Closed G_2-structures and Laplacian flow

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G_2-structures

Definition

A G_2-structure on a 7-manifold M is given by a 3-form φ with pointwise stabilizer isomorphic to G_2.

- Pointwise $\varphi = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}$.
- φ is non-degenerate: $i_X\varphi \wedge i_X\varphi \wedge \varphi \neq 0$, for every $X \neq 0$.

$\rightsquigarrow$ φ induces a metric g_φ with a volume form dV_φ:

$$g_\varphi(X, Y)dV_\varphi = \frac{1}{6} i_X\varphi \wedge i_Y\varphi \wedge \varphi.$$

Proposition (Fernández-Gray)

The following are equivalent:

(a) $\nabla^{LC}\varphi = 0$;

(b) $d\varphi = 0$ and $d(*\varphi) = 0$;

(c) $Hol(g_\varphi)$ is isomorphic to a subgroup of G_2.

A G_2-structure satisfying $(a), (b)$ or (c) is called parallel.

Remark

- The conditions $\nabla^{LC}\varphi = 0$ and $d(*\varphi) = 0$ are non-linear in φ.
- Metrics induced by parallel G_2-structures are Ricci-flat [Bonan].

Compact examples with holonomy G_2

Proposition (Joyce)

M compact with a parallel G_2-structure φ, then
$\mathrm{Hol}(g_\varphi) = G_2 \iff \pi_1(M)$ is finite.

Examples

- First compact examples using orbifold resolution together with a perturbative method [Joyce, 1996]
- Compact examples using geometric gluing [Kovalev, 2003; Corti-Haskins-Nordström-Pacini, 2013; Joyce-Karigiannis, 2017].

General idea: start with some space where the linear condition $d\varphi = 0$ is satisfied and then try to solve the nonlinear one $d(*\varphi) = 0$.

Closed G_2-structures

A G_2-structure φ is closed (or calibrated) if $d\varphi = 0$. Then

$$d * \varphi = \tau \wedge \varphi,$$

where $\tau \in \Lambda^2_{14} \cong \mathfrak{g}_2$, i.e. $\tau \wedge \varphi = - * \tau$ and $\tau \wedge *\varphi = 0$.

Remark

• $\tau = d^*\varphi \Rightarrow d^*\tau = 0 \Rightarrow d\tau = \Delta_\varphi \varphi$, where $\Delta_\varphi = dd^* + d^*d$ is the Hodge Laplacian.

• φ defines a calibration on M (i.e. $\varphi|_\xi \leq vol_\xi$, $\forall$ tg oriented 3-plane ξ) [Harvey-Lawson].

Associative 3-fold: $N^3 \subset M$ calibrated by φ, i.e. $\varphi|_{N^3} = dV_{N^3}$.
Coassociative 4-fold: $N^4 \subset M$ is coassociative if $\varphi|_{N^4} = 0$.

Ricci tensor

The Ricci tensor and the scalar curvature of g_φ can be expressed in terms of τ:

$$Ric(g_\varphi) = \frac{1}{4}|\tau|^2 g_\varphi - \frac{1}{4} j(d\tau - \frac{1}{2} * (\tau \wedge \tau)),$$

where $j : \Lambda^3 \to S^2$ is defined by

$$j(\beta)(X, Y) = *(\iota_X \varphi \wedge \iota_Y \varphi \wedge \beta)$$

The scalar curvature is given by

$$Scal(g_\varphi) = -\frac{1}{2}|\tau|^2 \leq 0$$

$\rightsquigarrow$ no restrictions on compact manifolds!

ERP condition

Theorem (Cleyton-Ivanov; Bryant)

If M is compact with a closed G_2-structure φ, then

1) g_φ Einstein $\Rightarrow \tau \equiv 0$, i.e. φ is parallel.

2) $\int_M [Scal(g_\varphi)]^2 dV_\varphi \leq 3 \int_M |Ric(g_\varphi)|^2 dV_\varphi$.

[Bryant]: equality in 2) holds if and only if

$$d\tau = \frac{|\tau|^2}{6}\varphi + \frac{1}{6} * (\tau \wedge \tau),$$

in such a case, φ is called extremally Ricci pinched (ERP).

