Nijenhuis Geometry: singularities and global issues

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(based on joint results with A. Bolsinov (Loughborough), and A. Konyaev (Moscow))

Geometry seminar
April the 5th, 2021

Philosophic question

WHAT IS THE (SIMPLEST) DIFFERENTIALGEOMETRIC CONDITION ON AN (1,1)-TENSOR (=OPERATOR)?

- DIFFERENTIALGEOMETRIC = can be written as a tensorial equation; essentially the same as that its fulfilment does not depend on the coordinate system.
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Though the subject of my talk is $(1,1)$-tensors, let me ask (and give known answers to) the same question $(2,0)$- and $(0,2)$-tensors. There are 4 cases,

- Symmetric (0,2)-tensors, skewsymmetric (0,2)-tensors
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On the next slides I review the situation in the cases indicated by red colour

Simplest geometric condition for skewsymmetric (2,0)-tensors

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$$\{f, h\} = \sum \tfrac{\partial f}{\partial x_i} P^{ij} \tfrac{\partial h}{\partial x_j}. \qquad (*)$$

  - The Jacobi identity is

$$\{g, \{f, h\}\} + \{f, \{h, g\}\} + \{h, \{g, f\}\} = 0. \qquad (**)$$

  - The condition that $(*)$ satisfies $(**)$ is a nonlinear (quadratic) system of PDEs of the first order on the components of P.

Simplest geometric condition on symmetric (0,2)-tensors

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$$R^i{}_{jk\ell} = 0$$
It is a nonlinear system of PDEs of second order on g.

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Each of these "simplest geometric conditions" started a science:

- Riemann introduced the Riemannian curvature tensor in 1861. This was the start of the Riemannian geometry.
  - Novosibirsk State University has extremely strong group (Taimanov, Mironov ,...) in Riemannian geometry
- Canonical Poisson bracket was used by Poisson in 1806 and in the modern form was introduced by Jacobi in 1836; the development of their ideas is now called "the theory of Hamiltonian systems".

Let us now concentrate on (1,1)-tensors:

As the simplest geometric condition on (1,1)-tensor L we suggest the condition that the Nijenhuis torsion of L vanishes:

$$\mathcal{N}_L = 0.$$

Definition

For two vector fields ξ, η , its Nijenhuis torsion is given by:

$$\mathcal{N}_L(\xi, \eta) = L^2[\xi, \eta] + [L\xi, L\eta] - L[L\xi, \eta] - L[\xi, L\eta].$$

Fact: $\mathcal{N}_L$ is actually a $(1, 2)$-tensor, in local coordinates it is given by the nonlinear formula

$$(\mathcal{N}_L)^i_{jk} = L^\ell_j \frac{\partial L^i_k}{\partial x^\ell} - L^\ell_k \frac{\partial L^i_j}{\partial x^\ell} - L^i_\ell \frac{\partial L^\ell_k}{\partial x^j} + L^i_\ell \frac{\partial L^\ell_j}{\partial x^k}.$$

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By Nijenhuis operators we understand $(1, 1)$-tensors $L = \big(L^i_j(x)\big)$ with vanishing Nijenhuis torsion.

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- A manifold M endowed with such an operator is called a Nijenhuis manifold.
- Nijenhuis geometry studies Nijenhuis manifolds

Let me draw the following analogy:

- Poisson geometry studies manifolds equipped with Poisson structure
- Nijenhuis geometry studies manifolds equipped with Nijenhuis operators, that is, (1,1)-tensors L with $\mathcal{N}_L = 0$.

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Why vanishing of $\mathcal{N}_L$ is the simplest geometric condition?

- **Theorem (Puninskii 2014; follows from Kolar-Michor-Slovak 1993)** The only nontrivial differentialgeometric operation from $(1,1)$-tensors to $(1,2)$-tensors that is homogeneous of degree 2 is the correspondence $L \mapsto \mathrm{const} \cdot \mathcal{N}_L$.
- **Remark.** "Trivial" operations of this type are just suitable algebraic expressions in L, $d\,\mathrm{trace}\,L$ and $d\,\mathrm{trace}\,L^2$.

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- Propose a program how to study them
- Demonstrate that program is realistic by presenting first local and global results.

In what situations Nijenhuis operators appeared

Corollary of the above mentioned result of Puninskii. For a geometric system of partial differential equations whose coefficients are linearly constructed by a $(1,1)$-tensor, the first differential compatibility conditions are related to $\mathcal{N}_L$, $L^* d \operatorname{trace} L$ and $d \operatorname{trace} L^2$.
Because of it, Nijenhuis operators appeared in many unrelated subjects:

- In theory of geodesically equivalent metrics (I will give definitions and discuss this case)

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- In the the theory of integrable systems, Nijenhuis operators occur as recursion operators (for both finite- and infinite-dimensional cases like systems of KdV equations) for compatible bi-Hamiltonian systems: Magri-Morosi 1984, Gelfand-Zakharevich 2001.

