

On the spectral problem of a three term difference operator

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Motivation: topological strings

Topological String/Spectral Theory (TS/ST) correspondence

$$\text{Toric CY 3-fold } M \xrightarrow{\text{Mirror Symmetry}} \rho_M \quad (\text{trace class operator})$$

The spectrum of ρ_M is expected to be related to enumerative invariants of M through the topological string partition functions. Suggested by [Aganagic–Dijkgraaf–Klemm–Mariño–Vafa \(2006\)](#) and materialized by [Grassi–Hatsuda–Mariño \(2016\)](#).

Example: the local \mathbb{P}^2

$$\rho_{\mathbb{P}^2}^{-1} = \mathbf{u} + \mathbf{v} + e^{i\frac{\hbar}{2}} \mathbf{v}^{-1} \mathbf{u}^{-1}$$

with [positive self-adjoint](#) operators \mathbf{u} and \mathbf{v} in a (separable) Hilbert space satisfying the Heisenberg–Weyl commutation relation

$$\mathbf{u}\mathbf{v} = e^{i\hbar} \mathbf{v}\mathbf{u}, \quad \hbar \in \mathbb{R}_{>0}.$$

Implications of the TS/ST correspondence

Fredholm determinant

$$\det(1 + \kappa \rho_M) = 1 + \sum_{N=1}^{\infty} Z(N, \hbar) \kappa^N \quad (\text{convergent series})$$

where the **fermionic spectral traces** $Z(N, \hbar) = e^{F(N, \hbar)}$ provide a non-perturbative definition of the topological string partition functions.

$$\hbar = \lambda N, \quad N \rightarrow \infty, \quad (\text{t'Hooft limit})$$

$$F(N, \lambda N) \simeq \sum_{g=0}^{\infty} \mathcal{F}_g(\lambda) (\lambda N)^{2-2g} \quad (\text{asymptotic series})$$

with the standard topological string genus g free energies $\mathcal{F}_g(\lambda)$ in the conifold frame where λ is a flat coordinate for the CY moduli space vanishing at the conifold point.

The spectral problem

Define a positive self-adjoint operator in $L^2(\mathbb{R})$

$$\mathbf{H} := e^{2\pi b \mathbf{p}} + e^{2\pi b \mathbf{x}} + e^{-2\pi b(\mathbf{p} + \mathbf{x})}, \quad b := \sqrt{\frac{\hbar}{2\pi}} \in \mathbb{R}_{>0}$$

with normalized Heisenberg's position and momentum operators

$$\langle x | \mathbf{x} = x \langle x |, \quad \langle x | \mathbf{p} = \frac{1}{2\pi i} \frac{\partial}{\partial x} \langle x |, \quad [\mathbf{p}, \mathbf{x}] = (2\pi i)^{-1}$$

Small \hbar limit

$$\mathbf{H} = 3 + \hbar\sqrt{3} \left(\mathbf{a}^* \mathbf{a} + \frac{1}{2} \right) + \mathcal{O}(\hbar^{3/2}),$$

$$\mathbf{a} := \frac{\sqrt{2\pi}}{\sqrt[4]{3}} \left(\mathbf{x} + \mathbf{p} e^{\frac{\pi i}{3}} \right), \quad [\mathbf{a}, \mathbf{a}^*] = 1$$

Power series expansion of eigenvalues $E_n(\hbar) = \sum_{k=0}^{\infty} E_{n,k} \hbar^{\frac{k}{2}},$

$$E_{n,0} = 3, \quad E_{n,1} = 0, \quad E_{n,2} = \sqrt{3} \left(n + \frac{1}{2} \right), \dots$$

Strongly coupled regime $|\hbar| = 2\pi$

Recall the parametrisation $\hbar = 2\pi b^2$.

$$b = e^{i\theta}, \quad 0 < \theta < \frac{\pi}{2} \Rightarrow \text{formally } [\mathbf{H}, \mathbf{H}^*] = 0 \text{ (Normal operator).}$$

Lemma

Let $\{\mathbf{a}_i\}_{i \in I}$ be a finite set of bounded operators such that $\mathbf{a}_i + \mathbf{a}_j$ is normal for any $i, j \in I$. Then $\mathbf{H} := \sum_{i \in I} \mathbf{a}_i$ is normal.

