Local and Global Inverse and Implicit Function Theorems

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Introduction

Let X, Y be Banach spaces, Σ be a topological space, $f : X \times \Sigma \to Y$ be a given mapping. Consider the equation

$$f(x, \sigma) = 0 \tag{1}$$

with the unknown $x \in X$ and the parameter $\sigma \in \Sigma$.

The classical implicit function theorem states that *if f is sufficiently smooth, $f(x_0, \sigma_0) = 0$ for certain $x_0 \in X$, $\sigma_0 \in \Sigma$ and the normality condition holds at the point (x_0, σ_0), i.e.*

$$\frac{\partial f}{\partial x}(x_0, \sigma_0)X = Y,$$

then there exists a neighbourhood O of the point σ_0 and a continuous mapping $g : O \to X$ such that $f(g(\sigma), \sigma) = 0 \quad \forall\, \sigma \in O, \quad g(\sigma_0) = x_0.$

Consider the following two questions:

1) If the normality assumption fails, then under what conditions does there exist an implicit function g?

2) Under what conditions there exists an implicit function g, defined not on a certain neighbourhood but on the entire space Σ?

Implicit function theorem in a neighborhood of an abnormal point

Let X, Y be Banach spaces, Σ be a topological space, $K \subseteq X$ be a closed convex cone, $f : X \times \Sigma \to Y$ be a given mapping. Consider the equation

$$f(x, \sigma) = 0, \quad x \in K \tag{2}$$

with the unknown $x \in X$ and the parameter $\sigma \in \Sigma$.

Let $x_0 \in K$, $\sigma_0 \in \Sigma$ be given, $f(x_0, \sigma_0) = 0$.

If f is sufficiently smooth, then Robinson's stability theorem implies that if

$$\frac{\partial f}{\partial x}(x_0, \sigma_0)\Big(K + \mathrm{span}\{x_0\}\Big) = Y, \tag{3}$$

((3) coincide with the normality condition when $K = X$), then there exists a neighborhood O of σ_0 such that $\forall\, \sigma \in O \quad \exists\, g(\sigma) \in K$:

$$f(g(\sigma), \sigma) \equiv 0, \quad \|g(\sigma) - x_0\| \le \mathrm{const}\|f(x_0, \sigma)\| \quad \forall\, \sigma \in O. \tag{4}$$

Moreover, it can be proved that the function $g(\cdot)$ can be chosen to be continuous.

Let us turn to implicit function theorems without a priori assumption of Robinson's conditions.

Assume that

- in a neighbourhood of (x_0, σ_0), the mapping f is twice continuously differentiable in x uniformly in σ, for each σ the mapping $\frac{\partial^2 f}{\partial x^2}(\cdot, \sigma)$, whose values are symmetric bilinear mappings, satisfies the Lipschitz condition in x and the Lipschitz constant does not depend on σ;
- the mappings $f(x_0, \cdot)$, $\frac{\partial f}{\partial x}(x_0, \cdot)$, $\frac{\partial^2 f}{\partial x^2}(x_0, \cdot)$ are continuous in a neighbourhood of σ_0.

Put

$$\mathcal{K} := K + \operatorname{span}\{x_0\}, \quad C := \frac{\partial f}{\partial x}(x_0, \sigma_0)(\mathcal{K}).$$

Assume that

- $\operatorname{span} C$ is closed and topologically complemented.

Let π be a continuous linear operator, that projects Y onto a topological complement of $\operatorname{span} C$.

Definition. Let

$$h \in \mathcal{K}, \quad \frac{\partial f}{\partial x}(x_0, \sigma_0)h = 0, \quad -\frac{\partial^2 f}{\partial x^2}(x_0, \sigma_0)[h, h] \in C. \tag{5}$$

The mapping f is said to be 2-regular at (x_0, σ_0) with respect to K in the direction h if

$$\frac{\partial f}{\partial x}(x_0, \sigma_0)\mathcal{K} + \frac{\partial^2 f}{\partial x^2}(x_0, \sigma_0)\left[h,\ \mathcal{K} \cap \ker \frac{\partial f}{\partial x}(x_0, \sigma_0)\right] = Y. \tag{6}$$

