Positive metric entropy in nearly integrable Hamiltonian systems

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Outline

Geometer's quick summary

This is a joint work with **Dmitri Burago** and **Dong Chen**: "An example of entropy non-expansive KAM-nondegenerate nearly integrable system" (2020), `arXiv.org:2009.11651`

- Main result: any Liouville integrable Hamiltonian system can be perturbed (in the class of Hamiltonian systems) to get a system with positive metric entropy and other interesting properties.
  The terms will be defined later.
- Hamiltonian systems include geodesic flows of Riemannian and Finsler metrics.
  Geodesics flows can be perturbed within the class of **Finsler** geodesic flows.
- Some examples of integrable systems: geodesic flows of the round metric on $\mathbb{S}^n$, flat metrics on $\mathbb{T}^n$ ($n \geq 2$), metrics of revolution on $\mathbb{T}^2$, etc.
- (Pesin, 1977) a geodesic flow has positive metric entropy $\iff$ there is a positive measure set of geodesics with exponentially growing Jacobi fields.

Outline

Hamiltonian systems

Hamiltonian system:

- A smooth manifold Ω^{2n} equipped with a non-degenerated closed 2-form ω (symplectic form). In suitable coordinates $q_1, \dots, q_n, p_1, \dots, q_n$ (Darboux coordinates), one has

$$\omega = dp_1 \wedge dq_1 + \dots dp_n \wedge dq_n.$$

  Main example: co-tangent bundle $\Omega = T^*M^n$

- A function $H\colon \Omega \to \mathbb{R}$ (Hamiltonian) defines Hamiltonian flow on Ω. The flow field X_H is the symplectic gradient of H: $\omega(\cdot, X_H) = dH$. In Darboux coordinates, the flow is given by equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

- For geodesic flows, $H(q,p) = \frac{1}{2}\|p\|_q^2$ $(p \in T_q^*M)$.
- The flow preserves H, ω, and the volume form ω^n.

Integrable systems

Integrable system = Liouville integrable Hamiltonian system = completely integrable Hamiltonian system.

The definition is technical. We only need the Liouville-Arnold theorem:

- In an integrable system (under some compactness assumptions), an open dense subset of Ω is foliated by invariant n-tori with quasi-periodic motion on them (Liouville tori).

- Moreover, in a neighborhood of each torus there is a Darboux coordinate system (action-angle coordinates) (q_i, p_i), $i = 1, \dots, n$, $q_i \in \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$, $p_i \in (-\varepsilon, \varepsilon)$, such that H depends only on p_i's: $H = H(p)$.

- The Liouville tori are $\{p = const\}$.
  On each of them, the motion has constant coordinate velocity $\dot{q}_i = \partial H/\partial p_i$, $\dot{p}_i = 0$.

KAM theory

We consider C^∞-small perturbations of H in an integrable system (Ω^{2n}, ω, H).
Perturbed integrable systems are vaguely referred to as nearly integrable systems.

Theorem (Kolmogorov-Arnold-Moser)

If the integrable system is KAM-non-degenerate (i.e. $(\partial H/\partial p_i)$ is a non-degenerate matrix) then a small perturbation of H preserves the invariant tori on a set of nearly full measure.

Question

What goes on between these tori?
How chaotic the perturbed system can be?

Some earlier results

- Arnold diffusion (1964): There are examples ($n \geq 3$) where a trajectory of a nearly integrable system starts near one of the KAM tori and travel far away from it.

  Note: This is not possible for $n = 2$ because the KAM tori divide the energy level into tiny components.

- (Newhouse, 1977) A C^2-generic Hamiltonian system has positive topological entropy.
- (Knieper-Weiss, 1994, Contreras, 2010) The geodesic flow of a C^2-generic Riemannian metric has a positive topological entropy.

Remark

Topological entropy $\geq$ metric entropy.

