand. 10 (1962), 111-118.

Iterated loop spaces

$$C(\mathbb{R}^n; X) \longrightarrow \Omega^n\Sigma^n X$$

is a (weak) homotopy equivalence if X is path-connected, and a group completion[2] in general[3].

In particular, the group completion of $C(\mathbb{R}^k, S^0) = \coprod_{n=0}^{\infty} F(\mathbb{R}^k, n)/\Sigma_n$ is $\Omega^k S^k$ up to homotopy.

Property:

$$F(\mathbb{R}^k, n)/\Sigma_n \text{ is } \begin{cases} K(B_n, 1) & \text{if} \quad k = 2 \\ K(\Sigma_n, 1) & \text{if} \quad k = \infty \\ \text{NOT a } K(\pi, 1) - \text{space} & \text{if} \quad 2 < k < \infty \end{cases}$$

[2] A group completion of a topological monoid M is ΩBM.

[3] (i) Segal, Graeme Invent. Math. 21 (1973), 213-221. (ii) May, J. P. The geometry of iterated loop spaces. Lectures Notes in Mathematics, Vol. 271, 1972. (iii) Cohen, Fred Bull. Amer. Math. Soc. 79 (1973), 1236-1241 (1974).

Quillen's Plus Construction

Using Quillen's plus construction[4],

$$\begin{array}{ccc} K(B_\infty,1) & \longrightarrow & K(\Sigma_\infty,1) \\ \downarrow & & \downarrow \\ K(B_\infty,1))^+ \simeq \Omega_0^2 S^2 & \longrightarrow & K(\Sigma_\infty,1)^+ \simeq \Omega_0^\infty S^\infty. \end{array}$$

This indicates that, philosophically, homotopy theory on S^2 is a kind of **braided interpretation** of stable homotopy theory on spheres.

Of course it is a **mystery** how $\pi_*(S^2)$ is related to braids and/or stable homotopy groups in some more concrete way.

[4] For a space X, Quillen's construction X^+, with a map $q\colon X \to X^+$ such that $q_*\colon H_*(X) \to H_*(X^+)$ is an isomorphism and $\pi_1(X^+) = \pi_1(X)/\text{maximal perfect subgroup}$.

Simplicial Sets

Simplicial set was originally called complete semi-simplicial complex[5] (c. s. s. complex) is a sequence of sets $X_* = \{X_n\}_{n\geq 0}$ with faces $d_i\colon X_n \to X_{n-1}$, $0 \leq i \leq n$, and degeneracies $s_j\colon X_n \to X_{n+1}$, $0 \leq j \leq n$, such that

$$d_j d_i = d_{i-1} d_j \tag{1}$$

for $j < i$,

$$s_j s_i = s_{i+1} s_j \tag{2}$$

for $j \leq i$ and

$$d_j s_i = \begin{cases} s_{i-1} d_j & j < i \\ \mathrm{id} & j = i, i+1 \\ s_i d_{j-1} & j > i+1. \end{cases} \tag{3}$$

[5] Eilenberg, Samuel; Zilber, J. A. Semi-simplicial complexes and singular homology. Ann. of Math. (2) 51 (1950), 499-513.

Simplicial Groups

A **simplicial group** is a simplicial set G_* so that each G_n is a group, and all faces and degeneracies are group homomorphisms.

The homotopy category of simplicial sets is equivalent to the homotopy category of CW-complexes.

The homotopy category of simplicial groups is equivalent to the homotopy category of loop spaces of CW-complexes.

Moore Theorem on Simplicial Groups. Let $N_nG = \bigcap_{i=1}^{n} \mathrm{Ker}(d_i \colon G_n \to G_{n-1})$. Then N_*G with $d_0| \colon N_nG \to N_{n-1}G$ is a chain complex (of possibly non-abelian groups) and $\pi_n(|G_*|) \cong H_n(N_*G)$.

What is the special of ΩS^2?

Let us look at the question in a geometric way: Let S^2 be a (geometric) 2-sphere. Let $q_0, q_1, q_2, \ldots \in \mathbb{R}^1 \rightarrowtail S^2$ with $q_0 < q_1 < q_2 < \cdots$. Let

$$\hat{F}_n = \pi_1(S^2 \smallsetminus \{q_0, q_1, \ldots, q_n\}), \quad n \geq 0.$$

The face $d_i \colon \hat{F}_n \to \hat{F}_{n-1}$ is induced by removing q_i, $0 \leq i \leq n$, and degeneracy $s_i \colon \hat{F}_n \to \hat{F}_{n+1}$ is induced by doubling q_i in a small neighborhood of q_i in the curve.

