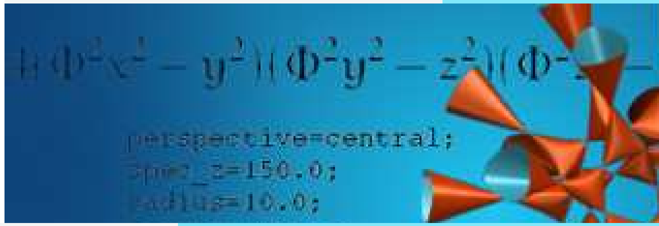
Philipps Universität Marburg

On the (non-)existence of complex structures on S^6



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– dedicated to Yurii Gagarin and Valentina Tereshkova for their contribution to manned space flight missions –

Background: Based on 'MAM 1'

Hopf problem: Does there exist a complex structure on S^6?

Resulted in a special issue of *Differential Geometry and its Applications*, Volume 57, pages 1-146 (April 2018) with a total of 9 articles

– printed copies are still available, if you would like to have one for your library, please contact me! –

The problem is still open. Recent preprints by Atiyah and Etesi did not bring the solution any closer.

Historical perspective

- 1936: Any smooth manifold admits a real-analytic structure (Whitney)
- 1947: Hopf: Is this also true for *complex structures*?

"We ask now: can one – in analogy to this situation – turn any orientable M^n of even dimension (which we may assume to be real-analytic), by introducing suitable local coordinates, into a complex manifold, or do only special topological types of manifolds allow the introduction of local complex-analytic coordinate systems?"

[Wir fragen nun: Kann man – in Analogie zu diesem Sachverhalt – jede orientierbare M^n gerader Dimension (die wir als reell-analytisch annehmen dürfen) durch Einführung geeigneter lokaler Koordinaten zu einer komplexen Mannigfaltigkeit machen, oder gestatten nur spezielle topologische Typen von Mannigfaltigkeiten die Einführung lokaler komplex-analytischer Koordinatensysteme?]

Hopf's first results (1947)

Negative answer: Exhibits infinitely many or. even-dim. mnflds that do not admit a complex structure, among them S^4 and S^8. Furthermore, he writes:

> "I was not able to determine whether or not the spheres S^{2m} with $m \neq 1, 2, 4$ are complex manifolds."

thus stating, implicitly, the problem for the sphere S^6.

Hopf's approach: • considers the sphere bundle over M

• introduces notion of $\mathcal{J}$*-manifold*: a mnfld whose sphere bundle admits a fibre-preserving map for which no direction is mapped to itself or its opposite

• As a complex structure on M induces a complex str. on each tangent space, it turns it into a $\mathcal{J}$-manifold

• derives a topological obstruction to the condition of being an $\mathcal{J}$-manifold

• In a footnote, Hopf mentions an alternative proof that S^4 is not a complex manifold by Ehresmann (published 1949)

The contribution by Kirchhoff (1947)

Observation: An almost complex structure on S^6 can be defined by interpreting S^6 as the purely imaginary Cayley numbers of norm 1.

Thm. For $n > 2$ and $n \neq 8r + 6$, the sphere S^n does not carry any almost complex structure.

His proof is elementary and combines [see Postnikov, *Diff. geometry*]

- if S^{2n} is almost complex, then S^{2n+1} is parallelizable (the $SO(2n+1)$-PFB over S^{2n+1} has a global section);
- as proved by Eckmann and Whitehead, if S^n is parallelizable, then either $n = 1$ or $n = 3$, or n is of the form $8r + 7$.

Of course, nowadays we know that the only parallelizable spheres are S^1, S^3 and S^7 (Adams, 1958), so S^2 and S^6 are the only candidates for an almost complex structure.

The last sentence of Kirchhoff's paper is

> "We do not know, however, if S^6 is a complex manifold."

Basic notions – in modern terms

Dfn. (M^{2n}, g, J) is called *almost Hermitian mnfd* if it's riemannian, $J : TM^{2n} \circlearrowleft$ is an almost complex structure ($J^2 = -\mathrm{Id}$) compatible with g:

$$g(JX, JY) = g(X, Y)\ \forall X, Y \in TM$$

$\Omega(X, Y) := g(JX, Y)$ is called *Kähler form*; M^{2n} is Kähler if $\nabla^g J = 0$.

