Localizations and completions of groups and spaces

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Outline

Reminder of rational homotopy theory

Rational homotopy theory

Theorem (Whitehead). *For any map of* 1*-connected spaces*

$$f : X \to Y$$

the following conditions are equivalent:

- $\pi_*(X) \otimes \mathbb{Q} \to \pi_*(Y) \otimes \mathbb{Q}$ *is an isomorphism;*
- $H_*(X, \mathbb{Q}) \to H_*(Y, \mathbb{Q})$ *is an isomorphism.*

A map satisfying these conditions is called a **rational homotopy equivalence**.

Rational homotopy theory is a homotopy theory that works with 1-c. spaces "up to rational homotopy equivalences".

Rational homotopy theory

$\mathcal{H}_1$ – homotopy category of 1-c. spaces.

$$\mathcal{H}_1^{\mathbb{Q}} = \mathcal{H}_1[\{\text{rational equivalences}\}^{-1}]$$

is the **rational homotopy category**.

Theorem (Quillen'69). *The category $\mathcal{H}_1^{\mathbb{Q}}$ is equivalent to the following categories*

- *homotopy category of reduced dg-Lie algebras over $\mathbb{Q}$;*
- *homotopy category of 2-reduced commutative dg-coalgebras over $\mathbb{Q}$.*

Rational homotopy theory

Sullivan constructed a (contravariant) functor to the category of 2-reduced commutative dg-algebras

$$A_{PL} : (\mathcal{H}_1^{\mathbb{Q}})^{\mathrm{op}} \to \mathsf{Ho}(\text{2-red. comm. dg-alg.})$$

which is known as the Sullivan-de Rham functor.

It induces a equivalence of subcategories with appropriate finitness conditions.

Each 2-red. commutative dg-algebra has a “minimal model”. Rational homotopy theory become very algebraic and computable using these minimal models of Sullivan.

Rational homotopy theory

A 1-c. space X is **rational** if $\pi_*(X)$ is a $\mathbb{Q}$-vector space.

For any 1-c. space X there is a universal map to a rational space, which is a rational equivalence

$$X \longrightarrow X_{\mathbb{Q}}.$$

Theorem. *The rationalization of a space induces an equivalence between $\mathcal{H}_1^{\mathbb{Q}}$ and the full subcategory of $\mathcal{H}_1$ consisting of rational spaces.*

Rational homotopy theory is equivalent to the ordinary homotopy theory of rational spaces.

Completions

Bousfield-Kan completions of spaces

Bousfield-Kan completions

Bousfield and Kan constructed a generalization of the functor $X \mapsto X_{\mathbb{Q}}$ which is called Bousfield-Kan completion.

The generalization in two directions:

1. 1-c. spaces are replaced by all spaces;
2. the ring $\mathbb{Q}$ is replaced by any commutative ring R.

For a commutative ring R the Bousfield-Kan completion is a functor

$$R_\infty : \mathcal{H} \longrightarrow \mathcal{H},$$

together with a natural transformation

$$X \to R_\infty X.$$

Bousfield-Kan completion

$f : X \to Y$ is called R**-equivalence** if

$$H_*(X, R) \xrightarrow{\cong} H_*(Y, R).$$

Properties of R_∞:

1. $f : X \to Y$ is an R-equivalence iff

$$R_\infty X \xrightarrow{\cong} R_\infty Y$$

is a homotopy equivalence.
2. If X is 1-c., then $\mathbb{Q}_\infty X \cong X_{\mathbb{Q}}$
3. There is a spectral sequence

$$E \Rightarrow \pi_*(R_\infty X)$$

whose E_1 depends naturally on $H_*(X, R)$. This is a generalisation of the unstable Adams spectral sequence.

Bousfield-Kan completion

A space X is called R**-good**, if the map

$$X \longrightarrow R_\infty X$$

is an R-equivalence.

X is called R**-complete**, if the map is a homotopy equivalence.

A space X is R-good $\iff$ $R_\infty X$ is R-complete.

The homotopy category of R-good spaces localized by R-equivalences is equivalent to the category of R-complete spaces.

*Homotopy theory of R-**good** spaces "up to R-equivalences" is the ordinary homotopy theory of R-**complete** spaces.*

Bousfield-Kan completion

All 1-c. spaces are R-good. All nilpotent spaces are R-good.

