Geometric facets of quantization

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Louis Ioos - David Kazhdan - L.P. (2020)

What is this lecture about?

Quantum classical correspondence: Quantum mechanics contains classical in the limit $\hbar \to 0$.

Not precise! (Groenewold- Van Hove)

- **quantum footprints** of symplectic geometry/Hamiltonian dynamics in phase space.
- **quantum errors** governed by Riemannian geometry (cf. Klauder)

- Optimal positive quantizations correspond to compatible almost complex structures;
- Classification of quantizations and Ulam stability for representations of $\mathfrak{su}(2)$ and quantum torus.

Deformation quantization

Star product: Associative (non-commutative) deformation of $(C^\infty(M), f\cdot g)$

$f * g = fg + \hbar c_1(f,g) + \hbar^2 c_2(f,g) + \dots,$

$\hbar$-formal parameter, $c_k(f,g)$ - bi-differential operators vanishing on constants

Bracket correspondence: $f * g - g * f = i\hbar\{f,g\} + \mathcal{O}(\hbar^2)$

F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, D. Sternheimer, 1977

Math. model of fin. volume quantum mechanics

H - finite dimensional Hilbert space over $\mathbb{C}$
$\mathcal{L}(H)$ - Hermitian operators on H
$\mathcal{S}$- density operators $\rho \in \mathcal{L}(H)$, $\rho \geq 0$, $Trace(\rho) = 1$.
$\hbar$-Planck constant.

Table: Quantum-Classical Correspondence

	CLASSICAL	QUANTUM
	Symplectic mfd (M, ω)	$\mathbb{C}$-Hilbert space H
OBSERVABLES	$f \in C^\infty(M)$	$A \in \mathcal{L}(H)$
STATES	Probability measures on M	Density ops $\rho \in \mathcal{S}$
BRACKET	Poisson bracket $\{f, g\}$	Commutator $\frac{i}{\hbar}[A, B]$

(M, ω) - closed symplectic, $\dim M = 2d$.

$H_\hbar$- family of Hilbert spaces of
$n_\hbar := \dim H_\hbar \sim (2\pi\hbar)^{-d}$, $\hbar = 1/k$, $k \to \infty$.

$T_\hbar : C^\infty(M) \to \mathcal{L}(H_\hbar)$- linear

Main features:

- **(positivity)** $f \geq 0 \Rightarrow T_\hbar(f) \geq 0$, $T_\hbar(1) = \mathbb{1}$;
- **(quasi-multiplicativity)** There exists a bi-differential operator $c : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ such that $T_\hbar(f)T_\hbar(g) = T_\hbar\left(fg + \hbar c(f, g)\right) + \mathcal{O}(\hbar^2)$
- **(reversibility)** $\mathcal{B}_\hbar := \frac{1}{n_\hbar} T_\hbar^* T_\hbar : C^\infty(M) \to C^\infty(M)$ satisfies $\mathcal{B}_\hbar(f) = f + \hbar Df + \mathcal{O}(\hbar^2)$, D - differential operator.
- **(bracket correspondence)** $[T_\hbar(f), T_\hbar(g)] = i\hbar T_\hbar(\{f, g\}) + \mathcal{O}(\hbar^2)$

Berezin-Toeplitz quantization-1

Extra axioms:
uniform norm $\leftrightarrow$ operator norm;
remainders (coefficients at $\hbar^p$) depend on N derivatives of f, g.

Positivity: for prob. measure $\alpha(x)$, coherent state $F_{x,\hbar} \in \mathcal{S}(H_\hbar)$

$$\boxed{T_\hbar(f) = n_\hbar \int_M f(x) F_{x,\hbar} d\alpha_\hbar(x)}$$

Trace correspondence: mean value $\leftrightarrow$ (normalized) trace.

$n_\hbar (2\pi\hbar)^d d\alpha_\hbar = \left(1 + \hbar r(x) + \mathcal{O}(\hbar^2)\right) d\mu$, μ-symplectic volume

Quasi-multiplicativity: In known examples, exists a star-product s.t. $T_\hbar(f) T_\hbar(g) = T_\hbar(f * g)$.
Here $T_\hbar$ extended to $C^\infty(M, \mathbb{C}) \to \mathcal{L}(H) \otimes \mathbb{C} = \mathrm{End}(H)$.

