Extra-twisted connected sum G_2-manifolds and analytic invariants

Johannes Nordström

1 February 2021

Joint work with Sebastian Goette, arXiv: 2010.16367

These slides available at
`http://people.bath.ac.uk/jlpn20/xxtcs.pdf`

1. Introduction
Jeopardy

Topic: Riemannian metrics on 7-manifolds with holonomy group G_2

A: An analytic invariant (defined in terms of eta invariants) of closed G_2-manifolds.

Q: How can one detect connected components of the G_2 moduli space, *ie* distinguish different holonomy G_2 metrics on the same smooth 7-manifold up to deformation?

A: Extra-twisted connected sum G_2-manifolds

Q: How can one construct examples of G_2-manifolds where the analytic invariant is computable and can be used to distinguished components of the G_2 moduli space?

Context: Berger's list

The *holonomy group* of a Riemannian n-manifold is the subgroup of $O(n)$ generated by parallel transport around closed loops.

Theorem (Berger)

The only possible holonomy groups of complete, simply connected Riemannian manifolds that are neither a product nor a symmetric space are

Holonomy group	dim	*Parallel spinors*	*Type*
$SO(n)$	n		*generic*
$U(k)$	$2k$		*Kähler*
$SU(k)$	$2k$	2	*Calabi-Yau (Ricci-flat)*
$Sp(\ell)$	4ℓ	$\ell + 1$	*hyper-Kähler (Ricci-flat)*
$Sp(\ell) \cdot Sp(1)$	4ℓ		*Quat. Kähler (Einstein)*
G_2	7	1	*exceptional (Ricci-flat)*
$Spin(7)$	8	1	*exceptional (Ricci-flat)*

Holonomy G_2 and G_2-structures

$G_2 = \mathrm{Aut}(\mathbb{O})$, automorphisms of the 8-dimensional octonion algebra.

$G_2 \subset SO(7)$ can also be defined as the stabiliser of a definite 3-form

$$\varphi_0 = dx^{123} + dx^{145} + dx^{167} + dx^{246} - dx^{257} - dx^{347} - dx^{356} \in \Lambda^3(\mathbb{R}^7)^*.$$

Therefore G_2-structure on $M^7 \leftrightarrow \varphi \in \Omega^3(M)$ pointwise equivalent to φ_0 (an open condition on φ).

A G_2-structure induces a metric. A metric has $\mathrm{Hol} \subseteq G_2$ if and only if it is induced by a G_2-structure that is torsion-free, *ie* satisfies

$$d\varphi = d^*\varphi = 0.$$

In particular, φ represents a de Rham cohomology class

$$[\varphi] \in H^3(M).$$

The G_2 moduli space

Let M be a closed 7-manifold. The moduli space

$$\mathcal{M} = \{\text{torsion-free } G_2\text{-structures on } M\}/\mathrm{Diff}(M)$$

is a locally finite quotient of the "Teichmüller space"

$$\mathcal{M}_0 = \{\text{torsion-free } G_2\text{-structures on } M\}/(\text{id component of } \mathrm{Diff}(M))$$

Theorem (Joyce)

$\varphi \to [\varphi]$ *induces a local homeomorphism* $\mathcal{M}_0 \to H^3(M)$.

So $\mathcal{M}$ is an orbifold, but little is understood about its *global* properties.

Two approaches to understanding connected components:

- Higher-dimensional gauge theory programme of Donaldson, Thomas, ... Try to define enumerative invariants from instantons and calibrated submanifolds.
- Use spinor interpretation of G_2-structures to define topological and analytic invariants, in particular an invariant $\bar{\nu}(\varphi) \in \mathbb{Z}$.

Constructions of closed G_2-manifolds

Closed G_2-manifolds cannot admit continuous symmetries (because $\mathrm{Ric} = 0$). All known examples come from gluing constructions.

