Deformed Donaldson-Thomas connections I

Kotaro Kawai
(joint with Hikaru Yamamoto)

Gakushuin University

For Calabi-Yau 3-folds,

- Strominger–Yau–Zaslow (SYZ conjecture):
  mirror symmetry of Calabi-Yau 3-folds would be explained in terms of special Lagrangian (SL) dual T^3-fibrations (including singular fibers).

$$
\begin{tikzcd}
X^6 \arrow[dr, "f"'] & & (X^6)^* \arrow[dl, "f^*"] \\
& B^3 &
\end{tikzcd}
$$

  For generic $b \in B$, $f^{-1}(b)$ and $(f^*)^{-1}(b)$ are "dual" SL T^3.

- Leung–Yau–Zaslow:
  If a SL dual T^3-fibration is given, SL submanifolds correspond to deformed Hermitian Yang–Mills (dHYM) connections via the real Fourier–Mukai transform.

By the similarity between Calabi-Yau 3-folds and G_2-manifolds ($\mathrm{SU}(3) \subset G_2$), we can consider analogous statements.

- Gukov–Yau–Zaslow:
  mirror symmetry of G_2-manifolds would be explained in terms of coassociative dual fibrations (including singular fibers).

$$\begin{tikzcd} X^7 \arrow[dr, "f"'] & & (X^7)^* \arrow[dl, "f^*"] \\ & B^3 & \end{tikzcd}$$

  For generic $b \in B$, $f^{-1}(b)$ and $(f^*)^{-1}(b)$ are "dual" coassociative T^4 or K3 surfaces.

- Lee–Leung:
  If a coassociative dual T^4-fibration is given, associative submanifolds correspond to deformed Donaldson–Thomas (dDT) connections via the real Fourier–Mukai transform.

$$\begin{tikzcd}[column sep=small] X^7 \arrow[dr, "f"'] & & (X^7)^* \arrow[dl, "f^*"] \\ & B^3 & \end{tikzcd}$$

The difference from the Calabi-Yau case: smooth fibers of fibrations can be T^4 or K3 surfaces.

In this talk, we focus on the T^4-fibrations.

- Remark for K3 fibrations:
  - By the study of K3 fibrations, it will be very interesting if we can observe phenomenons that cannot seen in the Calabi-Yau case.
  - "adiabatic limit" (volume of fiber $\to 0$) of coassociative K3 fibrations: intensively studied by Donaldson (Yang Li).
    - By taking the limit, the problems for G_2-manifolds will be simplified (more tractable). Solving these simplified problems, we may solve the original problems.

Definition

- X^7: a manifold with a G_2-structure $\varphi \in \Omega^3$,
- $(L, h) \to X$: a smooth complex Hermitian line bundle.

A Hermitian connection ∇ of (L, h) is called a deformed Donaldson–Thomas (dDT) connection (deformed G_2-instanton) if

$$\frac{1}{6} F_\nabla^3 + F_\nabla \wedge *\varphi = 0,$$

where $F_\nabla \in \sqrt{-1}\Omega^2$ is a curvature of ∇.

- As the name indicates, the dDT connection can also be considered as an analogue of the G_2-instanton (Donaldson–Thomas connection): $F_\nabla \wedge *\varphi = 0$.
- We expect that dDT connections will have similar properties to associative submanfolds and G_2-instantons.

$$
\begin{tikzcd}
& \{\text{dDT connections}\} \arrow[dl, leftrightarrow, "mirror"'] \arrow[dr, leftrightarrow, "analogue"] & \\
\left\{\begin{array}{c}\text{associative}\\ \text{submanifolds}\end{array}\right\} & & \{G_2\text{-instantons}\}
\end{tikzcd}
$$

We expect

- to study dDT connections using results of associative submanifolds and G_2-instantons.
- conversely, to study associative submanifolds and G_2-instantons by developing the theory for dDT connections. ($\rightsquigarrow$ lead to some breakthroughs?)

The real Fourier–Mukai transform

Now, we consider the real Fourier–Mukai transform for a T^4-fibration. Very roughly,

$$\text{real FM}: \{\text{Submanifolds}\} \longrightarrow \{\text{Connections}\}\,.$$

$$\text{real FM}: \left\{ \begin{array}{c} \text{associative} \\ \text{submanifolds} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{dDT} \\ \text{connections} \end{array} \right\}.$$

It is enough to consider the following case:

$$X = B^3 \times T^4, \qquad X^* = B^3 \times (T^4)^*,$$

where $B^3 \subset \mathbb{R}^3$ is an open set and

$$T^4 = \mathbb{R}^4/2\pi\mathbb{Z}^4, \qquad (T^4)^* = (\mathbb{R}^4)^*/2\pi(\mathbb{Z}^4)^*$$

and $(\mathbb{Z}^4)^* = \{\alpha \in (\mathbb{R}^4)^* \mid \langle \alpha, v\rangle \in \mathbb{Z} \text{ for } \forall v \in \mathbb{Z}^4\}$.