Then

- $|\hat{F}_*| \simeq \Omega S^2$.
- $\hat{F}_n = \langle x_0, x_1, \ldots, x_n : x_0 x_1 x_2 \cdots x_n = 1 \rangle$, free group of rank n.
- The braid group B_{n+1} acts on $\hat{F}_n$ via Artin representation, which is compatible with simplicial structure.

Combinatorial description of $\pi_*(S^2)$

- Let $R_i = \langle x_i \rangle^{\hat{F}_n}$ be the normal closure of x_i in $\hat{F}_n$ for $0 \leq i \leq n$. We can form a symmetric commutator subgroup

$$[R_0, R_1, \ldots, R_n]_S = \prod_{\sigma \in \Sigma_{n+1}} [\ldots [R_{\sigma(0)}, R_{\sigma(1)}], \ldots, R_{\sigma(n)}],$$

- **Theorem (Wu, 1994, published version[6]).** For $n \geq 1$, there is an isomorphism

$$\pi_{n+1}(S^2) \cong \frac{R_0 \cap \cdots \cap R_n}{[R_0, \ldots, R_n]_S}$$

This quotient group is isomorphic to the center of the group $F_n/[R_0, R_1, \ldots, R_n]_S$.

[6] Wu, J. Combinatorial descriptions of homotopy groups of certain spaces. Math. Proc. Cambridge Philos. Soc. 130 (2001), no. 3, 489-513.

Braid group actions

- There is an action of the braid group B_{n+1} on $F_n = \langle x_0, x_1, \ldots, x_n \mid x_0 x_1 \cdots x_n \rangle$ by the Artin representation, which induces an action of B_{n+1} on the quotient group $F_n/[R_0, R_1, \ldots, R_n]_S$.

- **Theorem**[7] The center of $F_n/[R_0, R_1, \ldots, R_n]_S$, which is isomorphic to $\pi_{n+1}(S^2)$, is exactly given by the fixed set of the pure braid group P_{n+1} action on $F_n/[R_0, R_1, \ldots, R_n]_S$ for $n \geq 3$.

[7] Wu, Jie A braided simplicial group. Proc. London Math. Soc. (3) 84 (2002), no. 3, 645-662.

Homotopy Groups and Brunnian braids

Let $\mathrm{Brun}_n(M)$ be the group of Brunnian braids on the surface M. Then inclusion D^2 into S^2 by regarding D^2 as the upper hemisphere induces a group homomorphism $f_*\colon\ \mathrm{Brun}_n(D^2) \to \mathrm{Brun}_n(S^2)$.

- **Theorem**[8] For $n \geq 5$, there is an exact sequence of groups

$$\mathrm{Brun}_{n+1}(S^2) \hookrightarrow \mathrm{Brun}_n(D^2) \xrightarrow{f_*} \mathrm{Brun}_n(S^2) \twoheadrightarrow \pi_{n-1}(S^2).$$

Roughly speaking $\pi_{n-1}(S^2)$ is given by the n-strand Brunnian braids on S^2 modulo the n-strand Bruunian on D^2.

[8] Berrick, A. J.; Cohen, F. R.; Wong, Y. L.; Wu, J. Configurations, braids, and homotopy groups. J. Amer. Math. Soc. 19 (2006), no. 2, 265-326.

Brunnian Braids on General Surfaces

Let M be a connected 2-manifold and let $n \geq 4$. Let

$$R_n(M) = \prod_{\sigma\in\Sigma_{n-1}} [[\langle\langle A_{\sigma(1),n}\rangle\rangle^P, \langle\langle A_{\sigma(2),n}\rangle\rangle^P], \ldots, \langle\langle A_{\sigma(n-1),n}\rangle\rangle^P]$$

be the symmetric commutator subgroup, where $\langle\langle A_{i,j}\rangle\rangle^P$ is the normal closure of the braid $A_{i,j}$ in $P_n(M)$.

Theorem[9]

1. If $M \neq S^2$ or $\mathbb{R}\mathrm{P}^2$, then
$$\mathrm{Brun}_n(M) = R_n(M).$$
2. If $M = S^2$ and $n \geq 5$, then there is a short exact sequence
$$R_n(S^2) \hookrightarrow \mathrm{Brun}_n(S^2) \twoheadrightarrow \pi_{n-1}(S^2).$$
3. If $M = \mathbb{R}\mathrm{P}^2$, then there is a short exact sequence
$$R_n(\mathbb{R}\mathrm{P}^2) \hookrightarrow \mathrm{Brun}_n(\mathbb{R}\mathrm{P}^2) \twoheadrightarrow \pi_{n-1}(S^2).$$

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[9] Bardakov, Valery G.; Mikhailov, Roman; Vershinin, Vladimir V.; Wu, Jie Brunnian braids on surfaces. Algebr. Geom. Topol. 12 (2012), no. 3, 1607-1648

Lie algebras of groups

We recall that for a group G the descending central series

$$G = \Gamma_1 \geq \Gamma_2 \geq \cdots \geq \Gamma_i \geq \Gamma_{i+1} \geq \ldots.$$

is defined by the formulae

$$\Gamma_1 = G, \;\; \Gamma_{i+1} = [\Gamma_i, G].$$

The descending central series of a discrete group G gives rise to the associated graded Lie algebra (over $\mathbb{Z}$) $L(G)$

$$L_i(G) = \Gamma_i(G)/\Gamma_{i+1}(G).$$

Yang-Baxter Lie algebra

Let $G = P_n$.

