Beijing - Novosibirsk seminar on geometry and mathematical physics

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Introduction to Gromov - Witten theory

M. Kazarian (HSE & Skoltech)

GW invariants

M complex projective manifold $\qquad \dim_{\mathbb{C}} M = m$

$H = H^{ev}(M, \mathbb{C})$

$a_1, \dots, a_n \in H \qquad a_i = P.D.(A_i) \qquad A_i \subset M$

$d \in H_2(M, \mathbb{Z})$

$GW_{g,n}(M; a_1, \dots, a_n; d) = \Big|$ # curves $C \subset M$ of genus g with n marked points $x_1, \dots, x_n$ such that $x_i \in A_i$ for $i = 1, \dots, n$ and $[C] = d$

$$\boxed{\sum \deg a_i = 2(m + c_1(d) + 3g - 3 + n)}$$

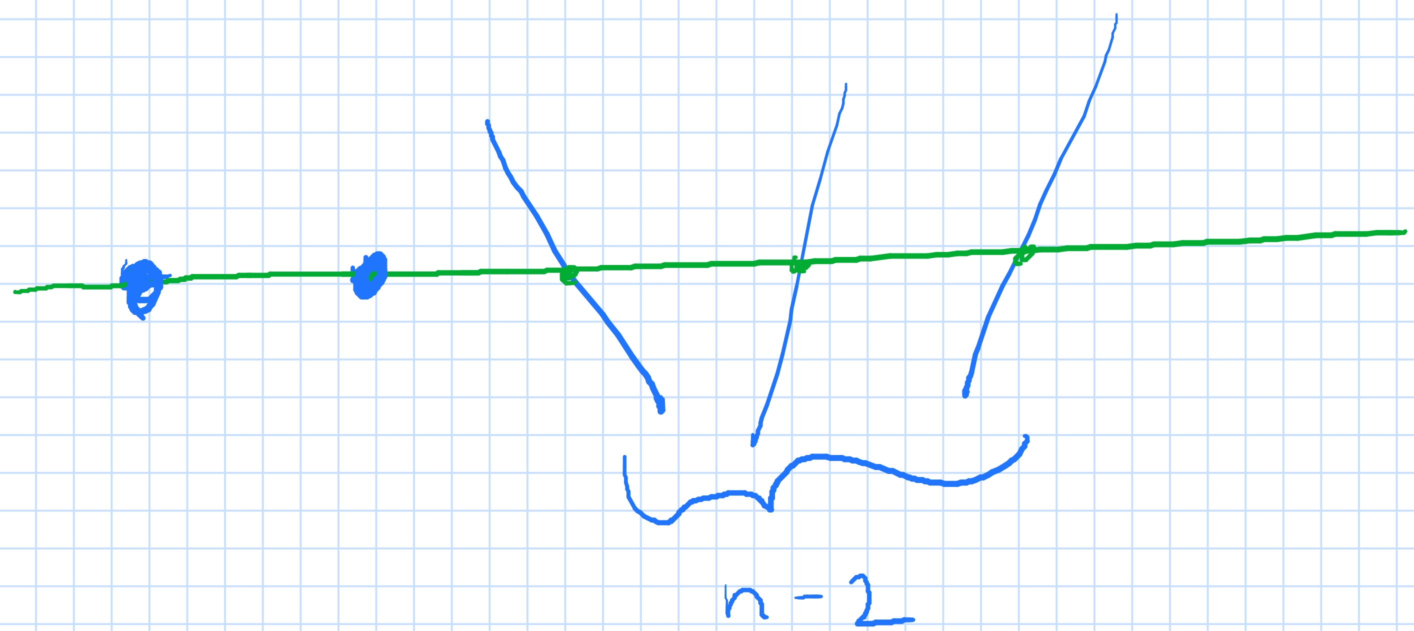
Expected to be a finite number (?!)