The idea of the real Fourier–Mukai transform:

1. For each $a \in T^4$, assign a connection ∇^a of $(T^4)^* \times \mathbb{C} \to (T^4)^*$.
2. Doing this for each fiber, we get

$$\left\{\text{graphs } B^3 \to T^4\right\} \longrightarrow \left\{\text{connections of } X^* \times \mathbb{C} \to X^*\right\}.$$

The correspondence (1) is given by the following identification:

$$\begin{aligned} T^4 &= \mathbb{R}^4/2\pi\mathbb{Z}^4 \\ &\cong H^1((T^4)^*, \mathbb{R})/2\pi H^1((T^4)^*, \mathbb{Z}) \\ &\cong \{\text{flat Hermitian connections of } (T^4)^* \times \mathbb{C} \to (T^4)^*\}/\text{gauge} \end{aligned}$$

Explicitly,

$$(a^1, \cdots, a^4) \mapsto \left[\sum_{j=1}^4 a^j dy^j\right] \mapsto \left[d + \sqrt{-1}\sum_{j=1}^4 a^j dy^j\right],$$

where $(y^1, \cdots, y^4)$ are coordinates on $(T^4)^*$.

(2) Given a graph $f = (f^1, \cdots, f^4) : B^3 \to T^4$, we obtain a Hermitian connection $\nabla := \left\{\nabla^{f(x)}\right\}_{x \in B^3}$:

$$\nabla = d + \sqrt{-1} \sum_{j=1}^{4} f^j(x) dy^j.$$

We call ∇ （up to gauge） the real Fourier–Mukai transform of $\mathrm{graph}(f)$.
The curvature $F_\nabla = \sqrt{-1} \sum_{j=1}^{4} df^j \wedge dy^j$ is independent of gauge.
By this explicit correspondence, a condition on graph f (say, being associative) is described in terms of F_∇.

$$\mathrm{graph}(f) : \text{associative} \iff \nabla : \mathrm{d}DT.$$

The real Fourier–Mukai transform can be applied only for torus fiber bundles, but it predicts many interesting geometric objects and properties. In the next talk, I will show that some predictions by the real Fourier-Mukai transform actually hold.

Calibrated geometry in G_2-manifolds

We list some properties of associative submanifolds, (which turn out to be true for dDT connections).

Definition (Harvey-Lawson, 1982)

Let (X^n, g) be a Riemannian manifold and $\xi \in \Omega^k(X)$ with $d\xi = 0$. ξ is called a calibration if for every oriented k-dim. submanifold N

$$\xi|_N \leq \mathrm{vol}_N. \qquad \left(\Leftrightarrow \begin{array}{c} \xi(e_1, \cdots, e_k) \leq 1 \\ \text{for oriented o.n.b. } \{e_i\} \text{ of } T_x N (\forall x \in N). \end{array} \right)$$

N is called a calibrated submanifold (ξ-submanifold) if $\xi_N = \mathrm{vol}_N$.

Lemma

Every compact calibrated submanifold N is volume-minimizing in its homology class. The volume is given topologically ($[\xi] \cdot [N]$).

Lemma

Every compact calibrated submanifold N^k is volume-minimizing in its homology class. The volume is given topologically $([\xi] \cdot [N])$.

Suppose that N' is any compact k-submanifold of X with $[N'] = [N] \in H_k(X, \mathbb{R})$. Then,

$$\mathrm{Vol}(N) = \int_N \mathrm{vol}_N = \int_N \xi = \int_{N'} \xi \leq \int_{N'} \mathrm{vol}_{N'} = \mathrm{Vol}(N').$$

Lemma

Let X^7 be a manifold with a G_2-structure $\varphi \in \Omega^3$ with $d\varphi = 0$. Then, φ is a calibration. The corresponding calibrated submanifolds are called associative submanifolds.

This follows from the associator equality.

Set $V = \mathbb{R}^7$. Define $\chi \in \Lambda^3 V^* \otimes V$ by

$$g(\chi(v_1, v_2, v_3), v_4) = *\varphi(v_1, v_2, v_3, v_4) \qquad \text{for } v_j \in V.$$

Lemma (associator equality)

$$|\varphi(v_1, v_2, v_3)|^2 + |\chi(v_1, v_2, v_3)|^2 = |v_1 \wedge v_2 \wedge v_3|^2 \qquad \text{for } \forall v_j \in V.$$