Positive topological entropy can live **on a set of zero measure** (usually on a Cantor-like set arising from a Smale horseshoe).

Outline

Metric entropy

- Metric entropy is also known as **measure-theoretic entropy** and **Kolmogorov-Sinai entropy**.
- It is an invariant $h_\mu(T)$ of a measure-preserving transformation $T\colon X \to X$ where $X = (X, \mu)$ is a probability space.
- For a system $\{\Phi^t\}$ with continuous time, the metric entropy is that of the time-1 map Φ^1.
- (Pesin entropy formula) If T is smooth, then

$$h(T) = \int_X \sum \lambda_i^+ \, d\mu$$

  where λ_i^+ are positive Lyapunov exponents ($\approx$ growth rate of $\|dT^k(x)\|$, $x \in X$, $k \to \infty$).
- (Corollary) $h_\mu(T) > 0$ iff there is a positive-measure set of $x \in X$ such that $\|dT^k(x)\|$ grows exponentially.

Our previous results

- (Burago-I., 2016) The standard metric of $\mathbb{S}^n$, $n \geq 4$, can be perturbed to a Finsler metric whose geodesic flow has positive metric entropy.

  Moreover, the perturbation can be made so that the resulting system is entropy-nonexpansive, i.e., a positive entropy is generated in an arbitrarily small invariant neighborhood of a periodic orbit.
- (Chen, 2017) The same, except non-expansiveness, holds for the standard metric on $\mathbb{T}^n$, $n \geq 3$.

Remark

The standard $\mathbb{S}^n$ is KAM-degenerate.
The standard $\mathbb{T}^n$ is non-degenerate, so KAM theorem applies.

Main Theorem

Theorem (Dmitri Burago, Dong Chen, I., 2020)

For every integrable system, near any Liouville torus, the Hamiltonian admits an arbitrarily C^∞-small perturbation such that the resulting system has positive metric entropy.

Notes:

- If the system is a geodesic flow, then the perturbation can be made in the class of Finslerian geodesic flows.
- The perturbation can be made so that the resulting system is entropy-nonexpansive.
- In the non-degenerate case, the perturbation can be made in a small neighborhood of one point.

Improvements:

- Dimension lowered to $n \geq 2$.
- Arbitrary integrable system.

Outline

Development of the proof

- The case of $\mathbb{S}^n$, $n \geq 4$ (Burago-I., 2016) was a by-product of a Finsler geometry result.

  Issues: (1) high dimension;
  (2) periodic and hence KAM-degenerate flow;
  (3) the method worked only for Hamiltonians arising from Finsler metrics.
- Lowering the dimension became possible after Berger-Turaev (2019) construction of symplectomorphisms of D^2 with positive metric entropy and close to the identity.
- Handling KAM-non-degenerate systems: construction of a perturbation featuring an invariant tube consisting of periodic orbits.
- Handling non-Finslerian Hamiltonians: found symplectic counterparts of all Finsler geometry construction

Plan of the proof

- Pick a periodic orbit $\gamma\colon [0, T] \to \Omega$ (if there is none, create one by perturbation).
- Pick at transverse section Σ^{2n-1} through $\gamma(0)$ and consider the Poincaré return map $R_H\colon \Sigma \to \Sigma$.
  It is a local diffeomorphism near $\gamma(0)$.
- Big step 1: Perturb R_H near $\gamma(0)$ so that the resulting map $\widetilde{R}\colon \Sigma \to \Sigma$ preserves a neighborhood U of $\gamma(0)$ and has positive metric entropy on U.

  In order for the next step to work, $\widetilde{R}$ must:
  (1) preserve level sets $H^{-1}(c) \cap \Sigma$ for all $c \in \mathbb{R}$;
  (2) preserve the symplectic form ω on each set $H^{-1}(c) \cap \Sigma$
- Big step 2: Given $\widetilde{R}$ satisfying (1) and (2), perturb H near the midpoint of γ so that the perturbed Hamiltonian $\widetilde{H}$ yields $\widetilde{R}$ as its Poincaré return map: $\widetilde{R} = R_{\widetilde{H}}$.

