On Lagrangian surfaces in the complex projective plane

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Contents

Background and motivation

- Differential Geometry originated in the study of curves (ODEs) and surfaces (PDEs).
- Of interest are surfaces stationary with respect to natural variational problems. Euler-Lagrangian equations are the natural non-linear PDEs.
- Surfaces geometry serves as a platform on which variational problems, integrable systems, mathematical physics, surface group representations, computer visualizations meet and interact.
- There have been lots of studies on surfaces in Euclidean spaces, real space forms, etc. Here we concern on Lagrangian surfaces in the complex projective plane.

Examples of Lagrangian surfaces in $\mathbb{C}P^2$

- $(\mathbb{C}P^2, g, J, \omega)$ the complex projective plane with the Fubini-Study metric of constant biholomorphic sectional curvature
- $f : \Sigma \to \mathbb{C}P^2$ Lagrangian surface
- "Lagrangian" $\underset{\text{def}}{\Longleftrightarrow} f^*\omega = 0 \Leftrightarrow Jf_* T\Sigma \perp f_* T\Sigma$

$$\pi : S^5 \subset \mathbb{C}^3 \to \mathbb{C}P^2 \text{ Hopf projection}$$

1. Totally geodesic $\mathbb{R}P^2$:
$$\mathbb{R}P^2 = \{\pi(z_1, z_2, z_3) \in \mathbb{C}P^2 | z_i = \bar{z}_i, 1 \leq i \leq 3\}.$$
2. Clifford torus:
$$T^2 = \{\pi(z_1, z_2, z_3) \in \mathbb{C}P^2 ||z_1|^2 = |z_2|^2 = |z_3|^2 = \frac{1}{3}\}.$$

Clifford torus

- $f : \mathbb{R}^2 \to S^5(1)$ defined by $f(x, y) = \frac{1}{\sqrt{3}}(e^{2iy}, e^{i(\sqrt{3}x-y)}, e^{-i(\sqrt{3}x+y)})$ is a flat, conformal, minimal Legendrian immersion in $S^5(1)$.
- f induces an embedded torus in $g : \mathbb{R}^2/\Lambda \to S^5(1)$, where $\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ with $\omega_1 = (0, 2\pi)$ and $\omega_2 = (\frac{\pi}{\sqrt{3}}, \pi)$.
- $f_0 := \pi \circ g : \mathbb{R}^2/\Lambda \to \mathbb{C}P^2$ is a minimal Lagrangian torus, where $\pi : S^5(1) \to \mathbb{C}P^2$ is the Hopf fibration.
- Since $f((x, y) + \frac{k}{3}\omega_1 + m\omega_2) = e^{-i\frac{2\pi}{3}k} f(x, y)$, with $k, m \in \mathbb{Z}$, f induces a map $f_0' : \mathbb{R}^2/\Lambda_C \to \mathbb{C}P^2$, satisfying $f_0' \circ \pi' = \pi \circ f$, where $\Lambda_C = \mathbb{Z}\frac{\omega_1}{3} \oplus \mathbb{Z}\omega_2$ and $\pi' : \mathbb{R}^2 \to \mathbb{R}^2/\Lambda_C$ is the natural projection.
- f_0' is an embedded minimal Lagrangian torus in $\mathbb{C}P^2$, which is called the Clifford torus.

Characterizations of classical examples

- Any minimal Lagrangian surface in $\mathbb{C}P^2$ of constant curvature is an open subset of $\mathbb{R}P^2$ or Clifford torus up to isometries of $\mathbb{C}P^2$.
- Any minimal Lagrangian immersion from a sphere is totally geodesic.
- Yau (1974): Let Σ be a minimal Lagrangian surface of $\mathbb{C}P^2(\frac{c}{4})$. If Σ is complete and $K \neq 0$ and if $c/4 - K \geq \alpha > 0$ for some constant α, then Σ is totally geodesic or flat.
- Chen-Ogiue (1974): Let Σ be a compact minimal Lagrangian surface in $\mathbb{C}P^2$. If $||A||^2 < 2$, then Σ is totally geodesic. (Consider $\frac{1}{2}\triangle||A||^2$)
- Ludden-Okumura-Yano (1975): If $||A||^2 = 2$, then Σ is the Clifford torus.