- **Joyce (1995)** Orbifold construction
  Resolve singularities of T^7/Γ using QALE Calabi-Yau spaces.
  Many examples, topology slightly complicated to pin down.
- **Kovalev (2003)**, **Corti-Haskins-N-Pacini (2014)**
  Twisted connected sums: glue asymptotically cylindrical Calabi-Yaus$\times S^1$.
  Many examples, topology computable and classification results often applicable, but limited variation.
- **Joyce-Karigiannis (2018)**
  Resolve singularities of $(CY^3 \times S^1)/\mathbb{Z}_2$.
  Not yet any new topological types.
- **Crowley-Goette-N (2018)** Extra-twisted connected sums
  Glue quotients of ACyl CY $\times S^1$ by finite groups.
  Limited (thousands?) but quite varied supply of examples.

Disconnecting the G_2 moduli space

By

- generating many extra-twisted connected sums
- computing their topological invariants
- computing $\bar{\nu}$
- applying smooth classification results

one can obtain many examples like the following one.

Example (Crowley-Goette-N)

Up to diffeomorphism there is a unique 2-connected smooth 7-manifold M with $H^4(M) \cong \mathbb{Z}^{77}$ such that $p_1(M) = 4x$ for a primitive $x \in H^4(M)$.

This M admits torsion-free G_2-structures with $\bar{\nu} = 0, 36$ and 48.
In particular, the G_2 moduli space on M has at least 3 components.

(Moreover, the torsion-free G_2-structures with $\bar{\nu} = 0$ and 48 can be connected to each other by a path of G_2-structures that are not torsion-free, but not to the one with $\bar{\nu} = 36$.)

2. The analytic invariant
G_2-structures and spinors

$Spin(7) \to SO(7)$ is a double cover, and $G_2 \hookrightarrow SO(7)$ has a lift $G_2 \hookrightarrow Spin(7)$.

The spin representation Δ of $Spin(7)$ is real of rank 8.

The image of G_2 in $Spin(7)$ is precisely the stabiliser of a non-zero $s_0 \in \Delta$ (unique up to scale).

Therefore a G_2-structure on M^7 is equivalent to

(orientation $+$) spin structure $+$ metric
$+$nowhere vanishing spinor field (up to scale)

Note: because spinor bundle of a spin M^7 has rank 8, nowhere-vanishing sections always exist.

M^7 admits G_2-structure $\leftrightarrow$ M is spin

Eta invariants

For an elliptic operator D_M on a closed M^7 and $s \in \mathbb{C}$ with $\operatorname{Re} s \gg 0$, define

$$\eta(D, s) := \sum_{\lambda \in \operatorname{Spec} D \setminus \{0\}} (\operatorname{sign} \lambda) |\lambda|^{-s}.$$

Let $\eta(D) \in \mathbb{R}$ be evaluation of the analytic continuation at $s = 0$.
"Measures asymmetry of spectrum if D"

Theorem (Atiyah-Singer)

For an elliptic operator D_X on a closed manifold X, $\operatorname{ind} D_X$ is determined by characteristic classes of X and the Chern character of D_X.

Theorem (Atiyah-Patodi-Singer)

Index of an elliptic operator D_W on a compact Riemannian manifold W with boundary M whose metric is a product in a collar neighbourhood can be expressed as Chern-Weil expression on W + eta invariant on M.

Atiyah-Patodi-Singer for Dirac operator

Let W^8 be a compact spin Riemannian 8-manifold with boundary M.
The Dirac operator $\not{D}^+_W$ has

$$\operatorname{ind} \not{D}^+_W = \int_W \widehat{A}(\nabla) - \tfrac{1}{2}(\eta + h)(\not{D}_M)$$

where $\widehat{A}(\nabla)$ is the Chern-Weil form for

$$\frac{7p_1^2 - 4p_2}{45 \cdot 2^7}$$

and $h(\not{D}_M)$ is the dimension of $\ker \not{D}_M$ (= parallel spinors if M is scalar flat).