If P is a perfect group, then BP is $\mathbb{Z}$-good and

$$\mathbb{Z}_\infty BP \cong BP^+.$$

If A is a ring, then $B\mathsf{GL}(A)$ is $\mathbb{Z}$-good and

$$\mathbb{Z}_\infty B\mathsf{GL}(A) \cong B\mathsf{GL}(A)^+.$$

In particular, $\pi_n(\mathbb{Z}_\infty B\mathsf{GL}(A)) = K_n(A)$ for $n \geq 1$.

Theorem(Bousfiled-Kan'72) *The infinite wedge of circles* $\overset{\infty}{\vee} S^1$ *is R-bad for any R.*

R-bad spaces

Theorem(Bousfiled-Kan'72) *The infinite wedge of circles* $\overset{\infty}{\vee} S^1$ *is R-bad for any R.*

Theorem(Bousfiled'77) *The wedge of two circles* $S^1 \vee S^1$ *is $\mathbb{Z}$-bad.*

Theorem(Bousfiled'92) *The wedge of two circles* $S^1 \vee S^1$ *is $\mathbb{Z}/p$-bad for any prime p.*

Until recently it was not known whether any finite CW-complex is $\mathbb{Q}$-good or not.

Theorem(-, Mikhailov'17) *The wedge of two circles* $S^1 \vee S^1$ *is $\mathbb{Q}$-bad.*

Completions

Completion of groups

Completion of groups

Assume $R \subseteq \mathbb{Q}$ or $R = \mathbb{Z}/m$.

R**-central extension** of G is a central extension

$$1 \to A \to E \to G \to 1$$

such that A is an R-module.

R**-nilpotent group of class** $\leq n$ is a group that can be obtained by n-fold R-central extension of the trivial group.

For any G there is a universal map to an R-nilponent group of class $\leq n$

$$G \to \nu_n^R(G)$$

Completion of groups

$\nu_n^{\mathbb{Z}}(G) = G/\gamma_{n+1}(G)$ is the quotient by $n+1$-st term of the lower central series.

$\nu_n^{\mathbb{Q}}(G) = \nu_n^{\mathbb{Z}}(G) \otimes \mathbb{Q}$ is the Malcev completion.

R-completion:

$$\hat{G}_R = \varprojlim \nu_n^R(G)$$

$\mathbb{Z}$-completion is known as the pro-nilponent completion.

If G is finitely generated, then $\hat{G}_{\mathbb{Z}/p}$ is isomorphic to the pro-p-completion.

B.-K. completion via completion of groups

There is an equivalence between homotopy categories of pointed connected spaces and simplicial groups.

$$\{\text{pointed connected spaces}\} \sim \{\text{simplicial groups}\}$$

The functor R_∞ can be described using the R-completion of groups and this equivalence.

$$\begin{tikzcd}
\{\text{pointed connected spaces}\} \arrow[r, leftrightsquigarrow] \arrow[d, "R_\infty"] & \{\text{simplicial groups}\} \arrow[d, "R\text{-compl.}"] \\
\{\text{pointed connected spaces}\} \arrow[r, leftrightsquigarrow] & \{\text{simplicial groups}\}
\end{tikzcd}$$

Completion of groups

The fact that $S^1 \vee S^1$ is $\mathbb{Q}$-bad follows from the following theorem.

Theorem (-, Mikhailov'17). *If F is a free group with ≥ 2 generators, then*

$$H_2(\hat{F}_{\mathbb{Q}}, \mathbb{Q}) \neq 0.$$

Moreover, $H_2(\hat{F}_{\mathbb{Q}}, \mathbb{Q})$ is uncountable.

Localizations

Localization of objects in categories

Localization of objects

Let $\mathcal{C}$ be a category and $\mathcal{W} \subseteq \mathsf{Mor}(\mathcal{C})$.

An object $L \in \mathcal{C}$ is **local** if

$$\begin{tikzcd} X \arrow[rr, "\forall w \in \mathcal{W}"] \arrow[dr, "\forall"'] & & Y \arrow[dl, dashed, "\exists !"] \\ & L & \end{tikzcd}$$

In other words,

$$w^* : \mathrm{Hom}_{\mathcal{C}}(Y, L) \to \mathrm{Hom}_{\mathcal{C}}(X, L)$$

is a bijection.

Localization of objects

A **localization** of an **object** $X \in \mathcal{C}$ is a morphism

$$X \xrightarrow{w} L,$$

where $w \in \mathcal{W}$ and L is local.

Localization is the universal map to a local object.

