Deformed Donaldson-Thomas connections II

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Review

- dDT connections are obtained from (co)associative submanifolds via the real Fourier-Mukai transform.
- dDT connection can also be considered as an analogue of the G_2-instanton: $F_\nabla \wedge *\varphi = 0$.

$$\begin{tikzcd} & \{\text{dDT connections}\} \arrow[dl, leftrightarrow, "\textit{mirror}"'] \arrow[dr, leftrightarrow, "\textit{analogue}"] & \\ \left\{\begin{array}{c}\text{associative}\\ \text{submanifolds}\end{array}\right\} & & \{G_2\text{-instantons}\} \end{tikzcd}$$

- We expect that dDT connections will have similar properties to associative submanfiolds and G_2-instantons.

Review (associative submanifolds)

associator equality $\Rightarrow$ G_2-structure is a calibration, associative submanifolds is characterized by the vanishing of $\chi \in \Omega^3(X^7, TX^7)$.

Properties of associative submanifolds:

- Homologically volume minimizing. The volume is given by topologically.
- The moduli space is 0-dimensional and orientable if we perturb the G_2-structure.
- If $X^7 := S^1 \times Y^6$, with Y^6 Calabi-Yau, every associative submanifold in $X^7 = S^1 \times Y^6$ representing the class $\gamma \in H_3(Y^6) \hookrightarrow H_3(X^7)$ is of the form $\{*\} \times L$ with $L \subset Y^6$ a SL submanifold.
- ciritical points of the Chern-Simons type functional.

(In fact, G_2-instantons have similar properties to the above.)

In this talk, I will show that similar properties to the above also hold for dDT connections. (The last one was done by Karigiannis-Leung.)

Review

Definition

- X^7: a manifold with a G_2-structure $\varphi \in \Omega^3$,
- $(L, h) \to X$: a smooth complex Hermitian line bundle.

A Hermitian connection ∇ of (L, h) is called a deformed Donaldson–Thomas (dDT) connection if

$$\frac{1}{6} F_\nabla^3 + F_\nabla \wedge * \varphi = 0,$$

where $F_\nabla \in \sqrt{-1}\Omega^2$ is a curvature of ∇.

Deformation theory

- X^7: a compact connected 7-manifold with a coclosed G_2-structure $\varphi \in \Omega^3$,
  - φ need not to be torsion-free.
    $\exists$ many explicit examples (e.x. S^7).
- $(L, h) \to X$: a smooth Hermitian complex line bundle.
- $\mathcal{M}_{G_2}$: the set of all dDT connections of L divided by the $U(1)$-gauge action (the moduli space).
- Suppose that $\mathcal{M}_{G_2} \neq \emptyset$.

The deformation of dDT connections is controlled by (a subcomplex of) an elliptic complex (as in common with other deformation theories).

Theorem (K.-Yamamoto)

For each dDT conneciton ∇, we can define the complex $(\#_\nabla)$.

1. $H^1(\#_\nabla) \cong T_{[\nabla]}\mathcal{M}_{G_2}$, *the space of infinitesimal deformations.*
2. *If* $H^2(\#_\nabla) = \{0\}$ *for* $[\nabla] \in \mathcal{M}_{G_2}$, $\mathcal{M}_{G_2}$ *is a* b^1*-dim. smooth manifold near* $[\nabla]$.
3. *Suppose that* $[\nabla] \in \mathcal{M}_{G_2}$ *and*
   - *the* G_2*-structure* φ *is torsion-free or nearly parallel, or*
   - ∇ *satisfies* $F_\nabla^3 \neq 0$.

   Then, the moduli space close to $[\nabla]$ *is a* b^1*-dim. smooth manifold if we perturb* φ *generically.*

The complex $(\#_\nabla)$

From

- the initial (coclosed) G_2-structure φ,
- F_∇ for a dDT connection ∇,

we can define a new coclosed G_2-structure $\tilde{\varphi}_\nabla$.