Theorem 1. *Let* $\mathrm{ri} C \neq \emptyset$ *and assume that there exists an* $h \in X$ *such that the mapping* f *is 2-regular at* (x_0, σ_0) *with respect to* K *in a direction* h*. Then for each* $l \in \mathrm{ri} C$ *there exists a neighbourhood* O *of the point* σ_0*,* $\delta > 0$*,* $c > 0$ *and a continuous mapping* $g : O \to K$ *such that*

$$f(g(\sigma), \sigma) = 0, \tag{7}$$

$$\|g(\sigma) - x_0\| \leq c\Big(\Delta_1(\sigma) + \Delta_2(\sigma) + \|f(x_0, \sigma)\| + \rho(-f(x_0, \sigma), C_\delta)^{1/2}\Big) \tag{8}$$

for all $\sigma \in O$*.*

Here,

$$\Delta_1(\sigma) = \sup\left\{ \left\| \pi \frac{\partial f}{\partial x}(x_0, \sigma) x \right\| : \; x \in \operatorname{span} K, \; \|x\| \leq 1 \right\},$$

$$\Delta_2(\sigma) = \sup\left\{ \left\| \frac{\partial f}{\partial x}(x_0, \sigma) x \right\| : \; x \in \ker \frac{\partial f}{\partial x}(x_0, \sigma_0) \cap \mathcal{K}, \; \|x\| \leq 1 \right\},$$

$C_\delta = \operatorname{cone}(B_\delta(l)) \cap \operatorname{span} C$, where cone stands for the conical hull.

Under the additional assumption that

$$\operatorname{cone} \; \frac{\partial f}{\partial x}(x_0, \sigma_0)(K) \;\; \text{is a subspace} \tag{9}$$

in the estimate (8), the term $\Delta_2(\sigma)$ can be omitted.

Consider Problem (2) without the constraint $x \in K$.

Corollary 1. *Let $K = X$ and all assumptions of Theorem 1 hold. Then there exists a neighbourhood O of the point σ_0, $c > 0$ and a continuous mapping $g : O \to X$ such that (7) and*

$$\|g(\sigma) - x_0\| \leq c \left(\|f(x_0, \sigma)\| + \|\pi f(x_0, \sigma)\|^{1/2} \right) \quad \forall\, \sigma \in O. \tag{10}$$

The classical implicit function theorem follows from Corollary 1.

Theorem 1 is valid only under the assumption of non-emptiness of the cone $\mathrm{ri}C$. However, if the space Y is infinite-dimensional, then the relative interior of the convex cone C can be empty. Let us present an implicit function theorem which is valid without the a priori assumption that $\mathrm{ri}C$ is non-empty.

Theorem 2. *Let the mapping f be 2-regular at (x_0, σ_0) with respect to K in a direction $h \in X$. Then there exists a neighbourhood O of the point σ_0, $c > 0$ and a continuous mapping $g : O \to K$ such that (7) holds and*

$$\|g(\sigma) - x_0\| \leq c\left(\Delta_1(\sigma) + \Delta_2(\sigma) + \|f(x_0, \sigma)\|^{1/2}\right) \quad \forall\, \sigma \in O.$$

Global implicit function theorems

As before, X and Y are Banach spaces, Σ is a topological space, $f : X \times \Sigma \to Y$ is a given mapping. Consider the equation

$$f(x, \sigma) = 0 \tag{1}$$

with the unknown $x \in X$ and the parameter $\sigma \in \Sigma$. The continuous function $g(\cdot)$, defined on the entire space Σ and satisfying the identity $f(g(\sigma), \sigma) \equiv 0$ is called the *global implicit function.*

The global inverse function theorem goes back to J. Hadamard who proved that *if $X = Y = \mathbb{R}^n$, a continuously differentiable mapping $F : X \to Y$ is uniformly regular, i.e. the linear operator $\dfrac{\partial F}{\partial x}(x)$ is invertible for each $x \in X$ and the function $\left\| \dfrac{\partial F}{\partial x}(x)^{-1} \right\|$ is bounded on X, then F is a diffeomorphism.*

If $X = \mathbb{R}^n$, $Y = \mathbb{R}^k$ and $k < n$ then existence of a diffeomorphism is out of question. However, if F is uniformly regular in a certain sense, then a natural question arises, is it possible to prove the existence of a smooth or continuous right inverse mapping $G : Y \to X$ (i.e. $F(G(y)) \equiv y$).

