Foliations of 3-manifolds with small module of mean curvature

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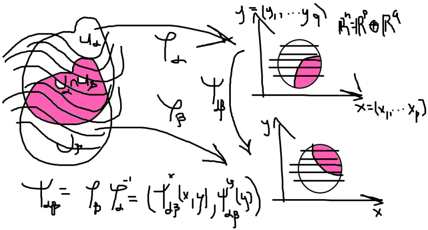
Foliation

Definition

For a smooth manifold M the foliation $\mathcal{F}$ of codimension q is a decomposition of M into immersed submanifolds $\{L_\alpha\}$ of codimension q that locally can be expressed as fibers of a submersion.

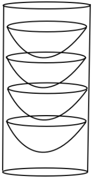


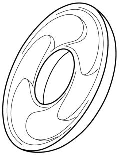
Everywhere we will assume $\dim M = 3$ and $co\dim \mathcal{F} = 1$. Also we suppose M is oriented and $\mathcal{F}$ is transversely oriented. In particular there is a transverse (or orthogonal if M is Riemannian) to $\mathcal{F}$ nondegenerate vector field.

Reeb component $\mathcal{R}$

The universal covering of Reeb foliation $\widetilde{\mathcal{R}}$ is a foliation of $D^2 \times \mathbb{R} : |z| \leq 1,\ z \in \mathbb{C}$, by the leaves:

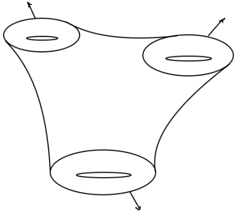
$$f_c = \frac{1}{1 - |z|^2} + c; |z| = 1$$





$M^3 = \mathcal{R} \bigcup_f \mathcal{R}$ $\qquad$ $M^3 = S^3; Lp/q; S^2 \times S^1$

Generalized Reeb component



A subset of a closed oriented 3-manifold M with smooth transversally oriented foliation $\mathcal{F}$ of codimension one is called *saturated set* if it consists of the leaves of $\mathcal{F}$. The saturated set $\mathcal{G}$ of $\mathcal{F}$ is called *generalized Reeb component* if $\mathcal{G}$ is a 3-manifold with boundary and the transversal to $\mathcal{F}$ vector field Z is directed inward or outward of $\mathcal{G}$. In particular the Reeb component is a generalized Reeb component. It is not difficult to prove that ∂G **is a family of tori**.

Taut foliation

Definition

A *taut* foliation is a codimension one transversaly oriented foliation of an oriented 3-manifold M with the property that for each leaf there is a transverse circle intersecting it.

D. Sullivan (1979) proved:

Theorem ([1])

The following properties of $\mathcal{F}$ is equivalent:

1. *$\mathcal{F}$ is taut;*
2. *each leaf is a minimal surface for some Riemannian metric on M;*
3. *$\mathcal{F}$ does not contain generalized Reeb components .*

Recall that a surface F is called *minimal* if the mean curvature H of F is identically zero.

Questions

Let (M, g) be a oriented Riemannian 3-manifold.

- Is there an upper bound on the modulus of mean curvature of the leaves of the foliation $\mathcal{F}$ to guarantee its tautness?
- How this estimate depends on the geometrical characteristics of the manifold?

Main theorem

Let (M, g) be a closed oriented Riemannian 3-Manifold sutisfying the following properties:

1. the volume $Vol(M, g) \leq V_0$;
2. the sectional curvature γ of (M, g) satisfies $\gamma \leq \gamma_0$ for the constant $\gamma_0 \geq 0$;
3. $\min\{inj(M, g), \frac{\pi}{2\sqrt{\gamma_0}}\} \geq i_0$.