Assume that any object has a localization. Then there exists a functor and a natural map

$$\mathcal{L} : \mathcal{C} \longrightarrow \mathsf{Loc}(\mathcal{W}), \qquad X \longrightarrow \mathcal{L}X$$

which is a localization of X.

Localizations of objects

Theorem. If any object has a localization, then the localization functor $\mathcal{L}$ induces an equivalence

$$\mathcal{C}[\mathcal{W}^{-1}] \simeq \mathsf{Loc}(\mathcal{W}).$$

Theory of objects of $\mathcal{C}$ "up to $\mathcal{W}$" is the theory of local objects.

Localizations of objects

Toy example. If we take $\mathcal{C} = \mathsf{Ab}$ and

$$\mathcal{W} = \{f : A \to B \mid f \otimes \mathbb{Q} \text{ is iso } \},$$

then local objects are $\mathbb{Q}$-vector spaces.

$$\mathsf{Loc}(\mathcal{W}) = \mathsf{Vect}(\mathbb{Q}).$$

The map

$$A \longrightarrow A \otimes \mathbb{Q}$$

is the localization.

$$- \otimes \mathbb{Q} : \mathsf{Ab} \to \mathsf{Vect}(\mathbb{Q})$$

is the localization functor.

Theory of abelian groups "up to $\mathcal{W}$" is the theory of $\mathbb{Q}$-vector spaces.

Localizations

Homological localization of spaces

Homological localization

R-homological localization

$$X \to L_R X$$

of a space X is the localization with respect to the class of R-equivalences.

Theorem (Bousfield'75). For any space there exists the R-localization.

The completion $R_\infty X$ is a R-local space. Hence, we have a map

$$L_R X \longrightarrow R_\infty X.$$

A space X is R-good if and only if the map is a homotopy equivalence.

Homological localization

Localization is better then completion because:

Homotopy theory of **all** *spaces "up to R-equivalences" is the ordynary homotopy theory of R*-**local** *spaces.*

$$\pi_1(L_R X) = ?$$

In order to answer this question we need the notion of HR-localization of groups

$$\pi_1(L_R X) = \ell_R(\pi_1(X)).$$

Localizations

HR-localization of groups

HR-localization of groups

A homomorphism $w : G_1 \to G_2$ is called **R-2-connected**, if

$$H_1(G_1,R) \xrightarrow{\cong} H_1(G_2,R) \;\; \text{is iso,}$$

$$H_2(G_1,R) \twoheadrightarrow H_2(G_2,R) \;\; \text{is epi.}$$

The localization with respect to R-2-connected homomorphisms is called **HR-localization**.

Theorem (Bousfield'75). For any group there exists an HR-localization

$$\ell_R : \mathsf{Gr} \longrightarrow \mathsf{Gr}.$$

Group theory "up to R-2-connected homomorphisms" is the group theory of HR-local groups.

Theorem (Bousfield'75). For any space X

$$\pi_1(L_R X) \cong \ell_R(\pi_1 X).$$

A problem of Bousfield

There is a natural map

$$\ell_R G \to \hat{G}_R.$$

Bousfield's problem: is it true that for any finitely presented group G and $R = \mathbb{Z}/p, \mathbb{Q}$ the map is an isomorphism?

Theorem (Mikhailov,-'14). For metabelian groups: "yes".

Theorem (Mikhailov,-'17). For the free group F with at least two generators: "no" in both cases $\mathbb{Q}$ and $\mathbb{Z}/p$.

In order to prove this we prove that $H_2(\hat{F}_{\mathbb{Q}}, \mathbb{Q}) \neq 0$ and $H_2(\hat{F}_{\mathbb{Z}/p}, \mathbb{Z}/p) \neq 0$.

Homology of pro-p-groups

Homology of pro-p-groups

A pro-p-group is a profinite group that can be presented as a limit of finite p-groups.

In the theory of profinite groups people usually consider continuous (co)homology.

Question (Fernandez-Alcober, Kazatchkov, Remeslennikov, Symonds): Is there a finitely presented pro-p-group $\mathcal{G}$ such that the comparison map

$$\varphi_2 : H^2_{\mathsf{cont}}(\mathcal{G}, \mathbb{Z}/p) \to H^2_{\mathsf{disc}}(\mathcal{G}, \mathbb{Z}/p)$$

is not an isomorphism?

We answer this question in the affirmative. Yes, there exists such a pro-p-group: the free pro-p-group.

Corollary of our Theorem. φ_2 is not an isomorphism for $\mathcal{G} = \hat{F}_{\mathbb{Z}/p}$.

Thank you for your attention!