Stability of Einstein metrics

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based on:

Stability of Compact Symmetric Spaces,
arXiv:2012.07328, (with Gregor Weingart)

Linear Instability of Sasaki Einstein and nearly parallel G_2 manifolds,
arxiv:2011.11965, (with Changliang Wang und McKenzie Wang)

On the linear stability of nearly Kähler 6-manifolds,
Ann. Global Anal. Geom. **57** no. 1, 15-22 (2020),
(with Changliang Wang and McKenzie Wang).

Definition

An Einstein metric is a Riemannian metric g such that $\mathrm{Ric}_g = Eg$ for some $E \in \mathbb{R}$, called the Einstein constant.

Einstein metrics are critical points of the Einstein-Hilbert functional:

$$g \in \mathcal{M}_1 \longmapsto \mathbf{S}[g] := \int_M \mathrm{scal}_g \, \mathrm{vol}_g$$

where $\mathcal{M}_1 \subset \mathcal{M}$ is the space of Riemannian metrics of volume 1.

Examples:

1. irreducible symmetric spaces, e.g. S^n, $\mathbb{C}P^n$
2. manifolds with Killing spinors: $\nabla_X \varphi = \lambda X \cdot \varphi$
   e.g. Einstein-Sasaki, 3-Sasaki, nearly Kähler, nearly G_2 mfds.

Einstein metrics are always saddle points of $\mathbf{S}$, but can be maxima for $\mathbf{S}$ restricted to a subspace $\mathcal{S} \subset \mathcal{M}_1$. There are three cases for $\mathbf{S}''$:

$$T_g\mathcal{M} = \Gamma(\mathrm{Sym}^2 M) = \mathrm{im}\,\delta_g^* \oplus \mathcal{C}^\infty(M)g \oplus [\ker \delta_g \cap \ker \mathrm{tr}]$$

if $M \neq S^n$ compact, $\delta_g h$ is the divergence of h and $\delta_g^* X = \frac{1}{2} L_X g$.

- $\mathrm{im}\,\delta^*$: tangent space of $\mathrm{Diff}(M)g$, here $\mathbf{S}'' = 0$
- $\mathcal{C}^\infty(M)g$: tangent space of the conf. class $[g]$, here $\mathbf{S}'' > 0$
- $\ker \delta \cap \ker \mathrm{tr}$: tangent space of $\mathcal{S} := \{g : \mathrm{vol}_g(M) = 1, \mathrm{scal}_g = c\}$
  = space of tt-tensors

$\mathbf{S}''(h,h) = -((\Delta_L - 2E)h, h)_{L^2}$ for all $h \in \ker \delta \cap \ker \mathrm{tr}$

where $\Delta_L = \nabla^*\nabla + q(R)$ is the Lichnerowicz Laplacian on $\Gamma(\mathrm{Sym}^2 M)$

◂ E

Let (M^n, g) be a compact Einstein manifold, with Einstein constant E.

Definiton

- g is stable if $\mathbf{S}'' < 0$ on tt-tensors $\leftrightarrow$ $\Delta_L > 2E$ on tt-tensors
- g is unstable if there is a tt-tensor h with: $\mathbf{S}''(h,h) > 0$

  $\leftrightarrow$ there is a tt-tensor h with: $\Delta_L h = \mu h$, $\mu < 2E$
- infinitesimal Einstein deformations: tt-tensors h with: $\mathbf{S}''(h,h) = 0$

  $\leftrightarrow \Delta_L h = 2Eh$

Remarks:

1. Stable metrics are local maxima of $\mathbf{S}$ restricted to $\mathcal{S} \subset \mathcal{M}_1$.
2. The space of unstable directions and infinitesimal Einstein deformations is a finite dimensional space.
3. Inf. Einstein deformations are solutions of the linearised Einstein equation, e.g. tangent vectors to curves of Einstein metrics.