Automorphism group

Remark

General results on the existence of closed G_2-structures on (compact) 7-manifolds are still not known.

$Aut(M, \varphi) := \{f \in Diff(M) | f^*\varphi = \varphi\} \Rightarrow$ when M is compact its Lie algebra is $\mathfrak{aut}(M, \varphi) = \{X \in \chi(M) | L_X \varphi = 0\}$.

Remark

If M is compact with a parallel G_2-structure φ and $Hol(g_\varphi) = G_2$, then $\mathfrak{aut}(M, \varphi) = \{0\}$.

Problem

What we can say about $Aut(M, \varphi)$ if $d\varphi = 0$ and $\tau \neq 0$?

Theorem (Podestá-Raffero)

M compact with φ closed non-parallel. If $X \in \mathfrak{aut}(M, \varphi)$, then the 2-form $i_X \varphi$ is harmonic. Consequently:

- $\dim \mathfrak{aut}(M, \varphi) \leq b_2(M)$;
- $\mathfrak{aut}(M, \varphi)$ *is abelian with dim* ≤ 6.

Consequences:

- There are no compact homogeneous examples with invariant (non-parallel) closed G_2-structures.
- Cohomogeneity one examples only occur for $M = \mathbb{T}^7$.

Known examples

• Compact locally homogeneous examples with invariant closed G_2-structures:

$(\Gamma\backslash G, \varphi)$, with φ invariant $\longleftrightarrow (Lie(G), \varphi)$

(the first example is a nilmanifold $\Gamma\backslash N$ [Fernández]).

• Complete closed G_2-structures invariant under the cohomogeneity one action of a compact simple Lie group [Cleyton-Swann].

• $(M^6 \times S^1, \varphi = \omega \wedge ds + \rho)$, where M^6 admits a symplectic half-flat $SU(3)$-structure (ω, ρ).

Classification results on Lie algebras

- Nilpotent Lie algebras [Conti-Fernández]

- Unimodular solvable (non-nilpotent) Lie algebras with non-trivial center [F-Raffero-Salvatore]

- Unimodular non-solvable Lie algebras [F-Raffero]

$\hookrightarrow$ $\mathfrak{g}$ must have Levi decomposition $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R}) \ltimes \mathfrak{r}$ with

$\mathfrak{r}$ centerless if $\mathfrak{g}$ is a product (3 classes of isomorphism)

$\mathfrak{r} \cong \mathbb{R} \ltimes \mathbb{R}^3$ if $\mathfrak{g}$ is not a product (1 class of isomorphism).

Formality

Conjecture

A simply-connected compact Riemannian manifold of special holonomy is a formal space, in the sense of Rational Homotopy Theory, i.e. its minimal model is formal.

Theorem (Amann-Taimanov)

A typical Joyce example of exceptional holonomy G_2 is formal.

An importal tool is

Theorem (Férnandez-Munoz)

Let M be a connected and orientable compact manifold of dimension $2n$ or $2n-1$. Then M is formal if and only if it is $(n-1)$-formal.

Problems

• Does there exist a compact formal manifold (not a product) M with a closed G_2-structure and $b_1(M) = 1$, which does not admit any parallel G_2-structure?

We will give a positive answer using orbifold resolutions.

• Does there exist a compact manifold M with a closed G_2-structure and $b_1(M) = 0$, which does not admit any parallel G_2-structure? Still open!

In particular, does S^7 admit a closed G_2-structure?

A compact example using orbifold resolution

Motivation: Compact examples by Joyce with $Hol(g_\varphi) = G_2$ obtained as orbifold resolutions of $\mathbb{T}^7/F$ with $F \subset G_2$ finite subgroup and by a perturbation argument.

Idea: instead of $\mathbb{T}^7$ start with a nilmanifold $M = \Gamma \backslash N$ with an invariant closed closed G_2-structure and $N \cong \mathbb{R}^7$ 3-step nilpotent with structure equations

$$[e_1, e_2] = -e_4, \ [e_1, e_3] = -e_5, \ [e_1, e_4] = -e_6, \ [e_1, e_5] = -e_7$$

and $\Gamma \cong 2\mathbb{Z} \times \mathbb{Z}^6$.