Then any transversally oriented foliation $\mathcal{F}$ of codimension one on M with the mean curvature of the leaves satisfying $|H| < H_0(V_0, \gamma_0, i_0)$, must be taut. The constant H_0 is defined as wollowing:

$$H_0 = \begin{cases} \min\{\frac{2\sqrt{3}i_0^2}{V_0}, \sqrt[3]{\frac{2\sqrt{3}}{V_0}}\}, & \text{if } \gamma_0 = 0, \\ \min\{\frac{2\sqrt{3}i_0^2}{V_0}, x_0\}, & \text{if } \gamma_0 > 0, \end{cases}$$

where x_0 is the positive root of the equation $\frac{1}{\gamma_0} \operatorname{arcctg}^2 \frac{x}{\sqrt{\gamma_0}} - \frac{V_0}{2\sqrt{3}} x = 0$.

Novikov theorem

Theorem (Novikov-1965)

Let M be a closed 3-manifold with a smooth codimension one foliation $\mathcal{F}$. Suppose any of the following conditions is satisfied:

1. $\pi_1(M)$ *is finite,*
2. $\pi_2(M) \neq 0$,
3. *there exists a leaf* $\mathcal{L} \in \mathcal{F}$ *such that the map* $\pi_1(\mathcal{L}) \to \pi_1(M)$ *induced by inclusion has a nontrivial rernel.*

Then $\mathcal{F}$ has a closed leaf of genus ≤ 1. In fact, exept in the case 2, wherethe closed leaf might be S^2 or $\mathbb{R}P^2$, in all other cases, the foliation contains a Reeb component.

The corollary

Corollary

Let (M, g) *be a Riemannian 3-manifold in Main theorem. If* $\pi_1(M) < \infty$ *or* $\pi_2(M) \neq 0$*, then excepting the foliation of* $S^2 \times S^1$ *by spheres,* (M, g) *does not admit a foliation with the mean curvature* H *of leaves satisfying the inequality* $|H| < H_0$*.*

Some facts

Let μ denote the volume 3-form on M and χ denote the volume 2-form tangent to $\mathcal{F}$

Lemma (Reeb)

$d\chi = 2H\mu$

Remark

The minimality of the leaves is characterized by the invariance of the volume relative to the flow determined by the (divergence-free) orthogonal vector field Z.

$$d\chi = di(Z)\mu = \Theta(Z)\mu = (divZ)\mu = 0$$

The following lemma is direct corollary of the definition, Reeb's lemma and Stokes theorem.

Lemma

Let (M, g) be a closed oriented Riemannian 3-Manifold with a transversaly oriented smooth foliation $\mathcal{F}$ of codimension one. Let $\mathcal{G}$ be a generalized Reeb component of the foliation $\mathcal{F}$. Then

$$2\int_{\mathcal{G}} H\mu = \pm Area(\partial\mathcal{G}).$$

Applying previous lemma to generalized Reeb components $A, \overline{M \setminus A}$ we obtain the following result:

Theorem

Let (M, g) be a closed oriented Riemannian 3-Manifold with a transversally oriented smooth foliation $\mathcal{F}$ of codimension one. Suppose $\mathcal{F}$ contains generalized Reeb component and the mean curvature of $\mathcal{F}$ satisfies $|H| < H_0$. Then $Area(\partial A) < H_0 Vol(M)$.

Systolic inequality for a Riemannian torus

Theorem (Lowner -1949)

Let (T^2, g) *be the Riemannian 2-torus. Denote by* l *the length of the shortest closed noncontractible geodesic on* T^2*. Then*

$$l^2 \leq \frac{2}{\sqrt{3}} Area(T^2).$$

About the proof

Let us suppose that $\mathcal{F}$ contains genarilazed Reeb component.