Moving frame equations for Lagrangian Surfaces in $\mathbb{C}P^2$

$f:\Sigma \to \mathbb{C}P^2$ an oriented Lagrangian surface with the induce metric $g = 2e^u dz d\bar{z}$

- One always can choose a local horizontal lift $\mathfrak{f}$ to S^5, i.e., $\mathfrak{f}_z \cdot \bar{\mathfrak{f}} = 0$.
- The metric g is conformal $\Longrightarrow \mathcal{F} = (e^{-\frac{u}{2}}\mathfrak{f}_z, e^{-\frac{u}{2}}\mathfrak{f}_{\bar{z}}, \mathfrak{f})$ Hermitian orthonormal moving frame

$$\mathcal{F}_z = \mathcal{F}\mathcal{U}, \qquad \mathcal{F}_{\bar{z}} = \mathcal{F}\mathcal{V},$$

$\mathcal{U}_{\bar{z}} - \mathcal{V}_z = [\mathcal{U}, \mathcal{V}] \Longleftrightarrow u, \phi, \psi$ satisfy

$$\begin{aligned} \phi_{\bar{z}} + \bar{\phi}_z &= 0, \\ u_{z\bar{z}} + e^u + |\phi|^2 - e^{-2u}|\psi|^2 &= 0, \\ e^{-u}\psi_{\bar{z}} &= \phi_z - u_z\phi. \end{aligned}$$

$$\Phi := e^{-u}\mathfrak{f}_{z\bar{z}} \cdot \overline{\mathfrak{f}_{\bar{z}}} dz := \phi dz, \qquad \Psi := \mathfrak{f}_{zz} \cdot \overline{\mathfrak{f}_{\bar{z}}} dz^3 := \psi dz^3.$$

Geometry of Φ and Ψ

$$\Phi := e^{-u}\mathfrak{f}_{z\bar{z}} \cdot \overline{\mathfrak{f}_{\bar{z}}} dz := \phi dz \qquad \Psi := \mathfrak{f}_{zz} \cdot \overline{\mathfrak{f}_{\bar{z}}} dz^3 := \psi dz^3$$

Bonnet theorem

A Lagrangian surface in $\mathbb{C}P^2$ is locally determined by $\{u, \phi, \psi\}$ satisfying the following equations

$$\begin{aligned} \phi_{\bar{z}} + \bar{\phi}_z &= 0, \\ u_{z\bar{z}} + e^u + |\phi|^2 - e^{-2u}|\psi|^2 &= 0, \\ e^{-u}\psi_{\bar{z}} &= \phi_z - u_z\phi. \end{aligned}$$

- f is minimal $\Longleftrightarrow \Phi \equiv 0 \Longrightarrow \Psi$ is holomorphic.
- f is Hamiltonian stationary Lagrangian (Hamiltonian minimal) $\Longleftrightarrow \Phi$ is holomorphic.
- f is twistor harmonic $\Longleftrightarrow \Psi$ is holomorphic.

Remark 1.

$\alpha_H := \omega(H, \cdot) = i(\Phi - \bar{\Phi})$.

Minimal Lagrangian surfaces in $\mathbb{C}P^2$

Minimal Lagrangian $\Longleftrightarrow \Phi \equiv 0 \Longrightarrow$

$$\begin{aligned} u_{z\bar{z}} &= e^{-2u}|\psi|^2 - e^u, \\ \psi_{\bar{z}} &= 0, \end{aligned}$$

which are invariant under the transformation $\Psi \to e^{it}\Psi$ for $t \in \mathbb{R}$.

$$\begin{tikzcd} C\tilde{\Sigma} \arrow[r, "\mathrm{SL}"] \arrow[d] & \mathbb{C}^3 \arrow[d] \\ \tilde{\Sigma} \arrow[r, "\text{min. Leg.}"] \arrow[d] & S^5 \arrow[d] \\ \Sigma \arrow[r, "\text{min. Lag.}"] & \mathbb{C}P^2 \end{tikzcd}$$

It gives rise to a local model of singular special Lagrangian 3-folds in Calabi-Yau threefolds.

CMC surfaces in $\mathbb{R}^3$ and mLi surfaces in $\mathbb{C}P^2$

$$u_{z\bar{z}} + e^{u} - e^{-u}|Q|^2 = 0, \qquad Q_{\bar{z}} = 0.$$

- Qdz^2 Hopf differential
- $g(\Sigma) = 0 \Rightarrow \Sigma =$ the round sphere
- normalize $Q = 1$:

$$u_{z\bar{z}} + e^{u} - e^{-u} = 0 \quad \text{(Sinh-Gordon equ.)}$$

- PDE $\to$ ODE (Abresch)
- Pinkall-Sterling
- CMC tori are constructed in terms of theta functions (Bobenko)
- CMC surfaces of higher genus by gluing constructions (Kapouleas)

$$u_{z\bar{z}} + e^{u} - e^{-2u}|\psi|^2 = 0, \qquad \psi_{\bar{z}} = 0.$$

- $\Psi = \psi dz^3$ cubic Hopf differential
- $g(\Sigma) = 0 \Rightarrow \Sigma$ totally geodesic
- normalize $\psi = -1$:

$$u_{z\bar{z}} = e^{-2u} - e^{u} \quad \text{(Tzitzéica equ.)}$$

- PDE $\to$ ODE (Castro-Urbano)
- MLi tori are constructed in terms of Prym-theta functions (M.-Ma, Caberry-McIntosh, Mironov)
- MLi surfaces of higher genus by gluing constructions (Haskins-Kapouleas)