It is enough to prove when $\{v_1, v_2, v_3\}$ is orthonormal. By the G_2-action, we may assume that $v_1 = e_1, v_2 = e_2, v_3 = ke_3 + \ell e_4$ with $k^2 + \ell^2 = 1$. ($\{e_i\}$: std basis of V with dual $\{e^i\}$.) Since

$$\varphi = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356},$$

$$*\varphi = e^{4567} + e^{2367} + e^{2345} + e^{1357} - e^{1346} - e^{1256} - e^{1247},$$

we have

$$\varphi(v_1, v_2, v_3) = k, \quad \chi(v_1, v_2, v_3) = -\ell e_7, \quad v_1 \wedge v_2 \wedge v_3 = ke^{123} + \ell e^{124}.$$

Hence, for an oriented 3-dim. vector space W, we have

$$\varphi|_W \leq \mathrm{vol}_W, \qquad \varphi|_W = \pm \mathrm{vol}_W \Leftrightarrow \chi|_W = 0.$$

We can define $\chi \in \Omega^3(X, TX)$ for a manifold X^7 with G_2-structure φ.

- An oriented 3-dim. submanifold N is associative with an appropriate orientation $\Leftrightarrow \chi|_N = 0$.

If fact, if X^7 is a manifold with a G_2-structure $\varphi \in \Omega^3$ with $d * \varphi = 0$, $*\varphi$ is also a calibration. The corresponding calibrated submanifolds are called coassociative submanifolds.

- There is a "coassociator equality", and
- an oriented 4-dim. submanifold N is coassociative with an appropriate orientation $\Leftrightarrow \varphi|_N = 0$.

Deformation theory

Let (X^7, φ) be a G_2-manifold and $L^3 \subset X^7$ be a compact associative submanifold. Denote by $\nu \to L$ the normal bundle of $L \subset X$.
Since $\exp : \nu \to X$ is a diffeomorphism near the zero section,

$$\exp : \Gamma(L, \mathcal{U}) := \{\text{small sections of } \nu\} \cong \{\text{submanifolds close to } L\}.$$

For $V \in \Gamma(L, \mathcal{U})$, Set $\exp_V : L \to X$ by $\exp_V(x) = \exp_x(V_x)$.

Define $F : \Gamma(L, \mathcal{U}) \to \Omega^3(L, \nu) \cong \Gamma(L, \nu)$ by

$$F(V) = \text{"}\phi_V\text{"}(exp_V^* \chi),$$

(where ϕ_V is a correction bundle isomorphism.)

$$\exp_V(L) \text{ :associative} \Leftrightarrow F(V) = 0 \quad \text{(:1st order nonlinear PDE)}.$$

$$\{\text{associative moduli}\} \overset{locally}{\cong} F^{-1}(0).$$

The linearization of F at 0

$$D = (dF)_0 : \Gamma(L, \nu) \to \Omega^3(L, \nu) \cong \Gamma(L, \nu)$$

is described explicitly (using local frame).

- [McLean] D is elliptic, self-adjoint $\Rightarrow \ker D = \mathrm{Coker} D$ $\Rightarrow$ expected dim. of the associative moduli is 0.

Proposition

The moduli space of associative submanifolds is 0-dimensional manifold if we perturb the G_2-structure.

- If we can compactify the moduli space, it will be a finite set. By "counting" them, can we find a invariant of G_2-manifolds?
- To define an enumerative invariant, it is believed that the moduli space needs to be oriented.
- The moduli space of associative submanifolds is shown to be oriented by the "flag structure" [Joyce].

- The behavior of associative submanifolds is similar to that of holomorphic curves.
- The behavior of coassociative submanifolds is similar to that of SL submanifolds. The moduli space of coassociative (resp. SL) submanifolds C (resp. L) is always a smooth manifold of dim. $b^2_+(C)$ (resp. $b^1(L)$).

Holonomy reduction

Let $(Y^6, \omega, g, J, \Omega)$ be a Calabi-Yau 3-manifold. Then, $X^7 := S^1 \times Y^6$ admits a G_2-structure φ given by

$$\varphi = dx \wedge \omega + \mathrm{Re}\Omega.$$

- $\{*\} \times L$ for SL submanifolds $L \subset Y^6$ $(\mathrm{Re}\Omega|_L = \mathrm{vol}_L)$
- $S^1 \times \Sigma$ for holomorphic curves $\Sigma \subset Y^6$ $(\omega|_\Sigma = \mathrm{vol}_\Sigma)$

are associative submanifolds.

Lemma

1. *Fix $\gamma \in H_3(Y^6) \hookrightarrow H_3(X^7)$. Every associative submanifold in $X^7 := S^1 \times Y^6$ representing the class γ is of the form $\{*\} \times L$ with $L \subset Y^6$ a SL submanifold.*
2. *Fix $\beta \in H_2(Y^6)$. Every associative submanifold in $X^7 := S^1 \times Y^6$ representing the class $[S^1] \times \beta$ is of the form $S^1 \times \Sigma$ with $\Sigma \subset Y^6$ a holomorphic curve.*

We prove only (1): Fix $\gamma \in H_3(Y^6) \hookrightarrow H_3(X^7)$. Every associative submanifold in $X^7 := S^1 \times Y^6$ representing the class γ is of the form $\{*\} \times L$ with $L \subset Y^6$ a SL submanifold.