Kohno'85:[10] The Lie algebra $L(P_n)$ is the quotient of the free Lie algebra $L[A_{i,j} \mid 1 \leq i < j \leq n]$ generated by elements $A_{i,j}$ with $1 \leq i < j \leq n$ modulo the "infinitesimal braid relations" or "horizontal $4T$ relations" given by the following three relations:

$$\begin{cases} [A_{i,j}, A_{s,t}] = 0, \text{ if } \{i,j\} \cap \{s,t\} = \emptyset, \\ [A_{i,j}, A_{i,k} + A_{j,k}] = 0, \text{ if } i < j < k, \\ [A_{i,k}, A_{i,j} + A_{j,k}] = 0, \text{ if } i < j < k. \end{cases} \tag{4}$$

[10] T. Kohno, Série de Poincaré-Koszul associée aux groupes de tresses pure, Invent. Math. 82 (1985) 57-75.

Braid Commutators and Vassiliev Invariants

Ted Stanford'96:[11] Let L and L' be two links which differ by a braid $p \in \Gamma_n(P_k)$. Let v be a link invariant of order less than n. Then $v(L) = v(L')$.

Here the meaning for two links to differ by a braid p is as follows: Let $\hat{x}$ denote the closure of a braid x. Let b and p be any two braids with the same number of strands. Then $\hat{b}$ and $\hat{pb}$ differ by p.

In brief, **lower central series of pure braid groups $\Longrightarrow$ Vassiliev Invariants.**

[11] T. Stanford, Braid commutators and Vassiliev invariants, Pacific J. Math. 174 (1996) 269-276.

Vassiliev Invariants on subgroups of pure braid groups

Let $G \leq P_k$ be a subgroup of P_k. The elements in G give a set of special type of braids. Then the set $\hat{G} = \{\hat{x} \mid x \in G\}$ gives a subset of special type of links.

For detecting the Vassiliev invariants on the special type of links given by $\hat{G}$, a natural way is to consider

$$G = \Gamma_1(P_k) \cap G \geq \Gamma_2(P_k) \cap G \geq \cdots \geq \Gamma_i(P_k) \cap G \geq \Gamma_{i+1}(P_k) \cap G \geq \ldots.$$

The resulting (relative) Lie algebra $L^{P_k}(G) = \bigoplus_{i=1}^{\infty} (\Gamma_i(P_k) \cap G)/(\Gamma_{i+1}(P_k) \cap G)$ is a sub Lie algebra of the Yang-Baxter Lie algebra $L(P_k)$.

Our Question

Our question is to determine the (relative) Lie algebra $L^{P_n}(\mathrm{Brun}_n)$.

Lie Monomial

A **Lie monomial** W on the letters $A_{1,n}, A_{2,n}, \ldots, A_{n-1,n}$ means $W = A_{i,n}$ for some $1 \leq i \leq n-1$ or a Lie bracket $W = [A_{j_1,n}, A_{j_2,n}, \ldots, A_{j_t,n}]$ under any possible bracket arrangements with entries taken from the letters $A_{i,n}$.

Recursive Construction

We recursively define the sets $\mathcal{K}(n)_k$, $1 \leq k \leq n$, in the reverse order as follows:

1) Let $\mathcal{K}(n)_n = \{A_{1,n}, A_{2,n}, \ldots, A_{n-1,n}\}$.
2) Suppose that $\mathcal{K}(n)_{k+1}$ is defined as a subset of Lie monomials on the letters

$$A_{1,n}, A_{2,n}, \ldots, A_{n-1,n}$$

with $k < n$. Let

$$\mathcal{A}_k = \{W \in \mathcal{K}(n)_{k+1} \mid W \text{ does not contain } A_{k,n} \text{ in its entries}\}.$$

3) Define

$$\mathcal{K}(n)_k = \{W' \text{ and } [\cdots[[W', W_1], W_2], \ldots, W_t]\}$$

for $W' \in \mathcal{K}(n)_{k+1} \smallsetminus \mathcal{A}_k$ and $W_1, W_2, \ldots, W_t \in \mathcal{A}_k$ with $t \geq 1$. Note that $\mathcal{K}(n)_k$ is again a subset of Lie monomials on letters $A_{1,n}, A_{2,n}, \ldots, A_{n-1,n}$.