If the dimension condition is not satisfied, we set GW invariant to be equal to 0

Example $\quad M = \mathbb{C}P^2 \qquad \ell = [\mathbb{C}P^1]$

$g=0, \; d=1$

$$GW_{0,n,1}(\mathbb{C}P^2; pt, pt, \overbrace{\ell, \dots, \ell}^{n-2}) = 1$$



$d>1$: $N_d = \#$ of rational plane curves through a given collection of $3d-1$ points

Quantum multiplication in cohomology

$H = H^{ev}(M, \mathbb{C})$ $\qquad (\, , \,)$ Poincaré pairing

define $* : H \otimes H \to H$ by

$$(a * b , c) = \sum_d GW_{0,3}(M; a, b, c)\, q^d$$

$(a \cdot b, c)$ counts intersection points $A \cap B \cap C$

$(a * b, c)$ counts "quantum" intersection points: rational curves intersecting A, B and C

$$\boxed{\text{\underline{Thm} quantum multiplication is commutative and associative}}$$

<u>Example</u> $H^*(\mathbb{CP}^n, \mathbb{C}) = \mathbb{C}[p] / (p^{n+1})$ $\qquad QH^*(\mathbb{CP}^n, \mathbb{C}) = \mathbb{C}[p] / (p^{n+1} - 1)$

<u>Degree</u>

$\operatorname{rk} H_2(M) = e$

$R = \mathbb{Q}[q_1^{\pm 1}, \dots, q_e^{\pm 1}]$

$$d = (d_1, \dots, d_e) \rightsquigarrow q^d = q_1^{d_1} \cdots q_e^{d_e}$$

$$GW_{g,n}(M; a_1, \dots, a_n) = \sum_{d_1 \dots d_e} GW_{g,n,d}(M, a_1, \dots, a_n) \, q^d \in R$$

<u>Remark</u>. In examples, we have $\ell = 1$, so we can set $q = 1$ and recover d from dimension restrictions

An equivalent definition:

define $e_{g,n}(a_1 \dots a_n) \in H^*(\overline{\mathcal{M}}_{g,n})$ such that

$$GW_{g,n}(M; a_1, \dots, a_n) = \int_{\overline{\mathcal{M}}_{g,n}} e_{g,n}(a_1, \dots, a_n)$$

$\overline{\mathcal{M}}_{g,n}(M, d)$ moduli space of stable maps

$\{(C; x_1, \dots, x_n; f)\}$

$(C; x_1, \dots, x_n)$ genus g curve with marked pts

$f : C \to M$ holomorphic map

$$\begin{tikzcd}
 & \overline{\mathcal{M}}_{g,n}(M,d) \arrow[dl, "p"'] \arrow[d, "ev_1"] \arrow[dr, "ev_n"] & \\
\overline{\mathcal{M}}_{g,n} & M \quad \cdots & M
\end{tikzcd}$$

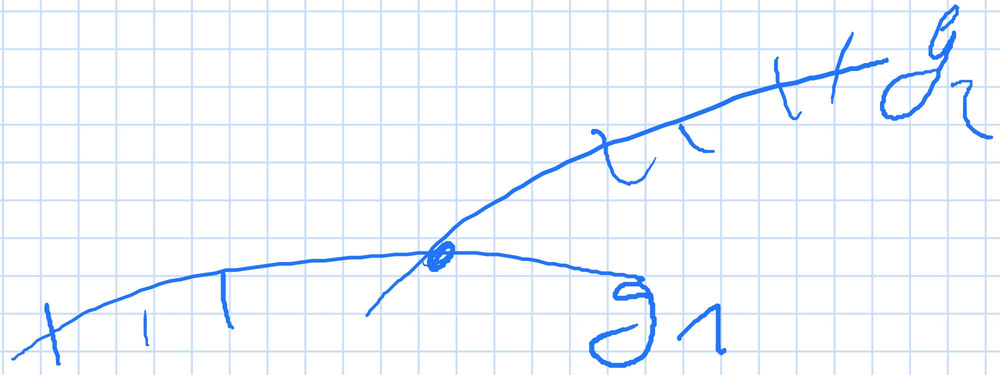
$$ev_i((C; x_1, \dots, x_n; f)) = f(x_i)$$