Proof. One deduces that $[\mathbf{a}_i, \mathbf{a}_j^*] = -[\mathbf{a}_j, \mathbf{a}_i^*] \quad \forall i, j \in I$. Thus,

$$[\mathbf{H}, \mathbf{H}^*] = \sum_{i, j \in I} [\mathbf{a}_i, \mathbf{a}_j^*] = - \sum_{i, j \in I} [\mathbf{a}_j, \mathbf{a}_i^*] = -[\mathbf{H}, \mathbf{H}^*] \quad \square$$

In our case $\mathbf{a}_1 = e^{2\pi b \mathbf{p}}$, $\mathbf{a}_2 = e^{2\pi b \mathbf{x}}$, $\mathbf{a}_3 = e^{-2\pi b(\mathbf{p} + \mathbf{x})}$ and

$$\mathbf{a}_1 + \mathbf{a}_2 = \mathbf{U} e^{2\pi b \mathbf{x}} \mathbf{U}^*, \quad \mathbf{U} := \Phi_b(\mathbf{p} - \mathbf{x}), \quad \Phi_b(x) := \frac{(-q e^{2\pi b x}; q^2)_\infty}{(-\bar{q} e^{2\pi b^{-1} x}; \bar{q}^2)_\infty}$$

$q := e^{\pi i b^2}$, $\bar{q} := e^{-\pi i b^{-2}}$, and similarly for two other pairs.

Principle of F-duality

The common spectral problem for \mathbf{H} and \mathbf{H}^* is equivalent to constructing an element $\langle x|\Psi\rangle := \Psi(x) \in L^2(\mathbb{R})$ admitting analytic continuation to a domain in \mathbb{C} containing the strip $|\Im z| < \cos \theta$ and satisfying two difference equations

$$\begin{aligned}\Psi(x - ib) + e^{-\pi ib^2} e^{-2\pi bx} \Psi(x + ib) &= (E - e^{2\pi bx}) \Psi(x), \\ \Psi(x - ib^{-1}) + e^{-\pi ib^{-2}} e^{-2\pi b^{-1}x} \Psi(x + ib^{-1}) &= (\bar{E} - e^{2\pi b^{-1}x}) \Psi(x)\end{aligned}$$

related to each other by the substitutions

$$(b, E) \leftrightarrow (b^{-1}, \bar{E}) \text{ (Faddeev's modular duality=F-duality)}.$$

In the general case of Baxter's TQ-equations, an approach for constructing solutions in the strongly coupled regime is suggested by [Sergeev \(2005\)](#).

A different approach through auxiliary non-linear integral equations is developed by [Babelon–Kozłowski–Pasquier \(2018\)](#).

F-dual asymptotics at $x \rightarrow +\infty$

There are two possibilities

$$\Psi(x)|_{x \rightarrow +\infty} \sim \psi_k(x) := e^{\pi i(1-3k)x^2 - 2\pi x \cos \theta}, \quad k \in \{0, 1\},$$

with exact solutions $\Psi_k(x) = \psi_k(x)\varphi_k(x)$,

$$\begin{aligned} \varphi_k(x - \epsilon_k) + e^{-(2x + \epsilon_k)3\pi b} \varphi_k(x + \epsilon_k) &= (1 - E e^{-2\pi b x}) \varphi_k(x), \\ \epsilon_k &:= (1 - 2k)ib, \end{aligned}$$

+ the F-dual equations $(b, E) \mapsto (b^{-1}, \bar{E})$ and the boundary conditions

$$\lim_{x \rightarrow +\infty} \varphi_k(x) = 1.$$

The factorisation Ansatz

The F-dual substitutions

$$\varphi_k(x) = \chi_k(e^{2\pi b x}) \bar{\chi}_k(e^{2\pi b^{-1} x}), \quad k \in \{0, 1\},$$

give rise to power series solutions ($q := e^{\pi i b^2}$)

$$\chi_k(z) = \phi_{q^{2k-1}, E}(1/z), \quad \phi_{q, E}(z) := \sum_{n=0}^{\infty} \frac{p_n(q, E)}{(q^{-2}; q^{-2})_n} z^n,$$
$$\bar{\chi}_k = \chi_k|_{(q, E) \mapsto (\bar{q}^{-1}, \bar{E})}$$

with the polynomials $p_n = p_n(q, E) \in \mathbb{Z}[q, q^{-1}][E]$ of degree n in E defined by

$$p_{n+1} = E p_n + (q^n - q^{-n})(q^{n-1} - q^{1-n}) p_{n-2}, \quad p_0 = 1.$$

$p_n(q, E)|_{n \rightarrow \infty} \sim q^{-n^2/3} \Rightarrow RC(\phi_{q, E}(z)) = \infty$ (radius of convergence) and $RC(\phi_{1/q, E}(z)) = 0$.