Let us formulate a global implicit function theorem in the case, when X and Y are Hilbert spaces without a priori assumption that $X = Y$.

Notation:

- $\mathcal{L}(X,Y)$ — space of linear bounded operators $A : X \to Y$;
- $\mathcal{SL}(X,Y) := \{A \in \mathcal{L}(X,Y) :\ AX = Y\}$;
- $B(x,r)$ — closed ball in space X of radius $r \geq 0$ with centre at the point $x \in X$ (the same notation is used for balls in Y).

For a linear operator $A \in \mathcal{L}(X,Y)$ put

$$\mathrm{cov}A := \sup\{\alpha \geq 0 :\ B(0,\alpha) \subset AB(0,1)\}.$$

Banach's open mapping theorem states that $\mathrm{cov}A > 0 \Leftrightarrow A \in \mathcal{SL}(X,Y)$.

Assume that the mapping $f(\cdot,\sigma)$ is differentiable for each $\sigma \in \Sigma$. For arbitrary continuous function $\varphi : \Sigma \to X$ and $t \geq 0$ put

$$\alpha_\varphi(t) := \inf\left\{\mathrm{cov}\frac{\partial f}{\partial x}(x,\sigma) :\ x \in B\big(\varphi(\sigma),t\big), \quad \sigma \in \Sigma\right\}.$$

Theorem 3. *Let X and Y be Hilbert spaces, the mapping $f(\cdot,\sigma)$ be twice continuously differentiable for every $\sigma \in \Sigma$, the mappings f, $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial^2 f}{\partial x^2}$ be continuous.*

Then for any continuous function $\varphi : \Sigma \to X$ for which

$$\int\limits_0^{+\infty} \alpha_\varphi(t)\,dt = +\infty \quad \text{or} \quad \sup_{\sigma\in\Sigma} \|f(\varphi(\sigma),\sigma)\| < \int\limits_0^{+\infty} \alpha_\varphi(t)\,dt, \tag{11}$$

there exists a continuous function $g = g_\varphi : \Sigma \to X$ such that

$$f(g(\sigma),\sigma) = 0, \qquad \int\limits_0^{\|g(\sigma)-\varphi(\sigma)\|} \alpha_\varphi(t)\,dt \le \|f(\varphi(\sigma),\sigma)\| \quad \forall\,\sigma \in \Sigma. \tag{12}$$

Note that, if

$$a := \inf\left\{ \operatorname{cov} \frac{\partial f}{\partial x}(x, \sigma) : \ x \in X, \quad \sigma \in \Sigma \right\} > 0, \tag{13}$$

then (11) holds automatically and inequality in (12) takes the form

$$\|g(\sigma) - \varphi(\sigma)\| \leq \frac{\|f(\varphi(\sigma), \sigma)\|}{a} \quad \forall \sigma \in \Sigma.$$

Below we present implicit function theorem for mappings of Banach spaces. In order to simplify the formulations we use the regularity assumption (13) instead of (11).

Let a continuous function $\pi : \Sigma \to \mathbb{R}_+$ be given. For $d > 0$ put

$$\Sigma(d) := \{\sigma \in \Sigma : \ \pi(\sigma) < d\}.$$

Assume that

(A1) $f(\cdot, \sigma)$ is differentiable for each $\sigma \in \Sigma$, f and $\dfrac{\partial f}{\partial x}$ are continuous.