Example

Let $n = 3$. The set $\mathcal{K}(3)_1$ is constructed by the following steps:

1) $\mathcal{K}(3)_3 = \{A_{1,3}, A_{2,3}\}$.
2) $\mathcal{A}_2 = \{A_{1,3}\}$,
$$\mathcal{K}(3)_2 = \{A_{2,3}, [[A_{2,3}, A_{1,3}], \ldots, A_{1,3}]\}.$$
3) $\mathcal{A}_1 = \{A_{2,3}\}$,
$\mathcal{K}(3)_1 = \{[\cdots[A_{2,3}, A_{1,3}], \ldots, A_{1,3}], A_{2,3}], \ldots, A_{2,3}]\}$, where the length of the entries $A_{2,3}$ in the brackets is ≥ 0.

Basis for $L^{P_n}(\mathrm{Brun}_n)$, Li-Vershinin-Wu'15[12]

Main Theorem 1. The Lie algebra $L^{P_n}(\mathrm{Brun}_n)$ is a free Lie algebra generated by $\mathcal{K}(n)_1$ as a set of free generators.

[12] Li, J. Y.; Vershinin, V. V.; Wu, J. Brunnian braids and Lie algebras. J. Algebra 439 (2015), 270-293.

Symmetric bracket sum of Lie ideals

Let L be a Lie algebra and $I_1, \ldots, I_n$ ideals of L. The **symmetric bracket sum** of these ideals is defined as

$$[[I_1, I_2], \ldots, I_l]_S := \sum_{\sigma \in \Sigma_l} [[I_{\sigma(1)}, I_{\sigma(2)}], \ldots, I_{\sigma(n)}],$$

where Σ_n is the symmetric group on n letters.

Symmetric Bracket Theorem, Li-Vershinin-Wu'15

Let us denote the ideal

$$L[A_{k,n}, [\cdots[A_{k,n}, A_{j_1,n}], \ldots, A_{j_m,n}] \mid j_i \neq k, n;\ j_i \leq n-1, i \leq m;\ m \geq 1]$$

by I_k.

Main Theorem 2.
The Lie subalgebra $L^{P_n}(\mathrm{Brun}_n)$ and the symmetric bracket sum $[[I_1, I_2], \ldots, I_{n-1}]_S$ are equal as subalgebras in $L(P_n)$:

$$L^{P_n}(\mathrm{Brun}_n) = [[I_1, I_2], \ldots, I_{n-1}]_S.$$

The rank of $L^{P_n}(\mathrm{Brun}_n)$, Li-Vershinin-Wu'15

Observe that $L^{P_n}(\mathrm{Brun}_n) = \oplus_{q=1}^{\infty} L_q^{P_n}(\mathrm{Brun}_n)$ is a graded Lie algebra, where each $L_q^{P_n}(\mathrm{Brun}_n)$ is a free abelian group of finite rank. The rank of $L_q^{P_n}(\mathrm{Brun}_n)$ can be determined.

Main Theorem 3.

$$\mathrm{rank}(L_q^P(\mathrm{Brun}_n)) = \sum_{k=0}^{n-1} (-1)^k \binom{n}{k} \mathrm{rank}(L_q(P_{n-k}))$$

for each n and q, where $P_1 = 0$ and, for $m \geq 2$,

$$\mathrm{rank}(L_q(P_m)) = \frac{1}{q} \sum_{k=1}^{m-1} \sum_{d|q} \mu(d) k^{q/d}$$

with μ the Möbius function.

Brunnian Lie algebra over S^2—Li-Vershinin-Wu, working progress

Let $\mathrm{BrunL}(S^2)_n = \bigcap_{i=1}^{n} \ker(d_i : L(P_n(S^2)) \to L(P_{n-1}(S^2)))$. Let J_i be the image of I_i under the projection $L(P_n) \to L(P_n(S^2))$.

Theorem. There is a short exact sequence

$$[[J_1, J_2], \ldots, J_{n-1}]_S \hookrightarrow \mathrm{BrunL}(S^2)_n \twoheadrightarrow \Lambda_{n-1}(S^2)$$

for $n \geq 5$, where $\Lambda(S^2)$ is the Λ-algebra. Moreover

$$[[J_1, J_2], \ldots, J_{n-1}]_S \leq L^P(\mathrm{Brun}_n(S^2)) \leq \mathrm{BrunL}(S^2)_n$$

with $|L^P(\mathrm{Brun}_n(S^2))/[[J_1, J_2], \ldots, J_{n-1}]_S| = |\pi_{n-1}(S^2)|$ for $n \geq 5$.

Thank You for Your Attention!