Atiyah-Patodi-Singer for signature operator

Let W^8 be a compact orientable 8-manifold with boundary M, and let $B_M : \Omega^{ev}(M) \to \Omega^{ev}(M)$ be the odd signature operator, *ie*

$$B_M = (-1)^k(*d - d*) \text{ on } \Omega^{2k}(M).$$

Then the signature of the intersection form on $H^4(W, M)$ is

$$\sigma(W) = \int_W L(\nabla) - \eta(B_M)$$

where $L(\nabla)$ is the Chern-Weil form for $\dfrac{7p_2 - p_1^2}{45}$.

Euler classes

Let W be a compact orientable $2n$-manifold with boundary M.

Proposition

The Euler characteristic of W is

$$\chi(W) = \int_W e(\nabla)$$

for $e(\nabla) \in \Omega^{top}(W)$ the Euler form of a connection on TW

Proof.

Follows from closed case by doubling. □

By the Poincaré-Hopf index theorem, $\chi(W)$ counts zeros (with signs) of a transverse section of TW that points outwards on the boundary.

Mathai-Quillen currents

To generalise the relation between zeros of sections of a bundle to integral of an Euler form to other bundles over a $2n$-manifold with boundary M, one can associate to any nowhere-section s of any rank $2n$ bundle on M with a connection a "Mathai-Quillen current"

$$MQ(s) \in \mathbb{R}$$

so that the following holds.

Theorem (Mathai-Quillen)

Let E be a rank $2n$ vector bundle on W with connection ∇. Then for any transverse section s of E

$$\#s^{-1}(0) = \int_W e(\nabla) + MQ(s_{|M}).$$

If $s_{|M}$ is parallel then $MQ(s_{|M}) = 0$.

Definition of the analytic invariant

Definition

For a torsion-free G_2-structure φ on a closed 7-manifold M, let $\not{D}_M$ and B_M be the associated Dirac and odd signature operators, and

$$\bar{\nu} := 3\eta(B_M) - 24\eta(\not{D}_M) \in \mathbb{R}.$$

Theorem

$\bar{\nu}(\varphi)$ is an integer, and it is invariant under deformations of the torsion-free G_2-structure.

Proof of continuity of $\bar{\nu}$:

Eta invariants change continuously under deformations of the operator, except that they jump by ± 1 if an eigenvalue becomes/ceases to be zero. Kernel of B_M consists of harmonic forms so has constant dimension, and $h(\not{D}_M) = 1 + b_1(M)$ for any torsion-free G_2-structure on M.

Relation of the invariant to coboundaries

Let W be any compact spin Riemannian manifold with boundary M.
The positive spinor bundle S^+W is a real rank 8 bundle.
It restricts to the spinor bundle SM of M.
A torsion-free G_2-structure φ on M defines a parallel section s_φ of SM.

Proposition

Let s be any transverse section of S^+W that restricts to s_φ. Then

$$\bar{\nu}(\varphi) = \chi(W) - 3\sigma(W) - 2\#s^{-1}(0) + 48\,\mathrm{ind}\,\not{D}^+_W - 24(1 + b_1(M)).$$

Because the bordism group $\Omega_7^{Spin} = 0$, every G_2-manifold has such a coboundary, proving that $\bar{\nu}$ is always an integer.

Dependence between characteristic classes

Proposition

Let s be any transverse section of S^+W that restricts to s_φ. Then

$$\bar{\nu}(\varphi) = \chi(W) - 3\sigma(W) - 2\#s^{-1}(0) + 48 \operatorname{ind} \not{D}^+_W - 24(1 + b_1(M)).$$

Proof.