- $\tilde{\varphi}_\nabla$ is described explicitly. $\tilde{\varphi}_\nabla$ is connected to φ in the space of G_2-structures. ($\tilde{*}_\nabla$: the Hodge star induced from $\tilde{\varphi}_\nabla$).

The complex $(\#_\nabla)$ is given by

$$0 \to \sqrt{-1}\Omega^0 \xrightarrow{d} \sqrt{-1}\Omega^1 \xrightarrow{\tilde{*}_\nabla \tilde{\varphi}_\nabla \wedge d} \sqrt{-1}d\Omega^5 \to 0. \qquad (\#_\nabla)$$

Remark

$(\#_\nabla)$ is considered to be a subcomplex of the canonical complex (an elliptic complex introduced by Reyes Carrión) w.r.t. $\tilde{\varphi}_\nabla$.

The canonical complex also appears as a deformation complex of G_2-instantons.

The strategy of the proof is standard. Roughly,

- Compute the linearization of the deformation map.
- Take the slice to the $U(1)$-gauge action.
- Apply the implicit function theorem.

Describing the linearization "nicely" would be nontrivial.
Naively, it is far from obvious to know that deformation is controlled by (the subcomplex of) an elliptic complex.
We solve this problem by introducing $\tilde{\varphi}_{\nabla}$.
(The proof was mainly given by pointwise computations, but this was surprising to us.)

Orientation of $\mathcal{M}_{G_2}$

Theorem (K.-Yamamoto)

If $H^2(\#_\nabla) = \{0\}$ for any $[\nabla] \in \mathcal{M}_{G_2}$, $\mathcal{M}_{G_2}$ is an orientable manifold.

- We show that $\Lambda^{\mathrm{top}} T\mathcal{M}_{G_2}$ is trivial.
- The key is that $\tilde{\varphi}_\nabla$ is connected to φ in the space of G_2-structures.

Set

$$\mathcal{G}_2 = \{(\text{not necessarily coclosed}) \ G_2\text{-structures on } X\}.$$

Since $\tilde{\varphi}_\nabla$ depends on F_∇ and φ, we can define a map

$$f : \mathcal{M}_{G_2} \to \mathcal{G}_2, \qquad f([\nabla]) = \tilde{\varphi}_\nabla.$$

We can show that there exists a line bundle $\mathcal{L} \to \mathcal{G}_2$ such that

$$\Lambda^{\mathrm{top}} T\mathcal{M}_{G_2} = f^*\mathcal{L},$$

(where we use $H^2(\#_\nabla) = \{ 0 \}$.)

- Recall $\tilde{\varphi}_\nabla$ is connected to φ in $\mathcal{G}_2$.
- $\Rightarrow \exists$ homotopy between f and the constant map f_φ.

By the homotopy property of vector bundles, we see that

$$\Lambda^{\mathrm{top}} T\mathcal{M}_{G_2} = f^*\mathcal{L} \cong f_\varphi^*\mathcal{L} = \mathcal{M}_{G_2} \times \mathcal{L}_\varphi.$$

"Volume" for connections

- X^7: a compact connected 7-manifold with a closed G_2-structure $\varphi \in \Omega^3$,
- $(L, h) \to X$: a smooth complex Hermitian line bundle,
- $\mathcal{A}_0$: the space of Hermitian connections of (L, h).

Define the "volume functional" $V : \mathcal{A}_0 \to \mathbb{R}$ by

$$V(\nabla) := \int_X v(\nabla) \mathrm{vol}_g,$$

$$v(\nabla) := \sqrt{1 + |F_\nabla|^2 + \left|\frac{F_\nabla^2}{2!}\right|^2 + \left|\frac{F_\nabla^3}{3!}\right|^2}.$$

- V is the mirror of the standard volume functional for submanifolds via the real FM.
- V is called the Dirac-Born-Infeld (DBI) action in physics.