N can be viewed as group of real matrices of the form

$$a = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},$$

with A_i real square matrices of order 6 $\hookrightarrow$ global coordinates $(x_1, \ldots, x_7)$ on N.

In terms of x_i:

$$\begin{aligned} e^j &= dx_j, j = 1, 2, 3, \\ e^4 &= dx_4 - x_2 dx_1, \quad e^5 = dx_5 - x_3 dx_1, \\ e^6 &= dx_6 + x_1 dx_4, \quad e^7 = dx_7 + x_1 dx_5. \end{aligned}$$

We consider the discrete subgroup Γ of N given by

$$\{a \in N \mid x_1 \in 2\mathbb{Z} \quad \text{and} \quad x_j \in \mathbb{Z}, j = 2, \ldots, 7\} \cong 2\mathbb{Z} \times \mathbb{Z}^6$$

$\hookrightarrow \Gamma \backslash N$ is compact and admits the invariant closed G_2-form

$$\varphi = e^{123} + e^{145} + e^{167} - e^{246} + e^{257} + e^{347} + e^{356}.$$

Remark

$M = \Gamma \backslash N$ is diffeomorphic to a mapping torus M_ν of $\mathbb{T}^6$ by a diffeomorphism ν of $\mathbb{T}^6$ induced by a linear automorphism of $\mathbb{R}^6$ with projection

$$p : [(x_1, \ldots, x_7)] \in M \mapsto x_1 + 2\mathbb{Z} \in S^1 = \mathbb{R}/2\mathbb{Z}.$$

Consider the action of $F = \mathbb{Z}_2$ generated by

$$\rho : (x_1, \dots, x_7) \in N \mapsto (-x_1, -x_2, x_3, x_4, -x_5, -x_6, x_7) \in N$$

Then

$$\rho(ab) = \rho(a)\rho(b),\ \forall a, b \in N, \quad \rho(\Gamma) = \Gamma$$

$\hookrightarrow \rho$ induces an action of $\mathbb{Z}_2$ on $M = \Gamma \backslash N$.

Remark

$$\rho^* e^i = -e^i, i = 1, 2, 5, 6, \quad \rho^* e^j = e^j, j = 3, 4, 7.$$

Proposition (Fernández-F-Kovalev-Munoz)

$\hat{M} = M/\mathbb{Z}_2$ is a compact 7-orbifold, $b_1(\hat{M}) = 1$ and $\hat{M}$ has an orbifold closed G_2-form $\hat{\varphi}$ (induced by φ).

• The action of $\mathbb{Z}_2$ is smooth and effective $\hookrightarrow$ $\hat{M} = M/\mathbb{Z}_2$ is a compact orbifold.

• By Nomizu's Theorem

$$H^1(M) = \langle [e^1], [e^2], [e^3] \rangle$$

$\hookrightarrow$ $H^1(\hat{M}) = H^1(M)^{\mathbb{Z}_2} = \langle [e^3] \rangle$.

• φ is $\mathbb{Z}_2$-invariant $\hookrightarrow$ φ induces an orbifold closed G_2-form on $\hat{M}$.

The singular locus S of $\hat{M} = M/\mathbb{Z}_2$

$\hat{\pi} : M \to \hat{M}$ projection $\Rightarrow$

$S = \hat{\pi}(S')$, where S' is the fixed point set in M by the $\mathbb{Z}_2$-action.

Remark

S consists of all the 3-dim spaces $S_{\mathbf{a}} = \hat{\pi}(S'_{\mathbf{a}}) = S'_{\mathbf{a}}/\mathbb{Z}_2$, where

$$S'_{\mathbf{a}} = \{(a_1, a_2, x_3, x_4, a_5, a_6, x_7) \mid (x_3, x_4, x_7) \in \mathbb{T}^3\} \subset M$$

and $\mathbf{a} = (a_1, a_2, a_5, a_6) \in \mathbb{A} = \{0, 1\} \times (\{0, \frac{1}{3}\})^3$.