The four characteristic classes $\widehat{A}$, L, e and e_+ have a linear dependence

$$2e_+ = 48\widehat{A} + e - 3L.$$

Writing $\chi(W) - 3\sigma(W) - 2\#s^{-1}(0) + 48 \operatorname{ind} \not{D}^+_W$ as Chern-Weil + boundary correction, the Chern-Weil terms all cancel, leaving

$$3\eta(B_M) - 24(\eta + h)(\not{D}_M) - 2MQ(s_\varphi).$$

Holonomy $\subseteq G_2$ implies $h(\not{D}_M) = 1 + b_1(M)$ and $MQ(s_\varphi) = 0$. □

3. Extra-twisted connected sums
Twisted connected sums

Kovalev (2003), **Corti-Haskins-N-Pacini (2014)**.
Ingredients:

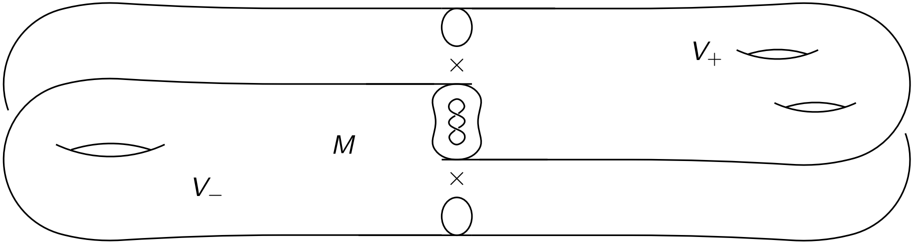
- Closed simply-connected Kähler 3-folds Z_+, Z_-
- $\Sigma_\pm \subset Z_\pm$ anticanonical K3 divisors ($[\Sigma_\pm] = c_1(Z_\pm)$) with trivial normal bundle
- $\mathfrak{r} : \Sigma_+ \to \Sigma_-$ diffeomorphism

Let $V_\pm := Z_\pm \setminus$ tubular neighbourhood $\Sigma_\pm \times \Delta$; so $\partial V_\pm = \Sigma_\pm \times S^1$.
Form simply-connected M^7 by gluing boundaries of $V_+ \times S^1$ to $V_- \times S^1$ by

$$\Sigma_+ \times S^1 \times S^1 \to \Sigma_- \times S^1 \times S^1,$$
$$(x, u, v) \mapsto (\mathfrak{r}(x), v, u)$$

Tian-Yau, **Haskins-Hein-N**:
$V_\pm$ admits asymptotically cylindrical Calabi-Yau metrics
$\rightsquigarrow$ metrics on $V_\pm \times S^1$ with holonomy $SU(3) \subset G_2$.
For carefully chosen $\mathfrak{r}$, these metrics glue to a holonomy G_2 metric on M.

Diagram of twisted connected sum



V_+, V_- asymptotically cylindrical Calabi-Yau threefolds with ends asymptotic to $\Sigma_\pm \times S^1 \times \mathbb{R}$, where $\Sigma_\pm$ are K3 surfaces.

Truncate ends and glue $V_- \times S^1$ to $V_+ \times S^1$, flipping the circles.

For large neck length, the G_2-structure on M obtained by gluing has $d\varphi = 0$ and $d^*\varphi$ "small", and can be perturbed to a torsion-free one.

Phenomena twisted connected sums exhibit

One can generate 10^8 examples by constructing the initial "building blocks" (Z, Σ) from *eg* Fano 3-folds or semi-Fano 3-folds, *ie* closed complex 3-folds with c_1 a Kähler class or "non-negative".

Many are 2-connected so that it is easy to apply smooth classification results, and many different constructions yield the same smooth manifold.

Can they be distinguished by $\bar{\nu}$? No.

Theorem

Any twisted connected sum has $\bar{\nu} = 0$.

However, one can use twisted connected sums to generate examples of

- G_2-manifolds that are homeomorphic but not diffeomorphic (**Crowley-N**)
- G_2-manifolds where components of the moduli space are distinguished by a different tool, the ξ-invariant (**Wallis**)

Tori

Recall:

From a building block (Z, Σ) we get an ACyl Calabi-Yau 3-fold $V := Z \setminus \Sigma$ with cylindrical end $\mathbb{R} \times S^1 \times \Sigma$. Think of this circle factor as "internal".