The Lichnerowicz Laplacian $\Delta_L = \nabla^*\nabla + q(R)$ on $\Gamma(\mathrm{Sym}^k M)$ is a Laplace type operator, i.e. in particular it is elliptic. Here $q(R)$ is a curvature term, e.g. $q(R) = \mathrm{Ric}$ for $k = 1$ and for $k = 2$ we have:

$$q(R) = 2\mathring{R} + Ric \qquad \text{with} \qquad (\mathring{R}h)(X, Y) = \sum h(R_{X,e_i}Y, e_i) \; .$$

On compact manifolds the Lichnerowicz Laplacian Δ_L has only finitely many negative eigenvalues, with lower bound:

$\Delta_L \geq 2q(R)$ on $\ker \delta$, with equality on divergence-free Killing tensors

Remarks: Let (M, g) be a compact Einstein manifold, then

1. $\Delta_L = \Delta = d^*d \geq 2E$ on co-closed 1-forms, equality on Killing v.f.
2. $\Delta_L = \Delta = d^*d \geq 2E$ for functions on Kähler-Einstein manifolds
3. On symmetric spaces Δ_L coincides with the Casimir operator. For the Killing form metric we have $2E = 1$, i.e. $2E$ is the Casmir eigenvalue of the adjoint representation.

Stable Einstein metrics

- Many compact irreducible symmetric spaces, e.g. S^n, $\mathbb{C}P^n$
- Einstein manifolds with $\sec < 0$ [Koiso, (1978)]
- Riem. manifolds with imaginary Killing spinors [Kröncke + Ch. Wang (2017)]
- Compact manifolds with parallel spinors, i.e. all known examples with $\mathrm{Ric} = 0$, (possibly with inf. Einstein deformations) [Dai et al., (2005)]

Unstable Einstein metrics

- Products of Einstein metrics with the same $E > 0$
- Jensen metric on S^{4n+3} (second Einstein metric in can. variation)
- Compact Kähler-Einstein mf. with $\mathrm{scal} > 0$ and $b_2 \geq 2$ [CHI, (2004)]

Remark: There is no known example of a non-symmetric stable Einstein metric with $\mathrm{scal} > 0$, (e.g. among mf. with Killing spinors).

E

Ricci flow motivation for studying unstable metrics

S-unstable $\Rightarrow$ ν-unstable $\Rightarrow$ dynamically unstable w.r.t Ricci flow

The Ricci flow, $\partial_t g_t = -2\mathrm{Ric}(g_t)$, can be considered as dynamical system on the space of Riemannian metrics (modulo diffeomorphismen and scaling). Einstein metrics evolve via homothetic scaling under the Ricci flow, i.e. can be considered as fixed points (up to scaling).

dynamically stable $\hat{=}$ stable as FP of the Ricci flow, i.e. the Ricci flow starting at a small perturbation of the metric will return to the original Einstein metric. Stability is investigated with Perelman's ν-entropy.

(M, g) compact Einstein, $\mathrm{scal} > 0$, then: g ν-stable $\Leftrightarrow$ $\nu'' \leq 0$

1. g is ν-stable $\Leftrightarrow$ (i) $\Delta_L \geq 2E$ on tt-tensors, (ii) $\Delta \geq 2E$ on $\mathcal{C}^\infty M$
2. g is a local maximum of ν $\Leftrightarrow$ g is dynamically stable

[(1): Cao, He (2015), (2): Kröncke (2015)]

Koiso determined the stability of compact irreducible symmetric spaces up to a few exceptions. All are stable, accept ($n \geq 3,\ r, s \geq 2$):

Symmetric spaces with infinitesimal Einstein deformations

$SU(n),\ SU(n)/SO(n),\ SU(2n)/Sp(n),\ U(r+s)/U(r)\times U(s),\ E_6/F_4$

Unstable symmetric spaces

$Sp(r), \quad Sp(n)/U(n), \quad SO(5)/SO(3)\times SO(2)$

Symmetric spaces with undecided stability status

$$Sp(r+s)/Sp(r)\times Sp(s), \quad \mathbb{H}P^2 \text{ (r=s=1)}, \quad \mathbb{O}P^2 = F_4/Spin(9) \qquad (*)$$

Remark: For $SU(n), E_6/F_4$ and the symmetric spaces in (*) Koiso already showed that there are Δ_L eigenvalues less than $2E$. However, it remained open whether the corresponding eigentensors can be realised as tt-tensors.

Theorem (S., Weingart, 2020)

The Cayley plane $\mathbb{O}P^2 = F_4/Spin(9)$ and the quaternionic projective plane $\mathbb{H}P^2$ are stable. For $r, s \geq 2$, the quaternionic Grassmannians $Sp(r+s)/Sp(r) \times Sp(s)$ are unstable.

Theorem (P. Schwahn, 2020)

The symmetric spaces $SU(n), n \geq 3$ and E_6/F_4 do not admit destabilising directions, i.e. tt-tensors which are Δ_L eigentensors for eigenvalues less than $2E$.