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Hamiltonian minimal (Hamiltonian stationary Lagrangian) submanifolds

Lawson-Simons: Any stable minimal submanifolds in $\mathbb{C}P^n$ is a complex submanifold.
$\Longrightarrow$ The minimal Lagrangian submanifolds in $\mathbb{C}P^n$ cannot be stable under general variations.

- (Y.G. Oh) Hamiltonian minimal (or called "Hamiltonian stationary Lagrangian"by Schoen-Wolfson) submanifolds of a Kähler manifold are the critical points of the volume functional on Lagrangian submanifolds under any (compactly supported) Hamiltonian variations.
- It is a generalization for minimal Lagrangian submanifolds.
- A Lagrangian immersion f is H-minimal/HSL $\Longleftrightarrow \delta\alpha_H = 0 \Longleftrightarrow \Phi$ is holomorphic.

HSL Surfaces in $\mathbb{C}P^2$

- A Lagrangian surface in $\mathbb{C}P^2$ is Hamiltonian stationary if and only if Φ is holomorphic.

$\Rightarrow$ The compatibility equations for HSL surfaces in $\mathbb{C}P^2$ are

$$\begin{aligned} \phi_{\bar{z}} &= 0, \\ u_{z\bar{z}} + e^u + |\phi|^2 - e^{-2u}|\psi|^2 &= 0, \\ e^{-u}\psi_{\bar{z}} &= \phi_z - u_z\phi, \end{aligned}$$

Remark 1.The above equations are invariant w.r.t. the transformation $\Phi \to \nu\Phi$, $\Psi \to \nu\Psi$ for $\nu \in S^1$.
2. $\alpha_H = i(\Phi - \bar{\Phi})$.
3. Any HSL sphere in $\mathbb{C}P^2$ is totally geodesic.

Hamiltonian stationary Lagrangian surfaces in $\mathbb{C}P^2$

- Hélein-Romon formulated HSL surfaces in 2-dim. Hermitian symmetric spaces are solutions of integrable systems as the vanishing of the curvature of a certain 1-parameter family of flat connections.
- Terng explained the HSL surfaces in $\mathbb{C}P^2$ as the second elliptic $SU(3)/SU(2)$-system.
- M., Hélein-Romon: Any HSL torus in $\mathbb{C}P^2$ can be constructed from a pair of commuting Hamiltonian ODEs on a finite dimensional subspaces of a certain loop Lie algebra, i.e., of finite type.
- M.-Schemies, Mironov: reduced PDE to ODE and gave new explicit examples for HSL for tori which are invariant under a one-parameter group of isometries of $\mathbb{C}P^2$.
- Hunter-McIntosh studied the spectral data of HSL tori in $\mathbb{C}P^2$.

Twistor harmonic surfaces

A surface in $\mathbb{C}P^2$ is called twistor harmonic if its twistor lift to $SU(3)/T^2$ is harmonic map.

Proposition(Castro-Urbano)

A Lagrangian immersion of an oriented surface is twistor harmonic if and only if Ψ is holomorphic.

Castro-Urbano classified all compact twistor harmonic non-minimal Lagrangian surfaces in $\mathbb{C}P^2$.

Classical Bonnet problem

Classical Bonnet Problem

Whether the metric e^u and the mean curvature function H are suffice to determine a surface in $\mathbb{R}^3$ up to rigidity motions?

- Around a non-umbilical point, a surface in $\mathbb{R}^3$ is determined uniquely by the metric e^u and the mean curvature function H, except three classes of surfaces: CMC surfaces, Bonnet surfaces and Bonnet pairs.
- Much less is known about Bonnet pairs.
- Lawson-Tribuzy showed that for compact oriented surfaces in $\mathbb{R}^3$ with nonconstant mean curvature, there are at most two surfaces with the given metric and mean curvature.
- There are no Bonnet pairs of genus zero in $\mathbb{R}^3$.
- It is open whether compact Bonnet pairs exist.