$$e_{g,n,d}(a_1 \dots a_n) = p_*\left(ev_1^* a_1 \cup \dots \cup ev_n^* a_n \cap [\overline{\mathcal{M}}_{g,n}(M,d)]^{vir}\right)$$
$$\in H^*(\overline{\mathcal{M}}_{g,n})$$

$$e_{g,n}(a_1 \dots a_n) = \sum_d e_{g,n,d}\, q^d \in H^*(\overline{\mathcal{M}}_{g,n}, R)$$

$e_{g,n}$ form a cohomological field theory (CohFT)

H vector space η_{ij} nondegenerate symmetric bilinear form
$\mathbb{1} \in H$ distinguished element



$$e_{g,n} : H \otimes \cdots \otimes H \longrightarrow H^*(\overline{\mathcal{M}}_{g,n})$$

- S_n-equivariant
- $p : \overline{\mathcal{M}}_{g,n+1} \longrightarrow \overline{\mathcal{M}}_{g,n}$ $\qquad p^* e_{g,n}(a_1 \dots a_n) = e_{g,n+1}(a_1, \dots, a_n, \mathbb{1})$
- $j : \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}$

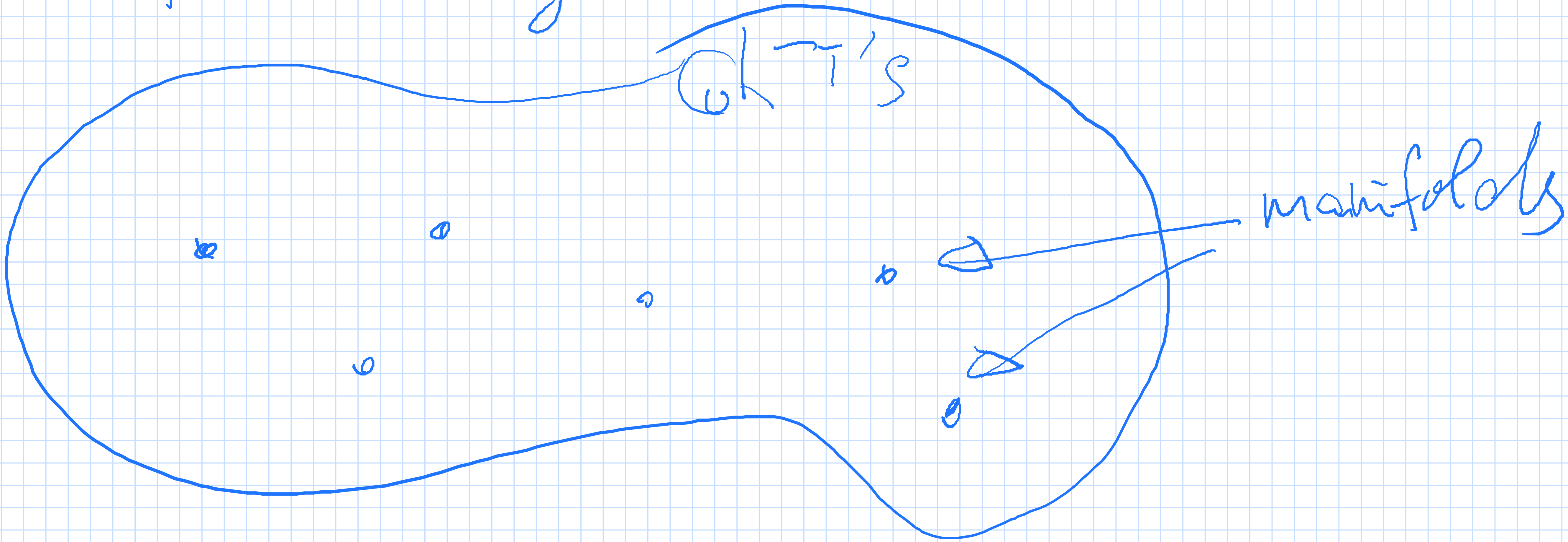
$$j^* e_{g_1+g_2,n_1+n_2}(a_1 \dots a_{n_1}, a'_1 \dots a'_{n_2}) = \sum_{i=1}^{\dim H} e_{g_1,n_1+1}(a_1, \dots, a_{n_1}, e_i)\, e_{g_2,n_2+1}(a'_1 \dots a'_{n_2}, e^i)$$