Reversibility: $\mathcal{B}_\hbar := \frac{1}{n_\hbar} T_\hbar^* T_\hbar$ - Berezin transform

composition of dequantization and quantization.

I. UNSHARPNESS METRIC

With Louis Ioos and David Kazhdan

Unsharpness cocycle

Bi-differential operator $c : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$:
$T_\hbar(f)T_\hbar(g) = T_\hbar\left(fg + \hbar c(f,g)\right) + \mathcal{O}(\hbar^2)$

$c_\pm(f,g) := (c(f,g) \pm c(g,f))/2$

Bracket correspondence $\Rightarrow c_-(f,g) = \frac{i}{2}\{f,g\}$

c_+ - symmetric unsharpness cocycle

Theorem (Ioos-Kazhdan-P., 2020)

(i) *Bi-differential operator c_+ is of order $(1,1)$.*

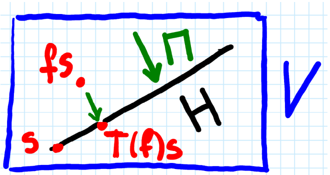
(ii) *There exists a bilinear symmetric form G on TM:*
$c_+(f,g) =: -\frac{1}{2}G(sgrad\, f, sgrad\, g)$
where $sgrad\, f$, $sgrad\, g$ Hamiltonian vector fields of $f,\ g \in C^\infty(M,\mathbb{R})$

(iii) $c_+(f,g) = -\frac{1}{2}\left(D(fg) - f\,D(g) - g\,D(f)\right)$, *where* $\mathcal{B}_\hbar(f) = f + \hbar Df + \mathcal{O}(\hbar^2)$*-Berezin transform.*

1. Kähler quantizaton Boutet de Monvel - Guillemin, 1981; Bordemann, Meinrenken and Schlichenmaier, 1994

(M, ω, J)- closed Kähler manifold, quantizable:
$[\omega]/(2\pi) \in H^2(M, \mathbb{Z})$
L- a holomorphic Hermitian line bundle over M
Curvature of Chern connection $= i\omega$.

$H_\hbar := H^0(M, L^{\otimes k}) \subset V_\hbar := L_2(M, L^{\otimes k})$.
$\Pi_\hbar : V_\hbar \to H_\hbar$ – the orthogonal projection.
The Toeplitz operator: $T_\hbar(f)(s) := \Pi_\hbar(fs)$, $f \in C^\infty(M)$, $s \in H_\hbar$.



Unsharpness tensor: $G(\xi, \eta) = \omega(\xi, J\eta)$ (Xu)

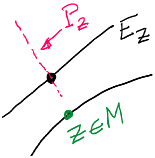
Kähler coherent states

$H_\hbar$ - holomorphic sections of L^k, $k = 1/\hbar$.

Hyperplane $E_z \subset H_\hbar$, $E_z := \{s \in H_\hbar \ : \ s(z) = 0\}$.

Kodaira embedding $M \to \mathbb{P}(H_\hbar^*)$, $z \mapsto E_z$

$P_{z,\hbar}$– orthogonal projector of $H_\hbar$ to $E_z^\perp$ coherent state projector



There exists Rawnsley function $R_\hbar \in C^\infty(M)$:

$$\boxed{T_\hbar(f) = \int_M f(x) R_\hbar(x) P_{x,\hbar} d\mathrm{Vol}(x)}$$

2. Almost - Kähler quantizaton Guillemin; Borthwick - Uribe; Schiffman - Zelditch; Ma - Marinescu; Charles; Ioos - Lu - Ma - Marinescu

(M, ω, J) , $G_J(\xi, \eta) = \omega(\xi, J\eta)$-Riemannian metric, $[\omega]$-quantizable: $[\omega]/(2\pi) \in H^2(M, \mathbb{Z})$

Modified construction: L - similar, H_k - spanned by eigenfunct. with "small" eigenvalues of Bochner Laplacian on L^k.
Unsharpness tensor: $G = G_J$ (Ioos,Lu,Ma,Marinescu)

3. Diffusion (M, ω, J) - Kähler or almost-Kähler, $T_\hbar$-quantization in Examples 1 or 2, Δ- (positive) Laplace-Beltrami.

Smearing by heat flow: $T_\hbar^{(t)}(f) := T_\hbar(e^{-t\hbar\Delta} f)$, $t > 0$.
Unsharpness tensor: $G^{(t)} := (1 + 4t)\, G_J$

Least Unsharpness Principle

(M, ω) - closed symplectic manifold
$T_\hbar$-Berezin-Toeplitz quantization.