Lagrangian Bonnet pairs

Lagrangian Bonnet Problem

Whether the metric e^u and the mean curvature form Φ are suffice to determine a Lagrangian surface in $\mathbb{C}P^2$ up to rigidity motions?

Lagrangian Bonnet pairs in $\mathbb{C}P^2$ are two non-congruent isometric surfaces with the same mean curvature 1-form Φ.

Theorem 1 (He-M.).

Let Σ be a compact oriented Lagrangian surface in $\mathbb{C}P^2$. If Σ is not twistor harmonic, then there exists at most two noncongruent isometric immersions of Σ in $\mathbb{C}P^2$ with the mean curvature form Φ.

- He-M.-Wang: The above result also holds in $\mathbb{C}^2$ and $\mathbb{C}H^2$

Further question on Lagrangian Bonnet pairs

Remark.

Certain S^1-invariant HSL tori in $\mathbb{C}P^2$ provide examples of compact Lagrangian Bonnet pairs in $\mathbb{C}P^2$.

The classical theory of Bonnet pairs in $\mathbb{R}^3$ is related to the theory of isothermic surfaces and belongs also the geometry described by integrable systems.

Question.

Does there exist a Lagrangian version of isothemic surfaces in the complex space forms? Do Lagrangian Bonnet pairs relate to integrable systems?

Index of minimal Lagrangian surfaces in $\mathbb{C}P^2$

- J. Simons: Any compact minimal surface Σ in $S^3(1)$ is unstable and $\mathrm{Ind}(\Sigma) \geq 1$. "="holds $\Longleftrightarrow$ Σ is totally geodesic.
- Urbano: Let Σ be a compact orientable nontotally geodesic minimal surface in $S^3(1)$. Then $\mathrm{Ind}(\Sigma) \geq 5$. "= "holds $\Longleftrightarrow$ Σ is the Clifford torus.

- Lawson-Simons: Any minimal Lagrangian surface in $\mathbb{C}P^2$ is unstable.
- Urbano: For any minimal Lagrangian compact orientable surface in $\mathbb{C}P^2$, $\mathrm{Ind}(\Sigma) \geq 2$. "= "holds $\Longleftrightarrow\Sigma$ is the Clifford torus.
- Urbano: For any minimal Lagrangian compact nonorientable surface of $\mathbb{C}P^2$, $\mathrm{Ind}(\Sigma) \geq 3$. "= "holds $\Longleftrightarrow$ Σ is the totally geodesic $\mathbb{R}P^2$.

Energy of a Lagrangian torus

$f: \Sigma \to \mathbb{C}P^2$ a Lagrangian torus with $ds^2 = 2e^{v(x,y)}(dx^2 + dy^2)$ is the image of $r: \mathbb{R} \to S^5 \to \mathbb{C}P^2$, where r is a horizontal lift.

- A $2D$ periodic Schrödingier operator is associated with f.

$$Lr = 0,\ L = (\partial_x - \frac{i\beta_x}{2})^2 + (\partial_y - \frac{i\beta_y}{2})^2 + V(x,y),\ V = 4e^v + \frac{1}{4}(\beta_x^2 + \beta_y^2) + \frac{i}{2}\Delta\beta,$$

  where β is the Lagrangian angle of Σ.

- M.-Mironov-Zuo (2018) introduced an energy functional on the set of Lagrangian tori.

$$E(\Sigma) = \int_\Sigma V dx \wedge dy.$$

- The energy of a Lagrangian torus is

$$E(\Sigma) = W^-(\Sigma) = 2\int_\Sigma d\sigma + \frac{1}{4}\int_\Sigma |\mathbf{H}|^2 d\sigma$$

  coincides with the Willmore functional W^- introduced by Montiel-Urbano (2002), where $\mathbf{H} = \mathrm{tr}h$.

Montiel-Urbano conjecture (Lagrangian version of Willmore conjecture)

The Clifford torus achieves the minimum of the Willmore functional W^- either amongst all tori in $\mathbb{C}P^2$ or amongst all Lagrangian tori in $\mathbb{C}P^2$.

- Homogeneous tori $T_{r_1,r_2,r_3} := \pi(\{(r_1 e^{i\alpha_1}, r_2 e^{i\alpha_2}, r_3 e^{i\alpha_3}) \in S^5 | \alpha_j \in \mathbb{R}\}) \subset \mathbb{C}P^2$
- $E(T_{r_1,r_2,r_3}) = \frac{\pi^2(1-r_1^2)(1-r_2^2)(1-r_3^2)}{r_1 r_2 r_3} \geq E(T_{Cl}) = 2\text{Area}(T_{Cl}) = \frac{8\pi^2}{3\sqrt{3}}$.
- Kazhymurat (2018): The value of the energy functional on any S^1-equivariant Hamiltonian minimal Lagrangian tori in $\mathbb{C}P^2$ given by Jacobi elliptic functions is strictly larger than $E(T_{Cl})$.
- The conjecture is still open.