Since f is differentiable in x, then for all $(x, \sigma) \in X \times \Sigma$, $\xi \in X$ we have

$$f(x + \xi, \sigma) = f(x, \sigma) + \frac{\partial f}{\partial x}(x, \sigma)\xi + o(x, \sigma; \xi).$$

Here $o : X \times \Sigma \times X \to Y$ is a mapping, for which

$$\forall\, (x, \sigma) \in X \times \Sigma, \quad \forall\, \varepsilon > 0 \quad \exists\, \delta > 0 : \quad \|o(x, \sigma; \xi)\| \leq \varepsilon \|\xi\| \quad \forall\, \xi \in B(0, \delta).$$

(A2) for all $d \geq 0$ the mapping f is uniformly differentiable in x in the following sense: $\forall\, \varepsilon > 0 \quad \exists\, \delta > 0 :$

$$\|o(x, \sigma; \xi)\| \leq \varepsilon \|\xi\| \quad \forall\, (x, \sigma) \in B(0, d) \times \Sigma(d), \quad \forall\, \xi \in B(0, \delta).$$

(A3) for all $d \geq 0$ the derivative of the mapping f with respect to x is uniformly bounded in the following sense:

$$\exists\, c = c(d) \geq 0 : \quad \left\| \frac{\partial f}{\partial x}(x, \sigma) \right\| \leq c \quad \forall\, (x, \sigma) \in B(0, d) \times \Sigma(d).$$

Note that if the spaces X and Y are finite dimensional, while Σ is compact, then **(A2)** and **(A3)** follow from **(A1)**.

Theorem 4. *Let*

$$a := \inf\left\{ \operatorname{cov}\frac{\partial f}{\partial x}(x, \sigma) : \ x \in X, \quad \sigma \in \Sigma \right\} > 0. \tag{14}$$

Then for any continuous function $\varphi : \Sigma \to X$ and for any $\gamma \in (0, a)$ there exists a continuous function $g = g_\varphi : \Sigma \to X$ such that

$$f(g(\sigma), \sigma) = 0, \quad \|g(\sigma) - \varphi(\sigma)\| \leq \frac{\|f(\varphi(\sigma), \sigma)\|}{\gamma} \quad \forall\, \sigma \in \Sigma. \tag{15}$$

In Theorem 4, the assumption (14) can be weakened by replacing it with the assumption (11). In this case, the proof of the theorem becomes quite complicated.

Theorem 4 implies the following global inverse function theorem.

Given a continuously differentiable mapping $F : X \to Y$, assume that the mapping $\dfrac{\partial F}{\partial x}$ is bounded on every bounded subset of X and F is uniformly differentiable, i.e.

$$\forall\, r > 0, \quad \forall\, \varepsilon > 0 \quad \exists\, \delta > 0 :$$
$$\left\| F(x+\xi) - F(x) - \frac{\partial F}{\partial x}(x)\xi \right\| \le \varepsilon\|\xi\| \quad \forall\, x \in B(0,r), \;\; \forall\, \xi \in B(0,\delta).$$

Corollary 2. *Let*

$$a := \inf\left\{ \operatorname{cov}\frac{\partial F}{\partial x}(x) \,:\; x \in X, \quad \sigma \in \Sigma \right\} > 0.$$

Then for every continuous function $\varphi : Y \to X$ and every $\gamma \in (0,a)$ there exists a continuous function $G = G_\varphi : Y \to X$ such that

$$F(G(y)) = y, \quad \|G(y) - \varphi(y)\| \le \frac{\|F(\varphi(y)) - y\|}{\gamma} \quad \forall\, y \in Y. \tag{16}$$

Corollary 2 guarantees the continuity of the inverse mapping, a smooth or Lipschitzian inverse mapping may not exist.

Hadamard's theorem obviously follows from Corollary 2.

Let us apply the global implicit function theorem to obtain an assertion on continuous extensions of implicit functions.

Theorem 5. *Let the topological space Σ be be Hausdorff and paracompact, the mapping f satisfy the assumptions* **(A1)–(A3)** *and (14).*

Then for any closed subset $C \subset \Sigma$ and any continuous function $\varphi : C \to X$ for which

$$f(\varphi(\sigma), \sigma) = 0 \quad \forall\, \sigma \in C,$$

there exists a continuous function $g : \Sigma \to X$ such that

$$g(\sigma) = \varphi(\sigma) \quad \forall\, \sigma \in C, \qquad f(g(\sigma), \sigma) = 0 \quad \forall\, \sigma \in \Sigma.$$

References

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Thank you for your attention!