Now suppose the building block (Z, Σ) has a cyclic automorphism group Γ that fixes Σ pointwise.
Then the action of Γ on V acts trivially on the Σ factor in the asymptotic end while rotating the S^1_{int} factor.

Next choose a free action of Γ on "external" circle S^1_{ext}.
Then $(S^1_{ext} \times V)/\Gamma$ is a smooth ACyl G_2-manifold. Its asymptotic end is of the form $\mathbb{R} \times T^2 \times \Sigma$, but the torus $T^2 := (S^1_{ext} \times S^1_{int})/\Gamma$ need *not* be a metric product of two circles.
The geometry of T^2 depends on the circumferences of S^1_{ext} and S^1_{int}, which can be chosen freely.

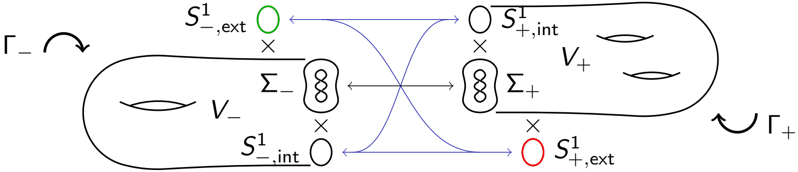
Adding the extra twist

To make an extra-twisted connected sum

- Find some building blocks $(Z_\pm, \Sigma_\pm)$ with automorphism groups $\Gamma_\pm$
- Choose circumferences so that there is an isometry $\mathfrak{t} : T^2_+ \to T^2_-$
- Find ACyl Calabi-Yau metrics so that there is $\mathfrak{r} : \Sigma_+ \to \Sigma_-$ that makes

$$(-1) \times \mathfrak{t} \times \mathfrak{r} : \mathbb{R} \times T^2_+ \times \Sigma_+ \to \mathbb{R} \times T^2_- \times \Sigma_-$$

an isomorphism of the asymptotic limits of the G_2-structures.

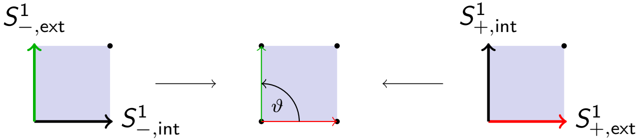


Inflexibility of the gluing angle for TCS

In the twisted connected sum construction we identify the asymptotic cross-sections $S^1_{+,ext} \times S^1_{+,int} \times \Sigma_+$ and $S^1_{-,ext} \times S^1_{-,int} \times \Sigma_-$ by the product of an isometry $\mathfrak{r} : \Sigma_+ \to \Sigma_-$ and the "flip" isometry

$$S^1_{+,ext} \times S^1_{+,int} \to S^1_{-,ext} \times S^1_{-,int}, \quad (u, v) \mapsto (v, u).$$

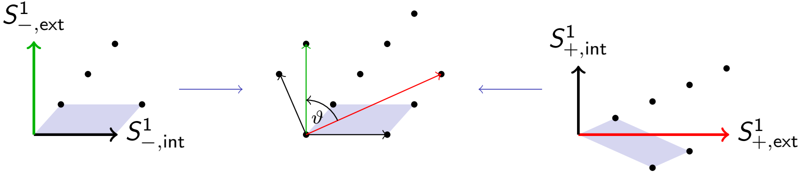
We can choose the circumferences of $S^1_{+,ext} = S^1_{-,int}$, and $S^1_{-,ext} \cong S^1_{+,int}$, but the angle ϑ between the external circle direction will *always* be $\frac{\pi}{2}$.



This "gluing angle" ϑ turns out to play a key role in the calculation of $\bar{\nu}$.

More exciting torus isometries

As soon as at least one of the tori T^2_+ and T^2_- is not simply an isometric product $S^1_{ext} \times S^1_{int}$, there are other possibilities for the gluing angle ϑ.