Ruh-Vilms theorem for minimal Lagrangian surfaces in $\mathbb{C}P^2$

Classical Theorem of Ruh-Vilms (1970)

Any immersed surface $\mathbf{x} : \Sigma^2 \to \mathbb{R}^3$ has CMC $\Longleftrightarrow$ its Gauss map $\mathbf{n} : \Sigma^2 \to S^2$ is harmonic.

Dorfmeister-Kobayashi-M. (2020)

A surface $\Sigma \to \mathbb{C}P^2$ is minimal Lagrangian $\Longleftrightarrow$ its Gauss map $\mathcal{G} : \Sigma \to SU(3)/U_1$, $p \mapsto (\mathfrak{f}(p), \mathrm{span}_{\mathbb{R}}\{\xi(p), \eta(p), \mathfrak{f}(p)\})$, is primitive harmonic relative to the automorphism σ of order 6.

Back to integrable systems

Passing to a gauge equivalent frame function

$$\mathbb{F}(\lambda) = \mathcal{F}(\nu) \begin{pmatrix} -i\lambda & 0 & 0 \\ 0 & \frac{1}{i\lambda} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $i\lambda^3\nu = 1$, normalized by $\mathbb{F}(z, \bar{z}, \lambda) \in SU(3)$.
Then

$$\mathbb{F}^{-1}\mathbb{F}_z = \begin{pmatrix} \frac{u_z}{2} & 0 & i\lambda^{-1} e^{\frac{u}{2}} \\ -i\lambda^{-1}\psi e^{-u} & -\frac{u_z}{2} & 0 \\ 0 & i\lambda^{-1} e^{\frac{u}{2}} & 0 \end{pmatrix} := \lambda^{-1} U_{-1} + U_0,$$

$$\mathbb{F}^{-1}\mathbb{F}_{\bar{z}} = \begin{pmatrix} -\frac{u_{\bar{z}}}{2} & -i\lambda\bar{\psi}e^{-u} & 0 \\ 0 & \frac{u_{\bar{z}}}{2} & i\lambda e^{\frac{u}{2}} \\ i\lambda e^{\frac{u}{2}} & 0 & 0 \end{pmatrix} := \lambda V_1 + V_0.$$

We will always interpret the family $\mathbb{F}_\lambda$ as a map into the twisted loop group

$$\Lambda SU(3)_\sigma = \{g : S^1 \to SU(3) \text{ smooth} \mid g(\epsilon\lambda) = \sigma g(\lambda)\}.$$

The associated family of a minimal Lagrangian immersion

Theorem

Let M be a non-compact Riemann surface

Let $f: M \to \mathbb{C}P^2$ a conformal minimal Lagrangian immersion

Let $\mathbb{F} : M \to SU(3)$ be the Hermitian moving frame of f

Let M_* be the universal cover of M (lift f and F to M)
Then there exists an S^1-family $f_\lambda = f(z, \bar{z}, \lambda)$ of conformal minimal Lagrangian immersions and a family of frames $\mathbb{F}_\lambda = \mathbb{F}(z, \bar{z}, \lambda)$ defined on M such that

- $f(z) = f(z, \lambda = 1)$
- $\mathbb{F}_\lambda : M_* \to SU(3)$ is the Hermitian frame of f_λ
- $\mathbb{F}_\lambda^{-1} d\mathbb{F}_\lambda : M_* \to su(3)$ is a one-parameter family of flat connections.

f_λ is called associated family of f
$\mathbb{F}_\lambda$ is called extended frame of f

Loop group method (DPW)

Loop group method (DPW):

- View such a one-parameter family of connections as a single connection for which the structure group is a loop group.
- Produce such connections by the Riemann-Hilbert factorization of loop groups.

Dorfmeister-Pedit-Wu develop a systematic scheme for the construction of all harmonic maps from a simply connected Riemann surface to a k-symmetric space

Consider

$$\Lambda G = \{h : S^1 \to G\},$$

the group of loops in G.

Actually, we always consider only loops, for which each matrix coefficient has an absolutely convergent Fourier series. Then ΛG is a Banach Lie group and all the operations which we will discuss are admissible.