Idea of the proofs

Let (M, g) be a spin manifold with a Killing spinor: $\nabla_X \varphi = \lambda X \cdot \varphi$.
Then g is Einstein with $scal > 0$ and M is compact (if $\lambda \in \mathbb{R} \setminus \{0\}$).
Moreover, $M = S^n$ or M is non-symmetric and belongs to one of:

- Einstein-Sasaki in dimension $2n + 1$
- 3-Sasaki in dimension $4n + 3$
- strict nearly Kähler in dimension 6
- proper nearly (parallel) G_2 in dimension 7

Remark: These four types of manifolds are characterised by the property that their metric cone ($\hat{M} = \mathbb{R}_+ \times M, \hat{g} = dr^2 + r^2 g$) has holonomy $SU(n), Sp(n), G_2$ or $Spin(7)$. In each case the Killing spinor corresponds to a parallel spinor on the metric cone.

⚠ In all cases ex. a metric connection $\bar{\nabla}$ with skew-sym. and parallel torsion.

Theorem (S., McKenzie Wang, Changliang Wang, 2020)

The following compact Einstein manifolds are unstable:

1. Einstein-Sasaki manifolds with $b_2 > 0$
2. Nearly Kähler manifolds (M^6, g, J) with $b_2 > 0$ or $b_3 > 0$
3. Nearly G_2 manifolds (M^7, g, φ) with $b_3 > 0$
4. The Berger space $SO(5)/SO(3)_{irr}$

Remark: Any 3-Sasaki mfd. is in particular Einstein-Sasaki and has $b_3 = 0$. The Berger space is a homology sphere, i.e. it does not have non-trivial harmonic forms. It also is a proper nearly G_2 manifold. There is no known example of a proper G_2 manifold with $b_3 > 0$. It is still unclear whether nearly G_2 manifolds with $b_2 > 0$ are unstable.

Idea of the proofs for (1),(2),(3)

$M^7 = G/K = SO(5)/SO(3)_{irr} = Sp(2)/Sp(1)_{irr}$, $\mathfrak{sp}(2) = \mathfrak{sp}(1) \oplus \mathfrak{m}$

$\mathfrak{m}^{\mathbb{C}} = \mathrm{Sym}^6 E$, with $E = \mathbb{C}^2$ the standard $Sp(1)$-representation

$$\Gamma(\mathrm{Sym}^2_0 TM^{\mathbb{C}}) = \overline{\bigoplus}_{k,l} V(k,l) \otimes \mathrm{Hom}_{Sp(1)}(V(k,l), \mathrm{Sym}^2_0\, \mathfrak{m}^{\mathbb{C}})$$

where $V(k,l)$ are irreducible complex $Sp(2)$ representations, with $k \geq l \geq 0, k, l \in \mathbb{N}$

$V(1,1)$: $\dim_{\mathbb{C}} V(1,1) = 5$ and $V(1,1) = \mathrm{Sym}^4 E$ as $Sp(1)$-repr.

$\underline{V(1,1) \subset \Gamma(\mathrm{Sym}^2_0 TM^{\mathbb{C}})}$ since: $\mathrm{Sym}^2_0 \mathfrak{m}^{\mathbb{C}} = \mathrm{Sym}^4 E \oplus \mathrm{Sym}^8 E \oplus \mathrm{Sym}^{12} E$

similarly: $V(1,1) \not\subset \Gamma(TM^{\mathbb{C}})$ and $V(1,1) \not\subset \Gamma(\mathrm{Sym}^3 TM^{\mathbb{C}})$

$\underline{V(1,1) \subset \ker \delta \cap \ker \delta^*}$, i.e. Killing + tt-tensors, (δ, δ^* are $Sp(2)$-inv. diff. op.)

$\Rightarrow$ $\Delta_L = 2q(R)$ on $V(1,1)$, $2q(R) = \frac{19}{30}\mathrm{Id}$ and $\frac{19}{30} < 2E = \frac{9}{10}$

since: (i) $q(\bar{R}) = \mathrm{Cas}_{\mathrm{Sp}(1)}$ and (ii) formula for $q(R) - q(\bar{R})$ [S., Alexandrov (2012)], [S., Moroianu (2010)],

$\Rightarrow$ $V(1,1)$ are destabilising directions, i.e. the Berger space is unstable

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Proof:

We have $\Delta_L - 2q(R) = \delta\delta^* - \delta^*\delta = \delta\delta^* \geq 0$ on $\ker\delta$ with equality on $\ker\delta \cap \ker\delta^*$, i.e. on the space of divergence-free Killing tensors.