The vector spaces $F_{p,c}$, $V_{p,\alpha,c}$, $T_{p,r}^m$

Let $\mathcal{O}_{\mathbb{C}_{\neq 0}}$ be the \mathbb{C} -vector space of holomorphic maps $f: \mathbb{C}_{\neq 0} \rightarrow \mathbb{C}$. For $c \in \mathbb{C}$, $p, r, \alpha \in \mathbb{C}_{\neq 0}$ and $m \in \mathbb{Z}$, define the following vector subspaces of $\mathcal{O}_{\mathbb{C}_{\neq 0}}$

- $F_{p,c} := \{f \mid f(z/p^2) + (zp)^3 f(zp^2) = (1 - cz)f(z)\};$
- $V_{p,\alpha,c} := \{f \mid \alpha z f(z/p^2) + z^2 p \alpha^{-1} f(zp^2) = (1 - cz)f(z)\};$
- $T_{p,r}^m := \{f \mid rz^m f(zp) = f(z)\}.$

Lemma

- (i) $|p|^m < 1 \Rightarrow \dim(T_{p,r}^m) = |m|$ (θ -functions of order $|m|$);
- (ii) $\dim(V_{q,\alpha,E}) = 1$;
- (iii) the multiplication of functions induces a linear map $V_{q,\alpha,E} \otimes T_{q^2,q^2\alpha}^1 \rightarrow F_{q,E}$;
- (iv) $\dim(F_{q,E}) = 3$ and $\phi_{q,E} \in F_{q,E}$.

Proof of (ii) $\dim(V_{q,\alpha,E}) = 1$

Consider a linear map

$$A: \mathcal{O}_{\mathbb{C} \setminus \{0\}} \rightarrow \mathcal{O}_{\mathbb{C} \setminus \{0\}}, \quad (Af)(z) = P_+(f\psi_{q,E})(1/\sqrt{-z})$$

where

$$\psi_{q,E}(z) := \sum_{n=0}^{\infty} \frac{p_n(q,E)q^{(1-n)n/2}}{(q^{-2}; q^{-2})_n} z^n = \psi_{1/q,E}(-z)$$

(∞ radius of convergence) and P_+ is the projection to the even part of a function:

$$P_+(f)(z) = (f(z) + f(-z))/2.$$

Then, the restriction $A|_{T_{q,-\alpha}^1}$ is a linear isomorphism between $T_{q,-\alpha}^1$ and $V_{q,\alpha,E}$.

First order matrix difference equation for $F_{p,c}$

For any $f \in F_{p,c}$, we have

$$\hat{f}(z) = L(z)\hat{f}(zp^2), \quad \hat{f}(z) := \begin{pmatrix} f\left(\frac{z}{p^2}\right) \\ f(z) \end{pmatrix}, \quad L(z) := \begin{pmatrix} 1 - cz & -z^3 p^3 \\ 1 & 0 \end{pmatrix}.$$

Defining

$$L_n(z) := L(z)L(zp^2) \cdots L(zp^{2n-2}) =: \begin{pmatrix} a_n(z) & b_n(z) \\ c_n(z) & d_n(z) \end{pmatrix}, \quad n \in \mathbb{Z}_{>0},$$

we have

$$L_{m+n}(z) = L_m(z)L_n(zp^{2m}), \quad \forall m, n \in \mathbb{Z}_{>0},$$

in particular,

$$L_{n+1}(z) = L(z)L_n(zp^2) = L_n(z)L(zp^{2n}), \quad \forall n \in \mathbb{Z}_{>0}.$$

Assuming $|p| < 1$ and taking the limit $n \rightarrow \infty$,

$$L_\infty(z) := \lim_{n \rightarrow \infty} L_n(z) = \begin{pmatrix} \phi_{p,c}(z/p^2) & 0 \\ \phi_{p,c}(z) & 0 \end{pmatrix}.$$