$\Gamma_- \cong \mathbb{Z}/3$ $\qquad\qquad$ $\Gamma_+ \cong \mathbb{Z}/4$

eg $\qquad \vartheta = \dfrac{3\pi}{4}, \ \dfrac{2\pi}{3}$ or $\arccos\left(\dfrac{1}{\sqrt{6}}\right)$.

Reducing the matching problem to periods

Given the size of Γ_+ and Γ_-, it is essentially a combinatorial problem to determine all possible torus isometries $\mathfrak{t} : T^2_+ \to T^2_-$.
Eg for $\Gamma_+ = \mathbb{Z}/3$ and $\Gamma_- = \mathbb{Z}/4$ there are 28 possibilities (up to symmetries).

Once we have chosen a torus isometry $\mathfrak{t}$, we need to find blocks (Z_+, Σ_+) and (Z_-, Σ_-) with those automorphism groups and diffeomorphism

$$\mathfrak{r} : \Sigma_+ \to \Sigma_-$$

satisfying a condition that depends only on the gluing angle ϑ of $\mathfrak{t}$, and on the periods and Kähler cones of $\Sigma_\pm$.
This works essentially because by the Calabi-Yau theorem a torsion-free $SU(n)$ structure is completely determined by the holomorphic n-form and the Kähler class, and the asymptotic limits of the torsion-free G_2-structures on $\mathbb{R} \times T^2_\pm \times \Sigma_\pm$ that need to be matched by $(-1) \times \mathfrak{t} \times \mathfrak{r}$ are determined by the torsion-free $SU(2)$-structure on $\Sigma_\pm$.
The topology of the resulting extra-twisted connected sum G_2-manifold depends not only on the topology of Z_+ and Z_- and on $\mathfrak{t}$, but also on in particular the action $\mathfrak{r}^* : H^2(\Sigma_-) \to H^2(\Sigma_+)$.

The matching problem and configurations

Call the image $N_\pm$ of $H^2(Z_\pm) \to H^2(\Sigma_\pm)$ the *polarising lattice* of $(Z_\pm, \Sigma_\pm)$ (because $\Sigma_\pm$ is always N-polarised: $N \subseteq \operatorname{Pic}\Sigma_\pm := H^{1,1}(\Sigma_\pm;\mathbb{R}) \cap H^2(\Sigma_\pm)$).

Then $\mathfrak{r}^*$ determines a pair of embeddings of N_+ and N_- into the K3 lattice $L := 3H \oplus 2(-E_8)$ (up to $O(L)$), which we call the *configuration* of $\mathfrak{r}$.

It is useful to set up the matching problem as

Given ϑ, a pair of deformation families of blocks and a configuration $N_+, N_- \hookrightarrow L$ of their polarising lattices, do there exist some members $(Z_\pm, \Sigma_\pm)$ of the families with a ϑ-matching $\mathfrak{r} : \Sigma_+ \to \Sigma_-$ realising that configuration?

because

- if there is such a matching then we can work out a lot of the topology of the resulting extra-twisted connected sum
- the Torelli theorem can be used to answer it, provided that we know enough about which N-polarised K3 surfaces appear as an anticanonical divisor in some block in the chosen families

A building block with $\mathbb{Z}/4$ action

Let

- $Q \subset \mathbb{P}^3$ any quartic K3 surface.
- $Y \to \mathbb{P}^3$ the fourfold cover branched over Q.
- $C \subset Q$ a hyperplane section ($=$ divisor of normal bundle of Q in Y).
- $Z \to Y$ the blow-up along C
- $\Sigma \subset Z$ the proper transform of Q (which is isomorphic to Q).

The blow-up ensures that the normal bundle of Σ in Z is trivial.
The deck transformation action of $\mathbb{Z}/4$ on Y lifts to Z.