Loop group formulations

Let σ be an automorphism of $SL(3,\mathbb{C})$ of order 6 defined by

$$\sigma : g \mapsto P(g^t)^{-1}P^{-1}, \quad P = \begin{pmatrix} 0 & \alpha & 0 \\ \alpha^2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \alpha = e^{2\pi i/3},$$

and τ is the conjugation of $SL(3,\mathbb{C})$ for the real form $SU(3)$ given by

$$\tau(g) := (\bar{g}^t)^{-1}.$$

Then the corresponding automorphism σ of order 6 and the anti-holomorphic automorphism τ of $sl(3,\mathbb{C})$ are

$$\sigma : \xi \mapsto -P\xi^t P^{-1}, \quad \tau : \xi \mapsto -\bar{\xi}^t.$$

Loop group formulation

Now consider the twisted loop groups and loop algebras

$$\begin{aligned}
\Lambda SL(3,\mathbb{C})_\sigma &:= \{g : S^1 \to SL(3,\mathbb{C}) | \sigma(g(\lambda)) = g(\epsilon\lambda)\},\\
\Lambda sl(3,\mathbb{C})_\sigma &:= \{\xi : S^1 \to sl(3,\mathbb{C}) | \sigma(\xi(\lambda)) = \xi(\epsilon\lambda)\},\\
\Lambda SU(3)_\sigma &= \{g \in \Lambda SL(3,\mathbb{C})_\sigma | \tau(g(\frac{1}{\bar\lambda})) = g(\lambda)\},\\
\Lambda su(3)_\sigma &= \{\xi \in \Lambda sl(3,\mathbb{C})_\sigma | \tau(\xi(\frac{1}{\bar\lambda})) = \xi(\lambda)\}.
\end{aligned}$$

Set

$$\Lambda_d = \{\xi \in \Lambda sl(3,\mathbb{C})_\sigma | \xi = \sum_{|k|\le d} \lambda^k \xi_k\}$$

for $d \in \mathbb{N}$.

Loop group formulation

In order to describe the symmetry of the flat connection, we need introduce additional symmetries s.t. we have twisted loop groups and twisted loop algebras.

Using loop group terminology, we have

Let $f\colon M \to \mathbb{C}P^2$ be a conformal parametrization of a simply connected Riemann surface. Then the following statements are equivalent:

1. f is minimal Lagrangian.
2. $\mathbb{F}^{-1}d\mathbb{F} = (\lambda^{-1}U_{-1} + U_0)dz + (\lambda V_1 + V_0)d\bar{z} \subset \Lambda su(3)_\sigma$ is a one-parameter family of flat connections.

Holomorphic extended frame – a $\bar{\partial}$-problem

Theorem (holomorphic extended frame)

If $\mathbb{F}(z, \bar{z}, \lambda)$ is the extended frame of a minimal Lagrangian surface $f\colon M \to \mathbb{C}P^2$. If M is non-compact, then F can be written on M_* in the form

$$\mathbb{F}(z, \bar{z}, \lambda) = C(z, \lambda)\, V_+(z, \bar{Z}, \lambda),$$

where C is holomorphic for $z \in M_*$ and $V_+ \in \Lambda^+ SL(3, \mathbb{C})_\sigma$ solves the $\bar{\partial}$-problem

$$\bar{\partial} V_+ V_+^{-1} = -(\alpha_{\mathfrak{k}}'' + \lambda \alpha_{\mathfrak{m}}''), \qquad V_+(0) = I$$

over M_*, so that $C \in \Lambda SL(3, \mathbf{C})_\sigma$ gives a holomorphic extended frame.

Moreover, for every $\gamma \in \pi_1(M)$,

$$\mathbb{F}(\gamma \cdot z, \overline{\gamma \cdot z}, \lambda) = \chi(\gamma, \lambda)\mathbb{F}(z, \bar{z}, \lambda),$$

$$C(\gamma \cdot z, \lambda) = \chi(\gamma, \lambda)\, C(z, \lambda)\, W_+(z, \lambda),$$

where $\chi(\gamma, \lambda) \in \Lambda SU(3)_\sigma$.