where $\{e_i\}$ and $\{e^i\}$ are dual bases wrt η

- similar property for the gluing map

$$\overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}$$

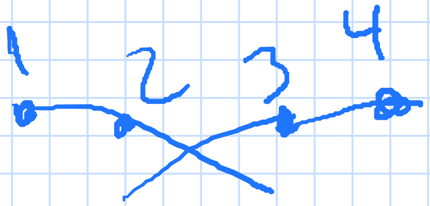
Most of the structures in GW theory are defined for arbitrary CohFT



<u>Example</u>: algebra structure on H (quantum multiplication)

$$(a * b, c) = \int_{\overline{\mathcal{M}}_{0,3}} c_{0,3}(a,b,c) \qquad \overline{\mathcal{M}}_{0,3} = pt$$

Associativity: $\Longleftrightarrow$ equality of classes of divisors



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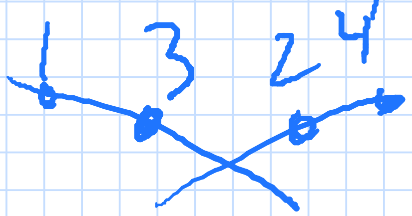

$\overline{\mathcal{M}}_{0,4} \cong \mathbb{CP}^1$ (both represent a point)

(Small) GW potential and Frobenius manifold structure

$\Phi : H \to \mathbb{C}$

$e_1 = 1, \dots, e_N$ basis in H

$$\Phi(t_1 \dots t_N) = \sum_{n=3}^{\infty} \frac{1}{n!} \sum_{k_1, \dots k_n = 1}^{N} \underbrace{\int_{\overline{\mathcal{M}}_{0,n}} e_{0,n}(e_{k_1}, \dots e_{k_n})}\, t_{k_1} \dots t_{k_n}$$

$\dim_{\mathbb{R}} \overline{\mathcal{M}}_{0,n} = 2(n-3)$

$$\eta_{ij} = (e_i, e_j) \equiv \frac{\partial^3 \Phi}{\partial t_1 \partial t_i \partial t_j} = \text{const.}$$

Multiplication on $T_t H \cong H$ for each $t \in H$, $e_i = \frac{\partial}{\partial t_i}$

$$(e_i *_t e_j, e_k) = \frac{\partial^3 \Phi}{\partial t_i \partial t_j \partial t_k}$$

Theorem t-deformed multiplication is commutative and associative

Associativity (WDVV) equation:

$$\boxed{\frac{\partial^3 \Phi}{\partial t_a \partial t_b \partial t_i}\, \eta^{ij}\, \frac{\partial^3 \Phi}{\partial t_j \partial t_c \partial t_d} = \frac{\partial^3 \Phi}{\partial t_a \partial t_c \partial t_i}\, \eta^{ij}\, \frac{\partial^3 \Phi}{\partial t_j \partial t_b \partial t_d}}$$

$$\Big(\Longleftrightarrow \quad ((e_a * e_b) * e_c, e_d) = ((e_a * e_c) * e_b, e_d) \Big)$$