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- Nijenhuis operators are essentially the same as Riemann invariants in the theory of infinite-dimensional integrable systems of hydrodynamic type (Dubrovin, Novikov, Olver, Fordi, Mironov, Tzarev, Ferapontov)

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Two metrics (on one manifold) are geodesically equivalent if they have the same unparametrized geodesics.

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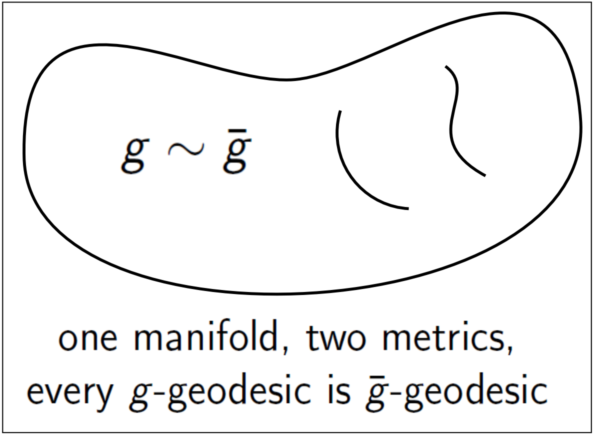

It is a natural topic started by Lagrange, Beltrami and Levi-Civita.

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  where
$$L_{ij} = \left| \frac{det \bar{g}}{\det g} \right|^{\frac{1}{n+1}} \bar{g}^{rs} g_{rj} g_{si} = \left| \frac{det \bar{g}}{\det g} \right|^{\frac{1}{n+1}} g \bar{g}^{-1} g.$$
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  Above we use g for covariant differentiation, and below we use g for index manipulations.

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Since the Christoffel symbols of the first kind

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Fact (Bolsinov-Matveev 2003). Compatibility conditions for this system are $\mathcal{N}_L = 0$.

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- The first results of my talk generalize these three Theorems for many Jordan blocks whose eigenvalues may be complex-valued.

Splitting theorem

Let $\chi_{L(p)}(t) = \chi_1(t)\,\chi_2(t)$ be a factorisation of the characteristic polynomial of L at a point $p \in M$ into two factors with no common roots (over $\mathbb{R}$). We call such factorisations *admissible*. This factorisation can be naturally extended to a neighborhood of p.

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Consider the distributions $\mathcal{D}_i = \operatorname{Ker} \chi_i(L)$ ($i = 1, 2$) that provide a natural decomposition of the tangent bundle $TM = \mathcal{D}_1 \oplus \mathcal{D}_2$.

Theorem

Let L be a Nijenhuis operator. The distributions $\mathcal{D}_i = \operatorname{Ker} \chi_i(L)$ are both integrable. Moreover, in any adapted coordinate system $(x_1, \dots, x_r, y_{r+1}, \dots, y_n)$:

$$L(x,y) = \begin{pmatrix} L_1(x) & 0 \\ 0 & L_2(y) \end{pmatrix} = \text{blockdiagonal}(L_1(x), L_2(y)). \qquad (1)$$

In other words, L splits into a direct sum of two Nijenhuis operators: $L(x,y) = L_1(x) \oplus L_2(y)$. By construction, $\chi_{L_1} = \chi_1$ and $\chi_{L_2} = \chi_2$

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Corollary

Every Nienhuis operator L locally splits into a direct sum of Nijenhuis operators $L = L_1 \oplus L_2 \oplus \cdots \oplus L_k$ each of which at the point $p \in M$ has either a single real eigenvalue or a single pair of complex eigenvalues.

Result of Haanties cited above and repeated in the box below is a direct corollary of our result

Theorem (Haanties 1955). Let L be a Nijenhuis operator such that it is diagonalisable at every point and such that all eigenvalues are real. Then, $L = \mathrm{blockdiagonal}\left(\lambda(X_1)\,\mathrm{Id}_{m_1}, ..., \lambda_k(X_k)\,\mathrm{Id}_{m_k}\right).$

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Proof. We consider the following decomposition of the characteristic polynomial:

$$\chi_L = \underbrace{(t - \lambda_1)^{m_1}}_{\chi_1} \cdot ... \cdot \underbrace{(t - \lambda_k)^{m_k}}_{\chi_k}.$$

The decomposition is admissible: the zeros of different χ_i's are different. Then, there exists a coordinate system such that

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where L_k has only one eigenvalue $\lambda_k(X_k)$.