We can understand quite precisely

- topology of (Z, Σ), *eg* $H^*(Z)$, $c_2(Z)$. The image N of $H^2(Z) \to H^2(\Sigma)$ is generated by a primitive element $x \in H^2(\Sigma)$ with $x^2 = 4$.
- which K3 surfaces Σ appear as anticanonical divisors in this family of blocks, namely any smooth quartic K3 (or equivalently any K3 with an ample class $x \in H^2(\Sigma)$ such that $x^2 = 4$ and no $v \in \operatorname{Pic}\Sigma$ such that $x.v = 0$ and $v^2 = -2$ or $x.v = 2$ and $v^2 = 0$).

Survey of “low-hanging fruit”

Using 25 similar blocks with involution (and rk $N \leq 2$) and 6 blocks with automorphisms of order 3 to 6 (and rk $N = 1$), one can make (at least)

- 305 matchings using only involutions blocks

  In many cases classifying diffeomorphism invariants can be worked out completely, and used to exhibit disconnected G_2 moduli space.

  $\bar{\nu}$ is always divisible by 3.

- 192 matchings using at least one block with automorphism of order ≥ 3 (thanks to greater variety of choices for the torus isometry)

  Greater variety in

  - topology, *eg* can get fundamental groups of order 2, 3, 4, 5, 6, 7, 8, 9, 10, 15 and 21 (but harder to work out full invariants)
  - values of $\bar{\nu}$ realised.

4. Computing $\bar{\nu}$
The gluing formula

Extra-twisted connected sums lend themselves naturally to "neck-stretching" formulas for eta invariants, *eg* of **Kirk-Lesch** and **Bunke-Ma**.

As $\ell \to \infty$ for an extra-twisted connected sum M_ℓ with neck length ℓ,

$$\bar{\nu}(M_\ell) \;\to\; \bar{\nu}(M_+, b_+) + \bar{\nu}(M_-, b_-) + m_{\bar{\nu}}(b_+, b_-)$$

where $\bar{\nu}(M_\pm, L_\pm)$ is a suitable interpretation of the invariant for ACyl G_2-manifolds involving boundary conditions $b_\pm$.

The inclusion of the "Maslov index" term m enables the boundary conditions to be different on the two halves.

Since $\bar{\nu}$ is invariant under deformations, $\bar{\nu}(M_\ell)$ is independent of ℓ, so actually

$$\bar{\nu}(M_\ell) \;=\; \bar{\nu}(M_+, b_+) + \bar{\nu}(M_-, b_-) + m_{\bar{\nu}}(b_+, b_-)!$$

Eta invariants on the halves

The boundary conditions $b_\pm$ amount to choices of Lagrangian subspaces of the spaces of harmonic spinors and harmonic forms on the asymptotic cross-section $T^2_\pm \times \Sigma_\pm$.

This determines function spaces on which the Dirac operator $\not{D}_\pm$ and odd signature operator $B_\pm$ act self-adjointly, so that their eta invariants are defined, and we can set

$$\bar{\nu}(M_\pm, b_\pm) := 3\eta(B_\pm) - 24\eta(\not{D}_\pm).$$

For $M_\pm := (S^1_{ext} \times V_\pm)/\Gamma_\pm$ as in the extra-twisted connected sum construction, there is natural choice for the boundary conditions (involving the distinguished direction of S^1_{ext} in the boundary) that allows the computation to be reduced to $V_\pm$.

If $\Gamma_\pm$ is trivial or $\mathbb{Z}/2$ then $M_\pm$ has an orientation-reversing involution from reflection of the S^1_{ext} factor. This distinguished $b_\pm$ is preserved by this involution, so the spectrum of $\not{D}_\pm$ and $B_\pm$ is symmetric. Thus

$$\bar{\nu}(M_\pm) = 0 \text{ if } |\Gamma_\pm| \leq 2 \ !$$

The Maslov index

There are neck-stretching formulas for each of $\eta(\not{D}_{M_\ell})$ and $\eta(B_{M_\ell})$.

The Maslov index term in $\lim_{\ell\to\infty}\eta(\not{D}_{M_\ell})$ turns out to always equal $\frac{\rho}{\pi}$, where $\rho := \pi - 2\vartheta$.