$S'_{\mathbf{a}}$ is either included in $p^{-1}(0 + 2\mathbb{Z})$ or in $p^{-1}(1 + 2\mathbb{Z})$

$\hookrightarrow$ $S'_{\mathbf{a}} \cong \mathbb{T}^3$ and S is a disjoint union of 16 copies of $\mathbb{T}^3$.

Local model around the singular locus

Using left translations L_a on N we obtain orbifold diffeomorphisms of $\hat{M}$ $\hookrightarrow$ we can focus on

$$S_{\mathbf{0}} = \{(0,0,x_3,x_4,0,0,x_7)\} \subset \hat{M}, \quad \text{where } \mathbf{0} = (0,0,0,0) \in \mathbb{A}.$$

Proposition (Fernández-F-Kovalev-Munoz)

$\exists$ neighboorhoods $U' \cong \mathbb{T}^3 \times B_\epsilon$ and $U'' \cong \mathbb{T}^3 \times B_{\frac{\epsilon}{2}}$ of $S'_{\mathbf{0}}$ in M with $U'' \subset U'$ and $\exists$ closed G_2-forms $\mathbb{Z}_2$-invariant $\phi \in \Omega^3(M)$ and $\psi \in \Omega^3(U')$ such that
i) $\phi = \varphi$ outside $U' \cong \mathbb{T}^3 \times B_\epsilon$;
ii) $\phi = \psi =$ standard G_2-form on $U'' \cong \mathbb{T}^3 \times B_{\frac{\epsilon}{2}}$.

$\hookrightarrow$ neighborhoods of $S_{\mathbf{0}}$ and orbifolds closed G_2-forms on $\hat{M}$.

We show that a neighborhood of each component of S is diffeomorphic to $\mathbb{T}^3 \times (B/\pm 1)$ and we replace $B/\pm 1$ with an appropriate neighborhood of the Eguchi-Hanson space.

Remark

Differently from the desingularization of Joyce's examples we need to modify the gluing to get a G_2-form on the desingularization.

Theorem (Fernández-F-Kovalev-Munoz)

- *$\exists$ a resolution $\pi : (\tilde{M}, \tilde{\varphi}) \to (\hat{M}, \hat{\varphi})$ with $\tilde{M}$ compact smooth manifold, $b_1(\tilde{M}) = 1$ and $\tilde{\varphi} = \pi^* \hat{\varphi}$ in the complement of a small neighborhood of the exceptional locus $E = \pi^{-1}(S)$.*
- *$\tilde{M}$ does not admit any parallel G_2-structure.*
- *$\pi_1(\tilde{M}) = \mathbb{Z}$ and $\tilde{M}$ is formal.*

Associative 3-folds of $(\tilde{M}, \tilde{\varphi})$

One can construct examples of associative 3-folds of $\tilde{M}$ applying

Proposition (Joyce)

Let (Y, φ) with a closed G_2 form φ and $\sigma \neq id_Y$ be an involution of Y such that $\sigma^* \varphi = \varphi$. Then the fixed point set P is an embedded associative 3-fold. Furthermore, if Y is cpt then so is P.

We use the involution on N:

$\sigma : (x_1, x_2, x_3, x_4, x_5, x_6, x_7) \to (-x_1, -x_2, x_3, x_4, -x_5, \frac{1}{2} - x_6, x_7)$

$\hookrightarrow$ σ induces an involution $\hat{\sigma}$ on the orbifold $\hat{M} = M/\mathbb{Z}_2$.

Fixed locus $\hat{P} = \hat{\pi}(P)$, where P is the fixed point set of $\sigma \hookrightarrow$ $\hat{P}$ is the disjoint union of 8 copies of $\mathbb{T}^3$ and $\hat{P} \cap S = \emptyset$.

Remark

Any compact associative 3-fold in a compact 7-manifold with a closed G_2-structure is either rigid, or has an infinitesimal associative deformations [Abkulut-Salur].