Fact. Every symplectic mnfd admits a compatible almost complex structure.

'Corollary'. If (M^{2n}, g) admits an almost complex structure, it admits an almost Hermitian structure relative to g.

$\longrightarrow$ already implicitly used by Kirchhoff, but obscure in the sources.

Finally, the following result was not yet available either:

Thm. Let (M^{2n}, J) be an almost complex mnfd. J is integrable (Nijenhuis tensor $N \equiv 0$) iff M is a complex mnfd. [Newlander-Nirenberg, 1957]

Approaching the almost Hermitian structure on S^6

It was not noticed immediately that (and how) this was related to the transitive action of G_2 on S^6!

Thm. The only compact connected simple Lie group which can be transitive on S^{2n} is $\mathrm{SO}(2n+1)$—*except for a a finite number of* $n's$. [Montgomery-Samelson, 1943]

Their method required the knowledge of the homology rings of simple Lie groups, which was not yet available for the five exceptional simple Lie groups; hence they couldn't give any further information on the exceptional cases.

Thm. The only even-dim. sphere with a transitive group G acting that is not orthogonal is S^6 with $G = G_2$. [Borel, 1949]

- Ehresmann, Libermann (1951): Studied the almost complex structure on S^6 in more detail, prove that it's not complex, conclude: 'The structure we considered is therefore locally equivalent to an almost hermitian structure on S^6 admitting G_2 as its group of automorphisms.'

Explicit description I: Hypersurfaces in $\mathrm{Im}\,\mathbb{O} \cong \mathbb{R}^7$ [Calabi, 1958]

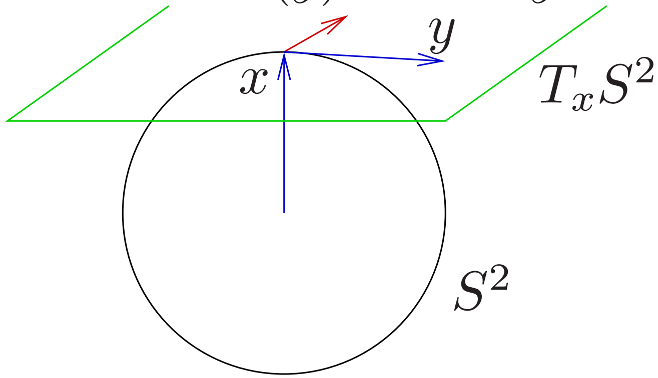
- M^6 a compact hypersurface in $\mathrm{Im}\,\mathbb{O}$
- N: normal vector field
- K: shape operator (Weingarten map)
- Define $J \in \mathrm{End}(TM)$ by

$$J(Y) = N \times Y, \quad Y \in TM$$

- $J^2 = -\mathrm{Id}$ is a non integrable almost complex structure satisfying

for $S^2 \subset \mathbb{R}^3$:

$J(y) := x \times y$



$$\langle(\nabla^g_X J)(Y), Z\rangle \;=\; \langle K(X) \times Y, Z\rangle$$

- For $M^6 = S^6$, $K = \mathrm{Id}$ and J satisfies the simpler eq. $\nabla^g_X J(X) = 0$

Such an almost Hermitian mnfd is called a ‘nearly Kähler manifold’.

Sketch of proof: The cross product on $\mathrm{Im}\,\mathbb{O}$ satisfies:

- $\langle A, (B \times C)\rangle = \langle (A \times B), C\rangle =: (ABC)$ (*scalar triple product identity*),
- $A \times (A \times B) = -|A|^2 B + \langle A, B\rangle A$ (*Malcev identity*),
- $(A \times B) \times C = A \times (B \times C) - \langle A, B\rangle C \quad$ for $C \perp A, B$.

- Prove $J^2 = -\mathrm{Id}$: Malcev id. implies $N \times (N \times X) = -|N|^2 X + \langle N, X\rangle N$ for any $X \in TS$. But $|N| = 1$ and $N \perp X \Rightarrow$ claim.
- Metric is J-compatible: Use scalar triple product and Malcev identity

$\langle J(X), J(Y)\rangle = \langle N \times X, N \times Y\rangle = \langle (N \times X) \times N, Y\rangle = -\langle N \times (N \times X), Y\rangle = \langle X, Y\rangle$

- Compute $(\nabla^g_X J)(Y)$:

$(\nabla^g_X J)(Y) = \nabla^g_X (J(Y)) - J(\nabla^g_X Y) = \nabla^g_X (N \times Y) - N \times \nabla^g_X Y = \nabla^g_X N \times Y + N \times \nabla^g_X Y - N \times \nabla^g_X Y = K(X) \times Y.$

In fact, more holds: J is integrable iff $K \circ J = -J \circ K$, which cannot hold on a closed hypersurface of Euclidian space.