Let $A^3 \subset X^7$ be associative with $[A] = \gamma \in H_3(Y^6) \hookrightarrow H_3(X^7)$.

$$\int_A \mathrm{Re}\Omega = \int_\gamma \mathrm{Re}\Omega = \int_\gamma (dx \wedge \omega + \mathrm{Re}\Omega) = \int_A \varphi = \int_A \mathrm{vol}_A.$$

Since $\mathrm{Re}\Omega$ is a calibration, we have $\mathrm{Re}\Omega|_N \leq \mathrm{vol}_N$ for any oriented 3-submanifold in X^7. Then, $\mathrm{Re}\Omega|_A = \mathrm{vol}_A$, which implies that $A = \{*\} \times L$ with $L \subset Y^6$ a SL submanifold.

Variational characterization

Let (X^7, φ) be a G_2-manifold. Fix a compact oriented 3-manifold L^3. Set

$$\mathcal{E} := \{f : L \to X \mid \text{embedding}\}.$$

Note that $T_f\mathcal{E} = \Gamma(L, f^*TX)$ for $f \in \mathcal{E}$. Define a 1-form Θ on $\mathcal{E}$ by

$$\Theta_f(V) = \int_L f^*(i(V) * \varphi) = -\int_L \langle f^*\chi, V\rangle$$

for $f \in \mathcal{E}$ and $V \in T_f\mathcal{E}$. Then

$$\Theta_f = 0 \quad \Longleftrightarrow \quad f \text{ is associative.}$$

In fact, Θ is closed. Let $\pi : \widehat{\mathcal{E}} \to \mathcal{E}$ be the universal conver of $\mathcal{E}$. $\exists CS : \widehat{\mathcal{E}} \to \mathbb{R}$ such that $dCS = \pi^*\Theta$. Hence associative submanifolds are ciritical points of CS (the Chern-Simons type functional).

Summary (mirror symmetry)

mirror symmetry of G_2-manifolds would be explained in terms of coassociative dual T^4 or K3 fibrations (including singular fibers).

$$X^7 \xrightarrow{f} B^3 \xleftarrow{f^*} (X^7)^*.$$

For $X = B^3 \times T^4$ and $X^* = B^3 \times (T^4)^*$,

real FM : $\left\{\text{graphs } B^3 \to T^4\right\} \longrightarrow \left\{\text{connections of } X^* \times \mathbb{C} \to X^*\right\}$,

$$f = (f^1, \cdots, f^4) \mapsto \nabla = d + \sqrt{-1} \sum_{j=1}^{4} f^j dy^j.$$

A condition on graph f (say, being associative) is described in terms of the curvature $F_\nabla = \sqrt{-1} \sum_{j=1}^{4} df^j \wedge dy^j$. This predicts many interesting geometric objects and properties.

Summary (mirror symmetry)

- dDT connections are obtained from (co)associative submanifolds via the real Fourier-Mukai transform.
- dDT connection can also be considered as an analogue of the G_2-instanton: $F_\nabla \wedge *\varphi = 0$.

$$\begin{tikzcd} & \{\text{dDT connections}\} \arrow[dl, leftrightarrow, "\textit{mirror}"'] \arrow[dr, leftrightarrow, "\textit{analogue}"] & \\ \left\{\begin{array}{c}\text{associative}\\ \text{submanifolds}\end{array}\right\} & & \{G_2\text{-instantons}\} \end{tikzcd}$$

- We expect that dDT connections will have similar properties to associative submanfiolds and G_2-instantons.
- conversely, it will be great if we can find something new for associative submanifolds and G_2-instantons by developing the theory for dDT connections. ($\rightsquigarrow$ lead to some breakthroughs?)

Summary (associative submanifolds)

associator equality $\Rightarrow$ G_2-structure is a calibration, associative submanifolds is characterized by the vanishing of $\chi \in \Omega^3(X^7, TX^7)$.

Properties of associative submanifolds:

- Homologically volume minimizing. The volume is given by topologically.
- The moduli space is 0-dimensional and orientable if we perturb the G_2-structure.
- If $X^7 = S^1 \times Y^6$, with Y^6 Calabi-Yau, every associative submanifold in $X^7 := S^1 \times Y^6$ representing the class $\gamma \in H_3(Y^6) \hookrightarrow H_3(X^7)$ is of the form $\{*\} \times L$ with $L \subset Y^6$ a SL submanifold.
- ciritical points of the Chern-Simons type functional.

In the next talk, I will show that similar properties to the above also hold for dDT connections. (The last one is already done by Karigiannis-Leung.)