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Since L is diagonalizable, $L_k = \lambda(X_k) \cdot \text{Id}$, $\square$

Important consequence of the Splitting Theorem

- In the local study of Nijenhuis operators, near a point p, one can assume that at p the operator L has one real eigenvalues, or a pair of complex-conjugated eigenvalues.

Generalization of the result of Thompson for blocks with nonconstant eigenvalues

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Theorem. Suppose that in a neighbourhood of a point $p \in M$, a Nijenhuis operator L is algebraically generic and similar to the standard Jordan block with a non-constant real eigenvalue $\lambda(x)$. Then there exists a local coordinate system $x_1, \dots, x_n$ in which the matrix $L(x)$ takes the following form:

$$L(x) = L_{\mathrm{can}} = \begin{pmatrix} \lambda(x_1) & & & & & \\ 1 & \lambda(x_1) & & & & \\ \xi_3 & 1 & \lambda(x_1) & & & \\ \vdots & & 1 & \ddots & & \\ \xi_{n-1} & & & \ddots & \lambda(x_1) & \\ \xi_n & & & & 1 & \lambda(x_1) \end{pmatrix} \quad \text{where} \quad \begin{aligned} \xi_3 &= -\lambda' x_3, \\ \xi_4 &= -\lambda'\, 2x_4, \\ &\vdots \\ \xi_{n-1} &= -\lambda'\,(n-3)\,x_{n-1}, \\ \xi_n &= -\lambda'\,(n-2)x_n, \end{aligned}$$

and $\lambda' = \frac{\partial \lambda}{\partial x_1}$. If $\mathrm{d}\,\lambda(p) \neq 0$, then we may set $\lambda(x_1) = x_1$ and $\lambda' = 1$.

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1. *M is a complex manifold w.r.t. a complex structure J canonically associated with L (in fact $J = f(L)$ where f is an analytic function on $\mathbb{C} \setminus \mathbb{R}$).*

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$$L^{\mathbb{C}} = \sum_{i,j=1}^{n} \ell^i_j(z)\, \mathrm{d}\, z^j \otimes \partial_{z^i}$$
with all the functions $\ell^i_j(z)$ being holomorphic in complex coordinates $z_1, \dots, z_n$.

Nijenhuis operators with complex eigenvalues: Generalized Nirenberg-Newlander theorem

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3. *The complex Nijenhuis torsion of L vanishes, i.e.*
$$\left(\mathcal{N}_L^{\mathbb{C}}\right)^i_{jk} = \ell^m_j \frac{\partial \ell^i_k}{\partial z^m} - \ell^m_k \frac{\partial \ell^i_j}{\partial z^m} - \ell^i_m \frac{\partial \ell^m_k}{\partial z^j} + \ell^i_m \frac{\partial \ell^m_j}{\partial z^k} = 0.$$

Let us compare the formula for $\mathcal{N}_L$ for the complex Nijenhuis torsion with the formula of the Nijenhuis torsion:

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Corollary: Nijenhuis operator L with only complex eingenvalues is given by the same matrix as for the real case, but in complex coordinates: if $dimM = 2n$, then $\ell \in \mathrm{Mat}(n \times n; \mathbb{C})$.

First "global" applications

Theorem

Let L be a Nijenhuis operator on a closed connected manifold M with a non-real eigenvalue $\lambda \in \mathbb{C} \setminus \mathbb{R}$ *at least at one point. Then this number* λ *is an eigenvalue of L with the same algebraic multiplicity at every point of* M*. Shortly: a Nijenhuis operator on a closed manifold may not have non-constant complex eigenvalues.*

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Corollary

The eigenvalues of a Nijenhuis operator on the 4-dimensional sphere S^4 are all real.

Success report and what question we study in the next part of the talk

- **Report.** We described or can describe local structure of Nijenhuis operators near almost every point
- **Next Goal.** What to do with other points?
  - What points are singular?
  - What are nondegenerate singular points?
- **Informal motivations.**
  - In theory of differential equations, a half of papers are on singular points
  - In algebraic geometry, a half of papers are on singular points
  - In topology, many results were proved or can be reproved using functions with nondegenerate singularities (e.g. Morse functions)

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- A singular point $p \in M$ is called *(C^k-) stable*, if for any perturbation
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such that $\widetilde{L}(x)$ is Nijenhuis and $R_k(x)$ has zero of order k at the point $p \in M$, there exists a local diffeomorphism $\phi : U(p) \to \widetilde{U}(p)$, $\phi(p) = p$, that transforms $L(x)$ to $\widetilde{L}(x)$.
- Singular points can be wild. Let us discuss a nondegeneracy conditions on singular points
  - A point $p \in M$ is called *differentially non-degenerate*, if the differentials $\mathrm{d}\,\sigma_1(x), \dots, \mathrm{d}\,\sigma_n(x)$ of the coefficients of the characteristic polynomial of $L(x)$ are linearly independent at this point.