Adjoint functions

Define the (skew-symmetric bilinear) *Wronskian pairing*

$$[\cdot, \cdot]: F_{q,E} \times F_{q,E} \rightarrow T_{q^2, q^3}^3, \quad [f, g](z) = f\left(\frac{z}{q^2}\right)g(z) - g\left(\frac{z}{q^2}\right)f(z),$$

and the *adjoint function* $\tilde{f}: U([\phi_{q,E}, f]) \rightarrow \mathbb{C}$

$$\tilde{f}(z) := \frac{f(z)}{[\phi_{q,E}, f](z)}, \quad \forall f \in F_{q,E}, \quad U(g) := \mathbb{C}_{\neq 0} \setminus g^{-1}(0).$$

Adjoint functions are analytic substitutes for the series $\phi_{1/q,E}(z)$.

Theorem

Let $f \in F_{q,E}$ be such that $U([\phi_{q,E}, f]) \neq \emptyset$. Then

$$z \in U([\phi_{q,E}, f]) \Rightarrow zq^{2\mathbb{Z}} \subset U([\phi_{q,E}, f]), \quad \lim_{n \rightarrow \infty} \tilde{f}(zq^{2n}) = 1,$$

and $\tilde{f}(z)$ admits an asymptotic expansion at small z in the form of the series $\phi_{1/q,E}(z)$.

The general Ansatz for $\Psi(x)$

$\Psi(x) = \Psi_0(x) + \xi \Psi_1(x)$ where $\xi \in \mathbb{C}$ and

$$\Psi_0(x) := \psi_0(x) \tilde{h}(e^{-2\pi b x}) \phi_{\bar{q}, \bar{E}}(e^{-2\pi b^{-1} x}), \quad h \in F_{q, E} \setminus \mathbb{C} \phi_{q, E},$$

$$\Psi_1(x) := \psi_1(x) \phi_{q, E}(e^{-2\pi b x}) \tilde{\bar{h}}(e^{-2\pi b^{-1} x}), \quad \bar{h} \in F_{\bar{q}, \bar{E}} \setminus \mathbb{C} \phi_{\bar{q}, \bar{E}},$$

sharing a common pole set (*Requirement(I)*).

Then, for any $\zeta, \sigma \in \mathbb{C}$, there exist a (multivalued) function

$E = E(q, \zeta, \sigma)$ and elements $f \in V_{q, q e^{-2\pi b \zeta}, E}$, $\bar{f} \in V_{\bar{q}, \bar{q} e^{-2\pi b^{-1} \zeta}, \bar{E}}$

with $\bar{E} := E|_{q \mapsto \bar{q}}$ such that

$$\Psi(x) = e^{-2\pi x \cos \theta} \frac{e^{\pi i x^2} f(z) \phi_{\bar{q}, \bar{E}}(\bar{z}) + \xi e^{-2\pi i(\zeta + 2 \sin \theta)x} \bar{f}(\bar{z}) \phi_{q, E}(z)}{\vartheta(z/s; q^2) \vartheta(zs e^{2\pi b \zeta}; q^2)}$$

where $z := e^{-2\pi b x}$, $\bar{z} := e^{-2\pi b^{-1} x}$, $s := e^{-2\pi b \sigma}$, $\bar{s} := e^{-2\pi b^{-1} \sigma}$,

$$\vartheta(u; p) := \sum_{n \in \mathbb{Z}} p^{(n-1)n/2} (-u)^n = (u, p/u, p; p)_{\infty}.$$

Theorem

Under the substitution

$$\xi = \xi(\theta, \zeta, \sigma) := -e^{\pi i \sigma(\sigma + 2\zeta)} \frac{f(s)\phi_{\bar{q}, \bar{E}}(\bar{s})\bar{s}}{\bar{f}(\bar{s})\phi_{q, E}(s)s}.$$

*all the poles of $\Psi(x)$ at $x = \sigma + ibm + ibn$, $m, n \in \mathbb{Z}$, are cancelled.
Furthermore, the equation*

$$\xi(\zeta, \theta, \sigma) = \xi(\zeta, \theta, \zeta - \sigma)$$

*ensures that all the remaining poles of $\Psi(x)$ are cancelled as well
([Requirement\(II\)](#)).*