Corollary. J is never integrable. For the sphere, $N(X, Y) = 4(\nabla^g_X J) JY$.

Interlude: What is a ‘good’ mathematical conjecture?

Maybe one that triggers further rich conceptional developments!

Outline:

1. Further results on the Hopf conjecture

2. Deeper investigation of the almost Hermitian structure on S^6 and the research it initiated

The early contributions of algebraic topology

Some subtle clarifications:

• All spheres S^{4k} are not almost complex. [Wu, 1949]

Based on explicit relations between the Chern and Pontryagin classes.

• If a mnfd *homeomorphic* to some S^n admits an almost complex structure, then $n = 2$ or $n = 6$. [Borel-Serre, 1953]

Uses Steenrod reduced p^{th} powers, newer simpler proofs are based on Bott periodicity and K-Theory.

• Infinite family compact 4-dim. mnfds admitting almost complex structures, but no complex structure (ex: $\mathbb{CP}^2\#(S^1 \times S^3)^2$). [van de Ven, 1966]

Uses the whole machinery: Based on Ehresmann-Wu (in a reformulation of Hirzebruch-Hopf), Milnor, Riemann-Roch theorem etc. [higher dimensions?]

Thm. Let M^4 be compact, $h \in H^2(M, \mathbb{Z})$. M admits an almost complex str. with first Chern class h iff $h \equiv w_2 \bmod 2$ and $h^2 = 3\tau + 2\chi$.

Orthogonal complex structures

Dfn. An (almost) complex structure J on (M, g) is called *orthogonal* (or g-orthogonal) if $g(JX, JY) = g(X, Y) \ \forall X, Y$.

Thm. (S^6, g_0) with the standard metric does not admit a g_0-orthogonal complex structure J_0. [Blanchard, 1953; rediscovered by Lebrun, 1987]

Proof strategy: By contradiction. Suppose J_0 exists.

1. Deduce a holomorphic embedding $\tau : \ S^6 \to M$ into some Kähler mnfd M. This makes S^6 into a a complex submnfd of a Kähler mnfd, and thus Kähler istelf.

2. But then $[\Omega] \neq 0$, in contradiction with $H^2_{\mathrm{dR}}(S^6, \mathbb{R}) = 0$.

Fix $x \in S^6$, view $T_x S^6$ as a subspace of $\mathbb{R}^7$. Using J_0, we complexify, decompose $T_{\mathbb{C}} S^6 = T^{1,0} \oplus T^{0,1} \subset \mathbb{C}^7$ and obtain

$$\tau : \ S^6 \to \mathrm{Gr}_3(\mathbb{C}^7) =: M, \quad \tau(x) = T^{0,1}_x.$$

Now check that all this works out.

A re-interpretation of the proof

Blanchard's proof uses what is now known as *twistor theory*. [Salamon, 1995]

Consider a Riemannian mnfd (N^{2n}, g).

$\mathcal{J}$:= { all possible g-orthogonal almost complex structures on M }

$\mathrm{SO}(2n)$ acts on $\mathcal{J}$ by conj. with stabilizer $\{h \in \mathrm{SO}(2n) : hJh^{-1} = J\}$

$\Rightarrow$ $\mathcal{J} \cong \mathrm{SO}(2n)/\mathrm{U}(n)$ and this is Kähler, the total space of the *twistor space of* M, and any orthogonal almost complex str. J on $U \subset M$ defines a local section $s_J : U \to \pi^{-1}(U) \subset \mathcal{J}$.

Thm. [Penrose / Atiyah-Hitchin-Singer, Salamon; OBrian-Rawsnley]

1) $\mathcal{J}$ admits an almost complex str. I s.t. $I \circ ds_J = ds_J \circ I$ iff J is integrable.