About Big Step 1

- The case $R = id$. E.g. the geodesic flow of $\mathbb{S}^2$. Use Berger-Turaev to construct $\widetilde{R}$.
- A non-degenerate example (a la $\mathbb{T}^2$): $R\colon \mathbb{R}^2 \to \mathbb{R}^2$,

$$R(q,p) = (q+p, p).$$

R is a time-1 map of the Hamiltonian flow with the Hamiltonian $H_0(p,q) = p^2/2$. Perturb H_0 to

$$\widetilde{H}_0(q,p) = p^2/2 + \varepsilon q^2/2, \qquad \varepsilon = (2\pi/N)^2$$

and let $\widetilde{R}$ be the time-1 map of the flow of $\widetilde{H}_0$.

System: $\dot{q} = p$, $\dot{p} = -\varepsilon q$.

Solution: $q(t) = c \sin \sqrt{\varepsilon} t$, $p(t) = c\sqrt{\varepsilon} \sin \sqrt{\varepsilon} t$

It is N-periodic. $\implies$ The problem reduces to the case $R = id$.

The general case is based on a similar idea

Outline

Lens maps

Goal: Prove Big Step 2 for Finsler geodesic flows.
Note: The problem is local.

Consider a simple Riemannian or Finsler disc $M = (D^n, F)$.
Simple means convex boundary and unique geodesics.

Notation: U_{in} and U_{out} are sets of unit tangent vectors at the boundary pointing inwards and outwards, resp.

Definition

Lens map of M is the map $\sigma_M \colon U_{in} \to U_{out}$ which send the incoming velocity vector of a geodesic to the outgoing vector of the same geodesic.

Our problem boils down to the following:

Question

Can a perturbation of the lens map can be realized by perturbations of the metric?

It suffices to consider perturbations supported near one vector from U_{in}.

Boundary distance

Definition

Boundary distance of $M = (D^n, F)$ is the map $bd_M : \partial M \times \partial M \to \mathbb{R}_+$ defined as follows: For $x, y \in \partial M$, $bd_M(x, y)$ is the length of the geodesic $[xy]$ in M.

Observations

For simple metrics, lens map and boundary distance function determine each other:

- The derivative of bd_M at (x, y) is determined by velocity of the geodesic $[xy]$ at x and y.
- σ_M is explicitly reconstructed from the first derivative of bd_M.
- bd_M is reconstructed from σ_M by integration. For this σ_M has to satisfy an integrability condition. This condition is equivalent to preservation of the symplectic form.

Realization of boundary distance

Let $M = (D^n, F)$ is a simple Finslerian disc.

Theorem (Burago-I., 2016)

- *For $n \geq 3$, any sufficiently small perturbation of bd_M supported away from the diagonal, can be realized by a pertubation of the Finsler metric in the interior.*
- *For $n = 2$, an additional condition is required: some cocycle must vanish.*
- *This condition is always satisfied if the perturbation is supported in a small neighborhood of a pair of points $x_0, y_0 \in bd_M$.*

Corollary: In our context, any symplectic perturbation of the lens map can be realized by a perturbation of the metric in the interior.

This proves Big Step 2 for Finsler geodesic flows.

Idea of proof of the boundary distance theorem

Plan of the proof:

- Consider the family of distance functions $F_x \colon M \to \mathbb{R}_+$, $x \in \partial M$, where $F_x(y) = d_M(x,y)$. This family uniquely determines the metric.
- Moreover, a sufficiently C^3-small perturbation of this family corresponds to a perturbation of the metric.
- Make a perturbation of $\{F_x\}$ to realize the prescribed restriction to the boundary and that's it (almost).
- One has to be careful not to change the metric near the boundary. This is doable.