The Maslov index term m_B for $\eta(B)$ is more interesting.

Let $R_\pm : H^2(\Sigma;\mathbb{R}) \to H^2(\Sigma;\mathbb{R})$ be the reflection in the polarising lattice $N_\pm := \mathrm{Im}(H^2(V_\pm) \to H^2(\Sigma))$, and define a unitary operator $U : H^2(\Sigma;\mathbb{C}) \to H^2(\Sigma;\mathbb{C})$ as $e^{\pm i\rho}R_+R_-$ on $H^{2,\pm}(\Sigma)$. Then

$$m_B = \frac{1}{\pi}\sum_{\substack{\lambda\in\mathrm{Spec}\,U\\ \lambda\neq -1}} \arg\lambda$$

where the branch of arg takes values in $(-\pi,\pi)$.

In particular, this depends only on the configuration of the polarising lattices!

$\bar{\nu} = 0$ for twisted connected sums

We have defined $U := e^{\pm i\rho} R_+ R_-$ on $H^{2,\pm}(K3; \mathbb{C})$.

If $\vartheta = \frac{\pi}{2}$ then $\rho = \pi - 2\vartheta = 0$, and U is the real orthogonal map $R_+ R_-$. Hence eigenvalues are ± 1 or occur in conjugate pairs, so $\sum \arg \lambda = 0$, and

$$m_{\bar{\nu}} = 0.$$

Since we already explained that $\bar{\nu}(M_\pm) = 0$ if $\Gamma_\pm$ are trivial, this proves that any ordinary twisted connected sum has

$$\bar{\nu} = 0.$$

Maslov index from configuration angles

We have defined $U := e^{\pm i\rho} R_+ R_-$ on $H^{2,\pm}(K3; \mathbb{C})$.

In general

$$\sum_{\substack{\lambda \in \mathrm{Spec}\, U \\ \lambda \neq -1}} \arg \lambda \;=\; \sum \pm\rho \;+\; \sum_{\substack{\lambda \in \mathrm{Spec}\, R_+ R_- \\ \lambda \neq -1}} \arg \lambda \;+\; \pi\, k \;=\; -16\rho + \pi\, k,$$

where $k \in \mathbb{Z}$ counts "half branch jumps" between λ and $e^{\pm i\rho}\lambda$. Then

$$m_{\bar{\nu}} = -72\frac{\rho}{\pi} + 3k.$$

If both $|\Gamma_\pm| \leq 2$, then the combinatorics of torus isometries forces ϑ to be an integer multiple of $\frac{\pi}{6}$, so $m_{\bar{\nu}}$ (and hence $\bar{\nu}$) is divisible by 3.

If one $|\Gamma_\pm| \geq 3$ then $m_{\bar{\nu}}$ need not even be rational!

Computing $\bar{\nu}$ of the ACyl pieces

$\bar{\nu}(M_\pm, b_\pm)$ can be computed as a sum of two contributions.

If one rescales $S^1_{\pm,ext}$ by a factor a, then

- $\lim_{a\to 0} \bar{\nu}(M^a_\pm, b_\pm) \in \mathbb{Q}$ depends only on the isolated fixed points of $\Gamma_\pm$ in $V_\pm$ (none if $\Gamma_\pm = \mathbb{Z}/2$).
- the variation $\displaystyle\int_0^1 \frac{d\bar{\nu}(M^a_\pm, b_\pm)}{da} da$ has been written by **Zagier** in terms (involving the Dedekind eta function) of the geometry of $T^2_\pm$, in particular the ratio of the circumferences of S^1_{ext} and S^1_{int}.

It seems magical at first that if the contribution $-24\frac{\rho}{\pi}$ in $m_{\bar{\nu}}$ is irrational then that must be compensated for by the variational terms.

It can be explained by thinking of the variational terms as integrals in the upper halfplane of a predifferential of the hyperbolic area 2-form, along two geodesics edges that meet at angle 2ϑ.