Definition:

A symmetric tensor $h \in \Gamma(\mathrm{Sym}^k M)$ is called a **Killing tensor**

$$\Leftrightarrow \quad \delta^* h = 0$$

$$\Leftrightarrow \quad (\nabla_X h)(X, \ldots, X) = 0$$

$$\Leftrightarrow \quad f_{h,\gamma}(t) := h_{\gamma(t)}(\dot\gamma(t), \ldots, \dot\gamma(t))$$

is constant along all geodesics γ

i.e. Killing tensors define polynomial first integrals

Example:

Killing tensors for $k = 1$ are Killing vector fields.

Definition

A strict nearly Kähler manifold is an almost Hermitian manifold (M, g, J) with J non-integrable and $X \lrcorner \nabla_X \omega = 0$, for all $X \in TM$, where ω is the Kähler form, i.e. $\omega(X, Y) = g(JX, Y)$.

Examples: 3-symmetric spaces, twistor spaces of quaternion Kähler manifolds. The homogeneous examples in dimension 6 are:

$$S^6, \quad \mathbb{CP}^3, \quad F(1,2), \quad S^3 \times S^3$$

The only other known examples are cohomogeneity one metrics on S^6 and $S^3 \times S^3$ found by L. Foscolo and M. Haskins in 2015.

Remark: Non-trivial harmonic forms on a nearly Kähler manifold (M^6, g, J) are sections of $\Lambda^{1,1}T$, $\Lambda^{2,2}T$ or $\Lambda^{(2,1)+(1,2)}T$ [Verbitsky, (2011)] .

Definition

A nearly (parallel) G_2-manifold is a Riemannian manifold (M^7, g) admitting a 3-form φ whose stabilizer at each point is isomorphic to the group G_2 and such that $d\varphi = \lambda * \varphi$ for some non-zero real number λ. The nearly G_2 manifold is proper if it does not admit a Einstein-Sasaki structure.

Remark: Nearly G_2 manifolds are either Einstein-Sasaki, 3-Sasaki or proper, corresponding to the dimension of the space of Killing spinors.

Examples:

- Einstein-Sasaki and 3-Sasaki manifolds in dimension 7
- Aloff-Wallach spaces $SU(3)/U(1)_{k,l}$ (proper if $(k, l) \neq (1, 1)$)
- the Berger space $SO(5)/SO(3)_{irr}$ (proper)
- the second Einstein metric in the canonical variation on a 7-dim. 3-Sasaki manifold (proper)

Definition

A Sasaki manifold is a Riemannian manifold with a unit length Killing vector field ξ satisfying $\nabla_X d\xi^* = -2X^* \wedge \xi^*$.

Examples: S^1-bundles over Kähler manifolds or Kähler orbifolds.

Definition

A 3-Sasaki manifold is a Riemannian manifolds with three Sasaki structures ξ_a, $_{a=1,2,3}$, satisfying $[\xi_1, \xi_2] = 2\xi_3$ and cyclic permutations.

Examples: S^3-bundles over quaternionic Kähler manifolds or quaternionic Kähler orbifolds.

Idea of the proofs: stability of symmetric spaces

Claim: $G/K = Sp(r+s)/Sp(r) \times Sp(s),\ r, s \geq 2$ is unstable.

$H := \mathbb{H}^r,\ E := \mathbb{H}^s,\ \ \mathfrak{sp}(r+s)^{\mathbb{C}} = [\mathfrak{sp}^{\mathbb{C}}(r) \oplus \mathfrak{sp}^{\mathbb{C}}(s)] \oplus \mathfrak{m}^{\mathbb{C}},\ \ \mathfrak{m}^{\mathbb{C}} = H \otimes E$

$\mathrm{Sym}^2_0\, \mathfrak{m}^{\mathbb{C}} = (\mathrm{Sym}^2 H \otimes \mathrm{Sym}^2 E) \oplus (\Lambda^2_0 H \otimes \Lambda^2_0 E) \oplus \Lambda^2_0 H \oplus \Lambda^2_0 E$