Theorem (Fernández-F-Kovalev-Munoz)

- $(\tilde{M}, \tilde{\varphi})$ *has eight associative (calibrated by* $\tilde{\varphi}$*) 3-tori.*
- *For each of those 3-tori,* $\exists$ *a smooth 3-dimensional family of non-trivial associative deformations.*

Laplacian flow

Idea: use a geometric flow to deform closed G_2-structures and eventually obtain a parallel one

Definition (Bryant)

Let φ_0 be a a closed G_2-structure on M^7. The Laplacian flow (LF) is

$$\begin{cases} \partial_t \varphi(t) = \Delta_{\varphi(t)} \varphi(t), \\ d\varphi(t) = 0, \\ \varphi(0) = \varphi_0. \end{cases}$$

where $\Delta_{\varphi(t)}$ is the Hodge Laplacian of $g_{\varphi(t)}$.

if $\varphi(t)$ solves the LF, then $\varphi(t) \in [\varphi_0] \in H^3_{DR}(M^7)$ and

$$\partial_t g_{\varphi(t)} = -2Ric(g_{\varphi(t)}) + l.o.t.$$

Remark

If M^7 is compact, then

- stationary points are parallel G_2-structures.
- the LF is the gradient flow of Hitchin's volume functional $\mathcal{V} : \varphi \in [\varphi_0] \mapsto \int_M \varphi \wedge *\varphi$.

$\mathcal{V}$ is monotonically increasing along the LF, its critical points are parallel G_2-structures and they are strict local maxima.

Theorem (Bryant-Xu)

Assume that (M^7, φ_0) is compact. Then the LF has a unique solution for short time $t \in [0, \epsilon)$, with ϵ depending on $\varphi_0 = \varphi(0)$.

Recent developments

- If φ_0 is near a torsion-free G_2-structure $\tilde{\varphi}$, then the LF converges to a torsion-free G_2-structure which is related to $\tilde{\varphi}$ via a diffeomorphism [Xu-Ye; Lotay-Wei].

- Shi-type derivative estimates for Rm and τ along the flow:

a bound on $\Lambda(x,t) := \left(\frac{1}{4} |\nabla \tau|^2_{g_{\varphi(t)}} + |\mathrm{Rm}(\mathrm{x},\mathrm{t})|^2_{\mathrm{g}_{\varphi(\mathrm{t})}} \right)^{\frac{1}{2}}$ will imply bounds on all covariant derivatives of Rm and τ. Then, the flow will exist as long as $\Lambda(x,t)$ remains bounded.

$\rightsquigarrow$ uniqueness and compactness theory [Lotay-Wei].

- Non-collapsing under the assumption of bounded Scal [G. Chen].

Solutions to the LF

Study of explicit solutions on

• simply connected Lie groups with left-invariant closed G_2-structure [Fernández-F-Manero; Lauret; F-Raffero].

• $\mathbb{T}^7$ with cohomogeneity one closed G_2-structure [Huang-Wang-Yao].

• $M^6 \times S^1$, with a warped closed G_2-structure $\varphi = f\, ds \wedge \omega + \rho$, $f \in \mathcal{C}^\infty(M^6)$, $f > 0$ and compact base (M^6, ω, ρ) [F-Raffero].

• M^7 with a S^1-invariant closed G_2-structure [Fowdar].

• $(M^4 \times T^3, \varphi)$, where φ is induced by a hypersymplectic structure $(\omega_1, \omega_2, \omega_3)$ on the compact M^4 [Fine-Yao].

Laplacian solitons

Self-similar solutions $\varphi(t) = \sigma(t) f_t^* \varphi$ of the LF $\Longleftrightarrow$ closed G_2-structures φ satisfying

$$\Delta_\varphi \varphi = \lambda \varphi + L_X \varphi$$

for some $\lambda \in \mathbb{R}$ and vector field X.

Definition

A Laplacian soliton φ is called

- shrinking if $\lambda < 0$,
- steady if $\lambda = 0$,
- expanding if $\lambda > 0$.

Theorem (Lin; Lotay-Wei)

On a compact manifold any Laplacian soliton φ (which is not torsion-free) must have $\lambda > 0$ and $X \neq 0$.

In particular, on a compact 7-manifold the only steady Laplacian solitons are given by parallel G_2-structures.

Open Problem

$\exists$ expanding Laplacian solitons on compact manifolds?

In the non-compact case:

- $\exists$ steady, shrinking and expanding (homogeneous) solitons [Lauret-Nicolini; F-Raffero; Ball].
- $\exists$ inhomogeneous complete steady gradient solitons [Ball; Fowdar].