Theorem ("Mirror" of associator equality, K.-Yamamoto)

For any $\nabla \in \mathcal{A}_0$, we have

$$\left(1 + \frac{1}{2}\langle F_\nabla^2, *\varphi\rangle\right)^2 + \left|*\varphi \wedge F_\nabla + \frac{1}{6}F_\nabla^3\right|^2 + \frac{1}{4}|\varphi \wedge *(F_\nabla)^2|^2 = v(\nabla)^2.$$

In particular,

$$\left|1 + \frac{1}{2}\langle F_\nabla^2, *\varphi\rangle\right| \leq v(\nabla)$$

for any $\nabla \in \mathcal{A}_0$. The equality holds if and only if ∇ is dDT.

- $*\varphi \wedge F_\nabla + \frac{1}{6}F_\nabla^3 = 0 \Rightarrow \varphi \wedge *(F_\nabla)^2 = 0$.
- This is predicted by real FM ([arXiv:2101.03984, Lemma 4.3]). The proof is given by pointwise (complicated) computations. We would not have found this without the prediction by real FM.
- This is the key theorem. This enables us to use arguments of calibrations/G_2-instantons.

Theorem (K.-Yamamoto)

For any $\nabla \in \mathcal{A}_0$, we have

$$\left|\mathrm{Vol}(X) + \left(-2\pi^2 c_1(L)^2 \cup [\varphi]\right) \cdot [X]\right|$$
$$= \left|\int_X \left(1 + \frac{1}{2}\langle F_\nabla^2, *\varphi\rangle\right) \mathrm{vol}_g\right| \leq V(\nabla),$$

where $c_1(L)$ is the first Chern class of L.

- *The equality holds if and only if ∇ is dDT.*
- *For any dDT connection ∇, $V(\nabla)$ is given topologically and ∇ is a global minimizer of V.*

This follows from the "mirror" of the associator equality and

$$\left|\int_X \left(1 + \frac{1}{2}\langle F_\nabla^2, *\varphi\rangle\right) \mathrm{vol}_g\right| \leq \int_X \left|1 + \frac{1}{2}\langle F_\nabla^2, *\varphi\rangle\right| \mathrm{vol}_g \leq V(\nabla).$$

Corollary

Suppose that L is a flat line bundle. Then, any dDT connection is a flat connection. In particular, the moduli space of dDT connections is $H^1(X,\mathbb{R})/H^1(X,\mathbb{Z})$.

Let ∇_0 be a flat connection and ∇ be any dDT connection. Then,

$$\int_X \sqrt{1+|F_\nabla|^2+\left|\frac{F_\nabla^2}{2!}\right|^2+\left|\frac{F_\nabla^3}{3!}\right|^2}\mathrm{vol}_g = V(\nabla) = V(\nabla_0) = \int_X \mathrm{vol}_g,$$

which implies that $F_\nabla = 0$.

Remark

We need to consider a non-flat line bundle to construct non-trivial examples of dDT connections on a compact 7-manifold with a closed G_2-structure.
cf. Lotay and Oliveira constructed nontrivial examples on the trivial complex line bundle over a 3-Sasakian 7-manifold.

Holonomy reduction

Let $(Y^6, \omega, g, J, \Omega)$ be a compact and connected Calabi–Yau 3-manifold and $(L, h) \to Y$ be a smooth complex Hermitian line bundle. $X^7 := S^1 \times Y^6$ admits a G_2-structure φ by

$$\varphi = dx \wedge \omega + \mathrm{Re}\,\Omega, \qquad *\varphi = \omega^2/2 - dx \wedge \mathrm{Im}\,\Omega,$$

Let $\pi : X^7 \to Y^6$ be a projection.

Lemma

For a dHYM connection ∇ of (L, h) (i.e. $F_\nabla^{0,2} = \mathrm{Im}\,(\omega + F_\nabla)^3 = 0$), the pullback $\pi^*\nabla$ is a dDT connection of $\pi^* L$.

This follows from

$$\begin{aligned} \frac{1}{6} F_\nabla^3 + F_\nabla \wedge *\varphi =& \frac{1}{6}\left(F_\nabla^3 + 3F_\nabla \wedge \omega^2\right) - dx \wedge F_\nabla \wedge \mathrm{Im}\,\Omega, \\ \sqrt{-1}\mathrm{Im}\,(\omega + F_\nabla)^3 =& F_\nabla^3 + 3F_\nabla \wedge \omega^2. \end{aligned}$$

We can prove the "converse".