<u>Example</u> (Kontsevich) $M = \mathbb{C}P^2$ $\quad \dim H^*(\mathbb{C}P^2) = 3$ $\quad \eta = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

$$\Phi = \frac{1}{2}\left(t_0 t_1^2 + t_0^2 t_2\right) + \sum_{d=1}^{\infty} N_d \frac{t_2^{3d-1}}{(3d-1)!} e^{t_1} \qquad \mathbb{1} = \frac{\partial}{\partial t_0}$$

$$\text{Associativity} \Rightarrow N_d = \sum_{k+\ell=d} N_k N_\ell \left(k^2 \ell^2 \binom{3d-4}{3k-2} - k^3 \ell \binom{3d-4}{3k-1} \right), \quad d \geq 2$$

n	1	2	3	4	5
N_d	1	1	12	620	87304

N_d = # rat curves of degree d through $3d-1$ pts

GW ancestor potential of a CohFT

encodes information on the classes $e_{g,n}$ of not necessarily top degree

t_{ik} $\quad i=1,\dots N \quad k=0,1,2,\dots$

$$F=\sum_{\substack{n\geq 0\\ g\geq 0}}\frac{\hbar^g}{n!}\sum_{\substack{i_1\dots i_n\\ k_1\dots k_n}}\int_{\overline{\mathcal{M}}_{g,n}}\psi_1^{k_1}\dots\psi_n^{k_n}\,\underbrace{e_{g,n}(e_{i_1},\dots,e_{i_n})\prod_{j=1}^{n}}\,t_{i_jk_j}$$

$$\psi_i\in H^2(\overline{\mathcal{M}}_{g,n})$$

Givental group action

Consider the space of all CohFT's with the fixed H, $(,)$, $\mathbb{1}$, and the fixed algebra structure at the origin

<u>Definition</u> The Givental's twisted loops group is formed by formal series

$$R(z) = 1 + R_1 z + R_2 z^2 + \dots, \quad R_k \in \mathrm{End}(H)$$

such that $R(z) R^T(-z) = \mathbb{1}$

The corresponding Lie algebra is formed by the series

$$r(z) = r_1 z + r_2 z^2 + \dots \qquad r_k \in \mathrm{End}(H)$$

such that $r(z) + r^T(-z) = 0$

The Givental group is acting on CohFT's

$N=1$

$$r_k z^k \rightsquigarrow r_k \frac{\partial}{\partial t_{k+1}} - \sum_{i=0}^{\infty} r_k t_i \frac{\partial}{\partial t_{k+i}} + \frac{\hbar}{2} \sum_{i+j=k-1} (-1)^i \frac{\partial^2}{\partial t_i \partial t_j}$$

acting on $e^{\frac{1}{\hbar}F}$

The action is given by an explicit formula expressing the modified classes $e_{g,n}$

<u>Example.</u> If $e_{g,n}$ is a CohFT, then $c(\Delta) e_{g,n}$ is also a CohFT

<u>Definition</u> TopFT is a CohFT taking values in $H^0(\dots) = \mathbb{C}$

TopFT $\Leftrightarrow$ algebra structure on $H \Leftrightarrow$ Frobenius structure is constant

$\Leftrightarrow \Phi$ is a qubic polynomial

The Givental group preserves the Top component of the CohFT

The TopFT is called semisimple if it is diagonalizable in some basis

The CohFT is called semisimple if its Top component is semisimple

Thm (Givental, Teleman) Any semisimple Frobenius structure is associated with certain CohFT.

All CohFT's with fixed Top component form one orbit of the Givental group action, therefore, it can be obtained from the corresponding TopFT by a suitable transformation

Given CohFT $\rightsquigarrow$ take $g=0$ component $\rightsquigarrow$ take the associated Frobenius struct

$\rightsquigarrow$ Apply Givental's algorithm to find $R(z)$ $\rightsquigarrow$ Apply $R(z)$ to TopFT $\rightsquigarrow$ The result is the original CohFT

Corollary

In particular, the $g>0$ components of a (semisimple) CohFT are determined by its $g=0$ component