ERP condition

Problem

Study the behaviour of the LF starting from an ERP φ.

Recall: a closed G_2-structure φ is ERP if

$$d\tau = \frac{|\tau|^2}{6}\varphi + \frac{1}{6} * (\tau \wedge \tau).$$

Proposition (Bryant)

M compact with φ ERP, then

- the norm $|\tau|$ is constant and $\tau^3 = 0$;
- τ^2 (resp. $*_\varphi(\tau^2)$) is a non-zero closed simple 4-form (resp. 3-form) of constant norm;
- $TM = P \oplus Q$ where

$$P := \left\{X \in TM \,|\, \iota_X(\tau^2) = 0\right\}, \quad Q := \left\{X \in TM \,|\, \iota_X *_\varphi (\tau^2) = 0\right\}.$$

Moreover, the P-leaves are associative submanifolds, while the Q-leaves are coassociative submanifolds.

- $\mathrm{Ric}(g_\varphi) = -\frac{1}{6}|\tau|^2 g_\varphi|_P$ non-positive with eigenvalues $-\frac{1}{6}\,|\tau|^2_\varphi$ of multiplicity three and 0 of multiplicity four.

Theorem (F-Raffero)

M compact with an ERP closed G_2-structure φ. Then the solution of LF with $\varphi(0)=\varphi$ is

$$\varphi(t) = \varphi + f(t)\, d\tau,$$

with $f(t) = \frac{6}{|\tau|^2_\varphi}\left(\exp\left(\frac{|\tau|^2_\varphi}{6}\,t\right) - 1\right).$

In particular:

- $\varphi(t)$ *is ERP for all* $t \in \mathbb{R}$ *with* $\tau(t) = \exp\left(\frac{|\tau|^2_\varphi}{6}\,t\right)\tau.$
- *LF has constant velocity* $|\Delta_{\varphi(t)}\varphi(t)|_{\varphi(t)} = \frac{1}{\sqrt{6}}|\tau|^2.$
- $\mathrm{Ric}(g_{\varphi(t)}) = \mathrm{Ric}(g_\varphi).$

Asymptotic behaviour

Remark

- If $\varphi(0)$ is not ERP, then $\varphi(t)$ cannot become ERP in finite time.
- The total volume $Vol_{g_{\varphi(t)}}(M) = \int_M dV_{\varphi(t)} = \exp\left(\frac{|\tau|^2}{3} t\right) Vol_{g_\varphi}(M)$.

Proposition (F, Raffero)

- when $t \to +\infty$, the the volume of the P-leaves goes to zero relative to the volume of the manifold, while the volume of the Q-leaves and the volume of the manifold grow at the same rate.
- when $t \to -\infty$, the volume of the Q-leaves and the volume of the manifold tend to zero at the same rate.

Examples of ERP's

Example (Bryant)

A compact quotient of the non-compact homogeneous $M = SL(2,\mathbb{C}) \ltimes \mathbb{C}^2/SU(2)$ by $\Gamma \subset Aut(M,\varphi)$. M can be also described by a non-unimodular solvable Lie group.

Example

The unimodular solvable Lie group S with structure equations

$$(0,0,0,-e^{14}-e^{24}-e^{34},-e^{15}+e^{25}+e^{35},e^{16}-e^{26}+e^{36},e^{17}+e^{27}-e^{37})$$

has a left-invariant ERP G_2-structure (steady Laplacian soliton).

- S is the only unimodular Lie group admitting a left-invariant ERP G_2-structure [F-Raffero].
- S admits a compact quotient by a lattice [Kath-Lauret].

Proposition (Lauret-Nicolini)

Any left-invariant ERP G_2-structure on a simply connected Lie group is always a steady soliton.

The converse does not hold!

Example (F-Raffero)

The solvable Lie group with structure equations

$$(0, 0, -e^{37}, e^{47}, 2e^{14} + e^{57}, -2e^{24} + e^{67}, 0).$$

admits a left-invariant steady Laplacian soliton which is not ERP.

THANK YOU VERY MUCH FOR THE ATTENTION!!