Since the boundary of generalized Reeb components is a family of torus, by the systolic theorem $l^2 \leq \frac{2}{\sqrt{3}} Area(T^2)$, where l is the length of the shortest closed geodesic α which is not homotopic to zero in T^2. From the theorem above follows

$$l^2 < \frac{2}{\sqrt{3}} H_0 V_0.$$

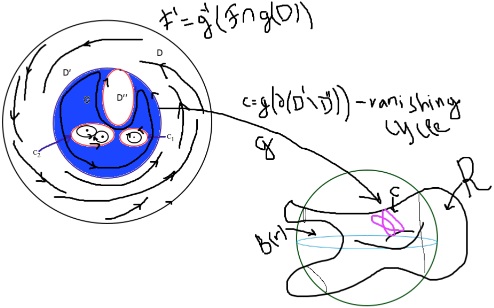
By the definition of H_0 we have $H_0 \leq \frac{2\sqrt{3} i_0^2}{V_0}$, therefore $l^2 < (2i_0)^2$. But this means that α is contained inside the ball $B(r)$ of radius r, where $l/2 < r < i_0$. Thus $i_*[\alpha] = [0] \in \pi_1(M)$.

Vanishing cycle

Jast as in the proof of the Novikov's theorem we pull the immersed disk $g(D) \subset B(r)$ on α and find the vanishing cycle $c \subset T = \partial\mathcal{R}$, where $\mathcal{R}$ is a Reeb component. The value r can be chosen such that

A) $2\sqrt{3}r^2 < H_0 V_0$

B) $T \cap S(r)$ is a family circles (or $\emptyset$). Here $S(r) := \partial B(r)$.



Theorem

There is a sphere $S(r)$, где $r < i_0$, which touches some leaf of $\mathcal{R}$ from inside the ball $B(r)$.

Denote by A connected component of $B(r) \cap \mathcal{R}$ containing c. Let $S \subset S(r) \cap \partial A$ be a connected surface. Since $S(r) \cap c = \emptyset$ each of the circle of $S(r) \cap T$ is homotopic to zero in $\mathcal{R}$ and the homomorphism $\pi_1(S) \to \pi_1(\mathcal{R})$ induced by inclusion $S \to \mathcal{R}$ is zero. We conclude : $p^{-1}(S) = \bigcup_i S_i$ is a family of copies S, where

$$p : \widetilde{\mathcal{R}} \to \mathcal{R}$$

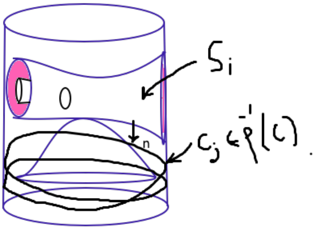
is an universal covering.

Case 1. $S(r) \cap T$ = family of circles homotopic to zero on T

In this case S_i bounds the compact domain W_i^c in $\widetilde{\mathcal{R}}$ and separate $\widetilde{\mathcal{R}}$ by compact and non-compact pieces W_i^c, W_i:

$$W_i^c \cup W_i = \widetilde{\mathcal{R}};\ W_i^c \cap W_i = S_i.$$

$p^{-1}(c) \not\subset W_i^c$ and the the internal (with respect to the ball B(r)) normal n of S_i is directed outside of W_i^c.



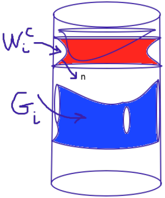
Case 2. $S(r) \cap T$ = family of circles containing at least one not homotopic to zero on T

In this case we can suppose that $G = \mathcal{R} \cap D' \subset D \subset B(r)$. We have : 1) $0 \neq [G] \in H_2(\mathcal{R}, T)$ 2) $p^{-1}G = \bigcup_k G_k$ are copies of G 3) G_k separate $\widetilde{\mathcal{R}}$ in two non-compact pieces.

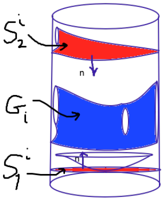
Case a) $S \subset \partial A \cap S(r)$ represent zero element of $H_2(\mathcal{R}, T)$. Then $S_i \subset p^{-1}(S)$ bounds the compact domain W_i^c in $\widetilde{\mathcal{R}}$ and separate $\widetilde{\mathcal{R}}$ by compact and non-compact pieces W_i^c, W_i:

$$W_i^c \cup W_i = \widetilde{\mathcal{R}};\ W_i^c \cap W_i = S_i.$$

We can prove $G_k \not\subset W_i^c$ and the the internal (with respect to the ball B(r)) normal n of S_i is directed outside of W_i^c and result follows immediately.