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Theorem

Assume that L is differentially non-degenerate at a point $p \in M$. Then there exists a local coordinate system $x_1, \dots, x_n$ in which L takes the following canonical form (universal for both semisimple and non-semisimple cases!):

$$L = \begin{pmatrix} x_1 & 1 & & & \\ x_2 & 0 & 1 & & \\ \vdots & \vdots & \ddots & \ddots & \\ x_{n-1} & 0 & \dots & 0 & 1 \\ x_n & 0 & \dots & 0 & 0 \end{pmatrix} \tag{2}$$

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At the most singular differentially non-degenerate point $(x_1, .., x_n) = (0, ..., 0)$, the operator is a Jordan block. Actually, at any other point, the is a sum of Jordan blocks with different eigenvalues. In particular, at the generic point the matrix L has n different eigenvalues.

Our vision and programme and where we are?

Poisson geometry	Nijenhuis geometry
< 1970	current situation
Local description at regular points	Local description at regular points
Tool in Hamiltonian Mechanics	Tool in integrable systems
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Shall we try?
Please, join !!!

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- Nijenhuis geometry makes the table multidimensional:

	geod. equiv.	bihamiltonian syst.	systems of hydrod. type
Haanties	Levi-Civita	M & M/G & Z	Mokhov/Ferapontov
Non-diagonal	I discuss this	Turiel	Active topic
Singular points	I discuss this	First Results	First results
Global Issues	First Results	First applications	First results

Local description of geodesically equivalent metrics

Equation for geodesic equivalence:

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- By Haanties Theorem, in the case when L has n different eigenvalues, $L = \mathrm{diag}(\lambda_1(x_1), ..., \lambda_n(x_n))$; when λ_i are not constant, one can even set $\lambda_i = x_i$). We plug it in the equation for geodesic equivalence and obtain (I give an answer in $n = 3$; natural generalisation of the formula holds in any dimension):

$$g = (x_1 - x_2)(x_1 - x_3)f_1(x_1)(dx_1)^2 + (x_2 - x_1)(x_2 - x_3)f_2(x_2)(dx_2)^2 + (x_2 - x_1)(x_3 - x_2)f_3(x_2)(dx_1)^2$$

This is the famous Theorem of Levi-Civita 1896!

Geodesic equivalence if L is a Jordan block

- In our theorem presented above, we described Nijenhuis operators L which are conjugate to a Jordan block with nonconstant eigenvalue. Plugging them in the equation for geodesic equivalence, we obtain

$$g = \begin{pmatrix} & & & 1 & & a_{n-1} \\ & & & & 0 & a_{n-2} \\ & & \iddots & & & \vdots \\ & 1 & & & & a_1 \\ a_{n-1} & a_{n-2} & \dots & a_1 & & \sum_{i=1}^{n-2} a_i a_{n-i-1} \end{pmatrix}$$

$$L = \begin{pmatrix} \lambda(x_n) & 1 & & & a_1 \\ & \lambda(x_n) & \ddots & & a_2 \\ & & \ddots & 1 & \vdots \\ & & & \lambda(x_n) & a_{n-1} \\ & & & & \lambda(x_n) \end{pmatrix}, \quad \text{where}$$

$$\begin{aligned} a_1 &= \lambda'_{x_n} x_1, \\ a_2 &= 2\lambda'_{x_n} x_2, \\ &\dots \\ a_{n-2} &= (n-2)\lambda'_{x_n} x_{n-2}, \\ a_{n-1} &= 1 + (n-1)\lambda'_{x_n} x_{n-1}, \end{aligned}$$

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- As you remember, we described Nijenhuis operators L near a differentially nondegenerate point; the formula is above.

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- **Theorem.** The following metric is geodesically compatible with this L

$$g^{-1} = \begin{pmatrix} 0 & \cdots & 0 & 0 & -1 \\ 0 & \cdots & 0 & -1 & x_1 \\ \vdots & \iddots & \iddots & \iddots & x_2 \\ 0 & -1 & x_1 & \iddots & \vdots \\ -1 & x_1 & x_2 & \cdots & x_{n-1} \end{pmatrix} \tag{3}$$

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- Moreover, in the real-analytic cathegory, every solution $\tilde{g}$ of the geodesic compatibility equation is given by $\tilde{g} = g\ f(L)$ for a real-analytic function f and g given by (3)

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Thank you for your attention!