Theorem (Ioos-Kazhdan-P., 2020)

(I) *Unsharpness tensor G is a Riemannian metric of the form $G_J + \rho$, where J is an ω-compatible almost complex structure, ρ is a non-negative symmetric bilinear form.*

(II) **Least unsharpness principle:** $\mathrm{Vol}(M, G) \geq \mathrm{Vol}(M, \omega)$, *with equality* $\Leftrightarrow G = G_J$, $\rho = 0$.

(III) *Assume (M, ω) is quantizable. Then every Riemannian metric as in (I) arises from some Berezin-Toeplitz quantization.*

Proof of (I): unsharpness (noise) of quantum measurements

(II): (cf. Gerhenstaber, 2007)

Proof of (III): almost Kähler quantization followed by diffusion.

Why unsharpness?

Dequantization of states: $\theta \in \mathcal{S}(H_\hbar)$.

Husimi measure μ_θ on M:

classical and quantum expectations coincide

$Exp(f,\mu_\theta) := \int f d\mu_\theta = \mathrm{tr}(T_\hbar(f)\theta) =: Exp(T_\hbar(f),\theta)$.

Unsharpness:
Increment of variances at coherent states (Ioos-Kazhdan-P.)

$$\boxed{Var(f,\mu_{F_{x,\hbar}}) = Var(T_\hbar(f),F_{x,\hbar}) + \frac{\hbar}{2}|\mathrm{sgrad} f(x)|_G^2 + \mathcal{O}(\hbar^2)}$$

Holds for weak reversibility $\mathcal{B}(f) = f + \mathcal{O}(\hbar)$.

II. Classifications of quantizations

Classification of quantizations

Star products: Change of variables $A : f \mapsto f + \sum_{m\geq 1} \hbar^m a_m(f)$
$a_m : C^\infty(M) \to C^\infty(M)$-differential operator.

Locally (in charts) star-products equivalent, globally classified by $H^2(M, \mathbb{R})[[\hbar]]$ (De Wilde - Lecompte,1983; Fedosov; Deligne; Nest-Tsygan; Gutt-Rawnsley)

Berezin-Toeplitz quantizations: Largely open.
Fix Hilbert spaces H_k,H'_k of the same dimension, $k \to \infty$, $\hbar = 1/k$

Quantizations $T_\hbar$, $T'_\hbar : C^\infty(M) \to \mathcal{L}(H_\hbar)$ are *m*-equivalent ($m \in \mathbb{N}$) if there exists unitaries $U_\hbar : H_\hbar \to H'_\hbar$ such that

$$\boxed{T'_\hbar(f) = U_\hbar T_\hbar(f) U^*_\hbar + \mathcal{O}(\hbar^m)}$$

Invariants of 2-equivalence

$$\boxed{T'_\hbar(f) = U_\hbar T_\hbar(f) U^*_\hbar + \mathcal{O}(\hbar^2)}$$

Observation 1: Unsharpness metric is an invariant of 2-equivalence.

Observation 2: Recall for prob. measure $\alpha(x)$, coherent state $F_{x,\hbar} \in \mathcal{S}(H_\hbar)$, $T_\hbar(f) = n_\hbar \int_M f(x) F_{x,\hbar} d\alpha_\hbar(x)$.

Rawnsley measure $r d\mu$, μ-symplectic volume

$$n_\hbar (2\pi\hbar)^d d\alpha_\hbar = \left(1 + \hbar r(x) + \mathcal{O}(\hbar^2)\right) d\mu \ ,$$

is an invariant of 2-equivalence.

Open problem: Are there other invariants of 2-equivalences?

Case study: 2-sphere, $SU(2)$-equivariant case

Assume H_k is the space of irrep of $SU(2)$ of dimension $k+1$.

Example: Spherical metric G of total area 2π. Standard Kähler quantization plus diffusion has unsharpness metric tG, $t \geq 1$.

Theorem

Any two $SU(2)$-equivariant quantizations with the same unsharpness metric coincide up to $\mathcal{O}(\hbar^2)$.

Tool: Representation theory.