Theorem (K.-Yamamoto)

Suppose that there exists a dHYM connection ∇_0 of L.

1. *For any dDT connection $\widetilde{\nabla}$ of $\pi^* L$, there exist a dHYM connection ∇ of L and a $\sqrt{-1}\mathbb{R}$-valued closed 1-form $\xi \in \sqrt{-1}\Omega^1(X^7)$ such that $\widetilde{\nabla} = \pi^*\nabla + \xi$.*
2. *Set*
$$\mathcal{M}_{G_2} = \text{the moduli space of dDT connections of } \pi^* L,$$
$$\mathcal{M}_{dHYM} = \text{the moduli space of dHYM connections of } L.$$

   Then $\mathcal{M}_{G_2} \cong S^1 \times \mathcal{M}_{dHYM}$.

The proof is almost parallel to the associative/SL case. Use

- Compute the integral, which is independent of $\nabla \in \mathcal{A}_0$.
- $\Rightarrow$ any dDT connection $\widetilde{\nabla}$ satisfies the equality in the "mirror" of SL equality (which follows from the "mirror" of associator equality), to show that $\widetilde{\nabla}$ is (essentially) dHYM.

Variational characterization (Karigiannis-Leung)

- (X^7, φ, g): a compact connected G_2-manifold,
- $(L, h) \to X$: a smooth complex Hermitian line bundle,
- $\mathcal{A}_0$: the space of Hermitian connections of (L, h).

Define a 1-form Θ on $\mathcal{A}_0$ by

$$\Theta_\nabla(\sqrt{-1}b) = \int_X \sqrt{-1}b \wedge \left(\frac{1}{6}F_\nabla^3 + F_\nabla \wedge *\varphi\right).$$

for $\nabla \in \mathcal{A}_0$ and $\sqrt{-1}b \in T_\nabla \mathcal{A}_0 = \sqrt{-1}\Omega^1$. Then

$$\Theta_\nabla = 0 \quad \Longleftrightarrow \quad \nabla \text{ is dDT}.$$

In fact, Θ is closed. Since $\mathcal{A}_0 = \nabla + \sqrt{-1}\Omega^1 \cdot \mathrm{id}_L$ ($\nabla \in \mathcal{A}_0$ is fixed) is contractible, $\exists CS : \mathcal{A}_0 \to \mathbb{R}$ such that $dCS = \Theta$. Hence dDT connections are ciritical points of CS (the Chern-Simons type functional).

Summary

dDT connections have the following common properties with associative submanifolds and G_2-instantons:

- The "mirror" of associator equality $\Rightarrow$ any dDT connection is a golobal minimizer of V and it is given by topologically.
- The moduli space is b^1-dimensional and orientable if we perturb the G_2-structure.
- If Y^6 Calabi-Yau, $(L, h) \to Y^6$ a complex Hermitian line bundle, $\pi : X^7 = S^1 \times Y^6 \to Y^6$, any dDT connection on $\pi^* L$ is dHYM (modulo closed 1-forms).
- ciritical points of the Chern-Simons type functional (Karigiannis-Leung).

We could also prove similar statements in the $\mathrm{Spin}(7)$-case (except for Chern-Simons).

Future work

- Can we find more analogies? (by the real Fourier–Mukai transform?)
- The "mirror" of mean curvature flow (MCF)
  - We can define the mean curvature as the negative gradient vector field of V.
  - We could show the short-time existence recently.
  - We may prove various results corresponding to the MCF for submanifolds.
- Construction of nontrivial examples of dDT connections on a compact 7-manifold with a closed G_2-structure (Lotay-Oliveira's examples are given on a manifold with a coclosed G_2-structure.)
- "Singular" dDT connections?
- Compactness theorem for dDT connections
  - associative Smith map [Cheng-Karigiannis-Madnick]
  - G_2-instanton [Tian]

  might be helpful?