Case b) ∂A does not contain the surfaces described in a). Then we can prove that $\partial A \cap S(r) = S_1 \cup S_2$ such that $0 \neq [S_i] \in H_2(\mathcal{R}, T), \ i = 1, 2.$ In this case the connected component of $p^{-1}(A)$ is compact and bounded by two copies of S_1^i, S_2^i with inverse directed internal normals (see below). This yields the result.



Case 3. $S(r) \cap T = \emptyset$
In this case decreasing r we touch $T = \partial\mathcal{R}$.

Completion of the proof

We have found the leaf $F \in \mathcal{F}$ touching the sphere $S(r)$, $r < i_0$, from the interior of the ball $B(r)$ at the some point x_0. Denote by H_r the mean curvature of the sphere $S(r)$, and by H_r^0 we denote the mean curvature of the sphere of radius r in the space of constant curvature γ_0.
Using comparision theorem we have (at the point x_0):

$$0 < H_r^0 \leq H_r \leq H < H_0. \tag{1}$$

In the case $\gamma_0 = 0$ we have $H_r^0 = 1/r$ and we obtain the system:

$$\begin{cases} 1/r < H_0 \\ 2\sqrt{3}r^2 < V_0 H_0 \end{cases} \tag{2}$$

Excluding r^2 we obtain: $\frac{1}{H_0^2} < \frac{V_0 H_0}{2\sqrt{3}}$. Thus $H_0 > \sqrt[3]{\frac{2\sqrt{3}}{V_0}}$. Taking into account $H_0 \leq \frac{2\sqrt{3}i_0^2}{V_0}$, the assumption $H_0 \leq \min\{\frac{2\sqrt{3}i_0^2}{V_0}, \sqrt[3]{\frac{2\sqrt{3}}{V_0}}\}$ give us a contradiction.

Let us consider the case $\gamma_0 > 0$. Since $r < i_0 \leq \frac{\pi}{2\sqrt{\gamma_0}}$, the mean curvature of the round sphere $S(r) \subset S^3(R)$ of radius r in a sphere of radius R of constant curvature $\gamma_0 = 1/R^2$, has the form $H_r^0 = \sqrt{\gamma_0}\ ctg\ (r\sqrt{\gamma_0})$. We obtain the following system:

$$\begin{cases} \sqrt{\gamma_0}\ ctg\ (r\sqrt{\gamma_0}) < H_0 \\ 2\sqrt{3}r^2 < V_0 H_0 \end{cases} \tag{3}$$

Excluding r^2 we obtain:

$$\frac{1}{\gamma_0} \operatorname{arcctg}^2 \frac{H_0}{\sqrt{\gamma_0}} - \frac{V_0}{2\sqrt{3}} H_0 < 0.$$

Left part of the inequality decrease as a function of H_0 in the domain of non-negative values H_0, and it is positive near 0 and negative at infinity. Thus there is a only one positive root x_0 of the equation

$$\frac{1}{\gamma_0} \operatorname{arcctg}^2 \frac{x}{\sqrt{\gamma_0}} - \frac{V_0}{2\sqrt{3}} x = 0.$$

Clearly, in the domain $H_0 \leq \min\{\frac{2\sqrt{3}i_0^2}{V_0}, x_0\}$ we obtain the contradiction.

CONJECTURE

Kronheimer, Mrowka [3] (see also Gabai ?) proved that there are only finitely many homotopic classes of distributions tangent to the tout foliations.
We conjecture that

There are only finitely many homotopy classes of distributions tangent to the foliations with $|H| < const$.

The bibliography

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THANK YOU!