1-equivalence

Class of quantizations: not necessarily positive

enhanced quasi-multiplicativity (mind c_2)

$$\boxed{T_k(f)T_k(g) = T_k\left(fg + \frac{1}{k}c_1(f,g) + \frac{1}{k^2}c_2(f,g)\right) + O(1/k^3)}$$

Fact:(L. Charles) Kähler quantizations with different complex structures are 1-equivalent.

Theorem (Ioos-Kazhdan-P.)

Let $M = S^2$ or (in progress) $M = T^2$.

- *Let $T_k : C^\infty(M) \to \mathcal{L}(H_k)$, $k \in \mathbb{N}$, be a quantization, and assume that* $\limsup_{k\to+\infty} \frac{\dim H_k}{k} < 2$. *Then* $\exists m \in \mathbb{Z} : \quad \forall k \gg 1 \ \dim H_k = k + m$.
- *Any two quantizations with the same m are 1-equivalent.*

Ulam problem for representations of $\mathfrak{su}(2)$.

Tool: Developing **approximate representations** for Lie algebras.
(for groups - Grove-Karcher-Ruh; Kazhdan; Lubotzky et al.)

Theorem (Ioos-Kazhdan-P.)

For every $c \in \mathbb{R}$ and $r > 0$, there exist $k_0 \in \mathbb{N}$ and $C > 0$ such that the following holds. Let H be a finite-dimensional Hilbert space, and assume that there exist $k \in \mathbb{N}$ with $k \geq k_0$ and a triple of operators $x_i \in \mathfrak{su}(H), i \in \mathbb{Z}/3\mathbb{Z}$, such that

- $\left\| x_1^2 + x_2^2 + x_3^2 + \left(\frac{k^2}{4} + \frac{kc}{2}\right) Id \right\|_{op} \leq r;$
- $\left\| [x_j, x_{j+1}] - x_{j+2} \right\|_{op} \leq r/k \quad \text{for} \quad j \in \mathbb{Z}/3\mathbb{Z}\,.$

Then $c \in \mathbb{Z}$. If in addition $\dim H < 2(k+c)$, then $\dim H = k + c$ and there exists an irreducible representation $\rho : \mathfrak{su}(2) \to \mathfrak{su}(H)$ such that $\|x_j - \rho(L_j)\|_{op} \leq C$, where $[L_j, L_{j+1}] = L_{j+2}\ \forall j$.

Sharp: $Irrep_k \oplus Irrep_k$, $\dim = 2k$.

Quantum torus

Theorem (Ioos-Kazhdan-P., in progress)

For every $c \in \mathbb{R}$ and $r > 0$, there exist $k_0 \in \mathbb{N}$ and $C > 0$ such that the following holds. Assume that for $k \in \mathbb{N}$ with $k \geq k_0$ there exist a finite-dimensional Hilbert space H with $\dim H < 2(k + c)$, and a pair of operators $x_1, x_2 \in End(H)$ such that

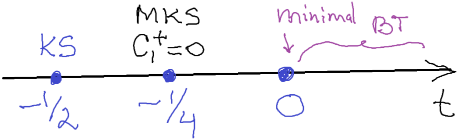
- $\left\| x_j x_j^* - \mathbb{1} \right\|_{op} \leq r/k^3 \quad$ *for all* $\quad j = 1, 2;$
- $\left\| \, x_1 x_2 - e^{2i\pi/(k+c)} x_2 x_1 \, \right\|_{op} \leq r/k^3.$

Then $c \in \mathbb{Z}$, $\dim H = k + c$, and there exist unitary operators $X_1, X_2 \in End(H)$, satisfying $X_1 X_2 = e^{2\pi i/(k+c)} X_2 X_1$ such that $\|x_j - X_j\|_{op} \leq C/k^{3/2} \quad$ for all $\quad j = 1, 2.$

Zoo of equivariant quantizations of S^2

$\dim H_k = k + m$, $m \in \mathbb{Z}$-fixed.
Consider quantizations
$T_{k+m-1}(e^{-t\hbar\Delta}f)$, $t \geq 0$ (Berezin-Topelitz and diffusion),

$T_{k+m-1}(f - t\hbar\Delta f)$, $t \geq 0$ ("anti-diffusion", non-positive).



$t = -1/4$- metaplectic Kostant-Souriau.
$c_1^+ = 0 \Rightarrow$ best quasi-multiplicativity
For $m = 0$ have $c_2^- = 0$, best bracket correspondence.

$t = -1/2$, Kostant-Souriau, best norm correspondence

The End

THANK YOU!