The loop group method for minimal Lagrangian immersions

Surface $\Rightarrow$ Potential

η will be called a "holomorphic potential"

$\eta = C^{-1} dC = \lambda^{-1}\eta_{-1} + \lambda^0\eta_0 + \lambda^1\eta_1 + \cdots$

$\Uparrow$

$C(z,\lambda)$ hol. extended frame

$\Uparrow$

Decompose $F_\lambda = CV_+$,

C holomorphic in z

V_+ holomorphic for $|\lambda| < 1$ ($\bar{\partial}$-problem)

$\Uparrow$

$F_\lambda : M_* \to \Lambda SU(3)_\sigma$ extended frame

$\Uparrow$

mLi surface $f : M_* \to \mathbb{C}P^2$

Potential $\Rightarrow$ Surface

Start from some "holomorphic potential" ξ, a holomorphic $(1,0)$-form

$\xi = \lambda^{-1}\xi_{-1} + \lambda^0\xi_0 + \lambda^1\xi_1 + \cdots$

$\Downarrow$

Solve the ODE $dC = C\xi$

$\Downarrow$

Decompose $C = F_\lambda (V_+)^{-1}$

Iwasawa splitting $\leftrightarrow$ Gram-Schmidt

$\Downarrow$

$F_\lambda : M_* \to \Lambda SU(3)_\sigma$ extended frame of a mLi

$\Downarrow$

$f_\lambda = [F_\lambda] : M_* \to \mathbb{C}P^2$ is mLi

Example

For the Clifford torus $f\colon \mathbb{C} \to \mathbb{C}P^2$, we have a horizontal lift $F\colon \mathbb{C} \to S^5(1)$ as follows

$$F(z,\bar{z}) = \frac{1}{\sqrt{3}}(e^{z-\bar{z}}, e^{\alpha z - \alpha^2 \bar{z}}, e^{\alpha^2 z - \alpha \bar{z}}),$$

where $\alpha = e^{\frac{2}{3}\pi i}$. It is easy to see that $\psi = F_{zz} \cdot \overline{F_{\bar{z}}} = -1$ and $e^u = 1$.

- The extended frame $\mathbb{F}(z,\bar{z},\lambda) = \begin{pmatrix} -i\lambda F_z & -i\lambda^{-1} F_{\bar{z}} & F \end{pmatrix}$.
- $\mathbb{F}^{-1} d\mathbb{F} = \lambda^{-1} \begin{pmatrix} 0 & 0 & i \\ i & 0 & 0 \\ 0 & i & 0 \end{pmatrix} dz + \lambda \begin{pmatrix} 0 & i & 0 \\ 0 & 0 & i \\ i & 0 & 0 \end{pmatrix} d\bar{z} =: U dz + V d\bar{z}$
- The holomorphic potential $\eta = iDdz = \begin{pmatrix} 0 & i\lambda & i\lambda^{-1} \\ i\lambda^{-1} & 0 & i\lambda \\ i\lambda & i\lambda^{-1} & 0 \end{pmatrix} dz.$
- $\mathbb{F} = \exp(Uz + V\bar{z}) = \exp(xiD) \cdot \exp(i(U - V)y) = \exp(ziD)\exp(2iyV)$
- The normalized potential $\eta = iDdz = \lambda^{-1} \begin{pmatrix} 0 & 0 & i \\ i & 0 & 0 \\ 0 & i & 0 \end{pmatrix} dz.$

Minimal Lagrangian immersions in $\mathbb{C}P^2$ with symmetries

Definition 6.1.

For a minimal Lagrangian immersion $f : M \to \mathbb{C}P^2$, a *symmetry* will always be a pair $(\gamma, \mathcal{R}) \in (\mathrm{Aut}(M), \mathrm{Iso}_0(\mathbb{C}P^2) = PSU(3))$, such that

$$f(\gamma.z) = \mathcal{R}f(z) \quad \text{for all} \quad z \in M$$

holds.

Dorfmeister-M. characterized minimal Lagrangian immersions $f : \mathbb{D} \to \mathbb{C}P^2$ defined on a contractible Riemann surface with

- a symmetry $(\gamma, \mathcal{R})$ of f, where γ has a fixed point in $\mathbb{D}$.
- a symmetry $(\gamma, \mathcal{R})$ of f minimal Lagrangian immersions, where $\mathcal{R}$ has finite order, but γ has no fixed point.

Homogeneous minimal Lagrangian surfaces in $\mathbb{C}P^2$

1. Every minimal Lagrangian immersion $f\colon S^2 \to \mathbb{C}P^2$ is homogeneous and $f(S^2)$ is, up to isometries of $\mathbb{C}P^2$, contained in $\mathbb{R}P^2$.
2. If $\mathbb{D}$ denotes the unit disk in $\mathbb{C}$, then there does not exist any homogeneous, minimal Lagrangian immersion $f\colon \mathbb{D} \to \mathbb{C}P^2$.
3. Every homogeneous minimal Lagrangian immersion $f\colon \mathbb{C} \to \mathbb{C}P^2$ is isometric to the Clifford torus.