$\Gamma(\mathrm{Sym}^2_0 TM^{\mathbb{C}}) = \overline{\bigoplus}_\gamma V_\gamma \otimes \mathrm{Hom}_K(V_\gamma, \mathrm{Sym}^2_0\, \mathfrak{m}^{\mathbb{C}})$

Consider: $V_\gamma = \Lambda^2_0\, \mathbb{H}^{r+s} = \Lambda^2_0(H \oplus E) = \mathbb{C} \oplus \Lambda^2_0 H \oplus (H \otimes E) \oplus \Lambda^2_0 E$

$\Rightarrow\ \dim \mathrm{Hom}_K(V_\gamma, \mathrm{Sym}^2_0 \mathfrak{m}^{\mathbb{C}}) = 2, \quad \dim \mathrm{Hom}_K(V_\gamma, \mathfrak{m}^{\mathbb{C}}) = 1$

Since the divergence δ is a G invariant diff. operator it restricts to a linear map

$\hat{\delta} : \mathrm{Hom}_K(V_\gamma, \mathrm{Sym}^2_0 \mathfrak{m}^{\mathbb{C}}) \to \dim \mathrm{Hom}_K(V_\gamma, \mathfrak{m}^{\mathbb{C}}), \quad \ker \hat{\delta} \neq \{0\}$

$\Rightarrow\ V_\gamma \subset \ker \delta$, i.e. sections in $V_\gamma = \Lambda^2_0\, \mathbb{H}^{r+s}$ are tt-tensors

$$\Delta_L|_{V_\gamma} = \mathrm{Cas}_{V_\gamma} = \tfrac{r+s}{r+s+1} \mathrm{Id}_{V_\gamma}, \quad \tfrac{r+s}{r+s+1} < 2E = 1$$

$\Rightarrow V_\gamma = \Lambda^2_0 \mathbb{H}^{r+s}$ are destabilising directions, i.e. G/K is unstable

(M^6, g, J) strict nearly Kähler with $b_2 > 0$

$\Rightarrow\ \exists \eta \in \Omega_0^{(1,1)}(M) :\ \Delta\eta = 0$ [Verbitsky, (2011)]

$\Rightarrow\ \bar{\Delta}\eta = 0$ where $\bar{\Delta} := \bar{\nabla}^*\bar{\nabla} + q(\bar{R})$, since $\Delta = \bar{\Delta}$ on $\Omega_0^{(1,1)}(M)$

[S., Moroianu, (2010)]

$\eta \overset{\Phi}{\mapsto} h \in \Gamma(\mathrm{Sym}_0^{2,+} M) :\quad h(X, Y) = \eta(JX, Y)$

$\delta h = 0$, i.e. h is a tt-tensor since $d^*\eta = 0$

$\Rightarrow 0 = \bar{\Delta}_L h = \bar{\nabla}^*\bar{\nabla} h + q(\bar{R})h$, since Φ is $\bar{\nabla}$-parallel [S., Weingart (2018)]

$= \Delta_L h - 6h$ because of comparison formulas from: [S., Moroianu (2010)]

$2E = 2\frac{\mathrm{scal}}{6} = 10 \ \Rightarrow\ 6 < 2E$

$\Rightarrow$ h is a destabilising direction, i.e. the NK metric g is unstable

Perelman's ν-entropy

<u>$\mathcal{W}$-functional</u>: $\mathcal{W}(g,f,\tau) := \int_M [\tau(\mathrm{scal}_g + |\nabla f|^2 - n)](4\pi\tau)^{-\frac{n}{2}} e^{-f} \mathrm{vol}_g,$

ν-entropy: $\nu[g] := \inf\{\mathcal{W}(g,f,\tau) : f, \tau > 0, (4\pi\tau)^{-\frac{n}{2}} \int_M e^{-f} \mathrm{vol}_g = 1\}$

- Einstein metrics with $\mathrm{scal} > 0$ occur among critical points
  (critical points = shrinking gradient Ricci solitons)
- The ν-entropy is monoton increasing under the Ricci flow

$\Rightarrow$ If the ν-entropy increases along some direction then the corresp. perturbation of the metric will never return under the flow.

The second variation ν'' has again three parts: [Cao, Hamilton, Ilmanen (2004)]

(i) $\nu'' = 0$ on $\mathrm{Diff}(M)g$ (ii) $\nu'' = \mathbf{S}''$ on tt-tensors

(iii) $\nu'' \leq 0$ on $C^\infty(M)g$ if $\Delta \geq 2E$ on $C^\infty(M)$