A minimal Lagrangian immersion $f\colon M \to \mathbb{C}P^2$ is *equivariant*, i.e., it admits a one-parameter group $(\gamma_t, R(t)) \in (\mathrm{Aut}(M), \mathrm{Iso}_0(\mathbb{C}P^2))$ of extrinsic automorphisms. Then, up to a biholomorphic change of domains and possibly a transition to the universal cover, one obtains exactly two types of immersions, all defined on contractible domains in $\mathbb{C}$.

1. rotationally equivariant minimal Lagrangian immersions, i.e. those, where the one-parameter group is the full group of rotations about a point $z_0 \in \mathbb{D}$,
2. translationally equivariant minimal Lagrangian immersions, i.e. those, where $\mathbb{D}$ can be realized as a strip and the one-parameter group as the full group of translations (without loss of generality, parallel to the real axis).

(Dorfmeister-M.)

Any minimal Lagrangian immersion f from $\mathbb{C}$ or S^2 into $\mathbb{C}P^2$ which is rotationally equivariant has a vanishing cubic Hopf differential, and therefore is totally geodesic in $\mathbb{C}P^2$ and its image is, up to isometries of $\mathbb{C}P^2$, contained in $\mathbb{R}P^2$.

(Dorfmeister-M.)

For any translationally equivariant minimal Lagrangian immersion, the extended frame $\mathbb{F}$ can be chosen such that $\mathbb{F}(0, \lambda) = I$ and

$$\mathbb{F}(t + z, \lambda) = \chi(t, \lambda)\mathbb{F}(z, \lambda),$$

holds, where $\chi(t, \lambda)$ is a one-parameter group in $SU(3)$.

By Burstall-Kilian,

Theorem

A minimal Lagrangian surface in $\mathbb{C}P^2$ is translationally equivariant if and only it is generated by a degree one constant potential Ddz, where D has form of

$$D = i \begin{pmatrix} 0 & \lambda\bar{b} & \lambda^{-1}a \\ \lambda^{-1}b & 0 & \lambda\bar{a} \\ \lambda\bar{a} & \lambda^{-1}a & 0 \end{pmatrix} \in \Lambda su(3)_\sigma$$

where a is nonzero purely imaginary.

Entire radially symmetric minimal Lagrangian surfaces

Thereom (Dorfmeister-M.)

If the normalized potential $\eta(z,\lambda) = \lambda^{-1} D(z,\lambda)\, dz$ of a minimal Lagrangian immersion $f\colon \mathbb{C} \to \mathbb{C}P^2$ satisfies

$$\eta(p_t z, q_t \lambda) = T\eta(z,\lambda)\, T^{-1},$$

with $p_t z = e^{ip_0 t} z$, $q_t \lambda = e^{iq_0 t}\lambda$, p_0, q_0, $t \in \mathbb{R}$ and $T = \mathrm{diag}(\tau, \tau^{-1}, 1)$ with $\tau = e^{it_0 t}$, (called an entire radially symmetric minimal Lagrangian surface,) then the metric only depends on the radius and the Hopf differential ψ is of the form $\psi(z) = \psi_0 z^{2k+n}$ with ψ_0 a complex number.

Theorem (Dorfmeister-M.)

The metric of an entire radially symmetric minimal Lagrangian surface with metric $g = 2e^u dz d\bar{z}$ and cubic form $\psi(z) dz^3 = \psi_0 z^{2k+n} dz^3$ only depends on r and satisfies the Painlevé equation PIII of type D_7, with $h = h(s) = e^{u(r(s))} s^{\frac{1-2k-n}{2k+n+3}}$ satisfying

$$\ddot{h} = \frac{(\dot{h})^2}{h} - \frac{\dot{h}}{s} - \frac{16}{(2k+n+3)^2}\frac{h^2}{s} + \frac{16|\psi_0|^2}{(2k+n+3)^2}\frac{1}{h}.$$

Theorem(Dorfmeister-M.)

Let $f\colon \mathbb{C} \to \mathbb{C}P^2$ be an entire radially symmetric minimal Lagrangian surface with metric $e^{u(r)}$ and cubic form $\psi(z)dz^3 = \psi_0 z^{2k+n} dz^3$ and let

$$h(s) = e^{u(r(s))} s^j, \tag{6.1}$$

with $s = r^l$, where $l = \frac{1}{2}(2k + n + 3)$and $jl = \frac{1}{2}(1 - 2k - n)$. Then for $s \to 0$ the function $h(s)$ has the asymptotic behaviour

$$\log(h(s)) \approx \frac{2k - n + 1}{2k + n + 3} \log s + 2 \log |a_k| + o(s). \tag{6.2}$$

Examples of entire radially symmetric minimal Lagrangian surfaces in $\mathbb{C}P^2$ can be given.

Thanks for your attention!