2) $(\mathcal{J}, I)$ is a complex mnfd iff M is conformally flat ($n \geq 3$) resp. anti-selfdual ($n = 2$).

Hence, the (hypothetical) complex conformally flat S^6 would have a holomorphic embedding to $\mathcal{J}$, contradiction.

• A modification of the proof (idea: curvature operator $R = \mathrm{Id}$ of S^6 is positive) yields the following refinement:

Thm. (S^6, g_0) with the standard metric does not admit an orthogonal complex structure for a metric 'close to g_0'. [Bor, Hernandez-Lamoneda, 1999]

Several refinements exist for how 'close' this would be.

• Observe:

- $F_{1,2}$ is the twistor space of $\mathbb{CP}^2$,

- $\mathbb{CP}^3$ is the twistor space of S^4,

- S^4 and $\mathbb{CP}^2$ are the only 4-dim. anti-selfdual compact mnfds [Hitchin, Friedrich-Kurke, 1991]

Hence, $F_{1,2}$ and $\mathbb{CP}^3$ carry, besides their 'obvious' Kähler structures both another almost complex structure—which turns out to be *nearly Kähler*.

. . . so, it's time to return to the previously defined almost complex structure on S^6!

Explicit description II: $S^6 = G_2/\mathrm{SU}(3)$ as naturally reductive space

[Fukami, Ishihara, 1955]

(M, g) a Riemannian mnfld, $M = G/H$ s. t. G is a group of isometries acting transitively and effectively

Dfn. $M = G/H$ is *naturally reductive* if $\mathfrak{h}$ admits a reductive complement $\mathfrak{m}$ in $\mathfrak{g}$ s. t.

$$\langle [X,Y]_{\mathfrak{m}}, Z\rangle + \langle Y, [X,Z]_{\mathfrak{m}}\rangle \;=\; 0 \;\text{ for all } X, Y, Z \in \mathfrak{m}, \qquad (*)$$

where $\langle -, -\rangle$ denotes the inner product on $\mathfrak{m}$ induced from g.
The PFB $G \to G/H$ induces a metric connection ∇ with torsion

$$T(X,Y,Z) := g(\nabla_X Y - \nabla_Y X - [X,Y], Z) = -\langle [X,Y]_{\mathfrak{m}}, Z\rangle,$$

the so-called *Ambrose-Singer connection*. It always satisfies $\nabla T = \nabla \mathcal{R} = 0.$

Observations:

- If G/H is symmetric, then $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$, hence $T = 0$ and $\nabla = \nabla^g$
- condition $(*)$ $\Leftrightarrow$ T is a 3-form, i. e. $T \in \Lambda^3(M)$.

The characteristic connection of a nearly Kähler mnfd

Thm. $S^6 = G_2/\mathrm{SU}(3)$ is naturally reductive and the torsion 3-form of the Ambrose-Singer connection is given by the formula

$$T^c(X, Y, Z) \;=\; -\langle J(X \times Y), Z\rangle \;=\; -\langle N, (X \times Y) \times Z\rangle \;.$$

The existence of this connection is no coincidence. Gray started the systematic investigation of nearly Kähler mnfds in the early 70ies and proved:

Thm. Let (M, g, J) be a nearly Kähler mnfd. There exists a unique metric connection preserving J and with skew symmetric torsion, and its torsion 3-form is given by the formula

$$T^c(X, Y, Z) \;=\; \langle (\nabla^g_X J)(JY)\,,\, Z\rangle \,.$$

Thm. The only homogeneous nearly Kähler 6-manifolds are S^6, $S^3 \times S^3$, $\mathbb{CP}^3$ and the flag manifold $F_{1,2}$. They are all naturally reductive and their characteristic connection coincides with the Ambrose-Singer connection. [Butruille, 2005]

A nearly Kähler manifold is said to be of constant type if $\exists\ \alpha > 0$ such that for all vector fields

$$\|(\nabla^g_X J)(Y)\|^2 = \alpha[\|X\|^2\|Y\|^2 - g(X,Y)^2 - g(JX,Y)^2].$$

Thm. Let (M, g, J) be a 6–dimensional nearly Kähler manifold that is not Kähler. Then

- M is of constant type,
- g is an Einstein metric on M,
- the first Chern class of M vanishes and hence it is spin. [Gray, 1976]

Thm. The characteristic torsion of a nearly Kähler 6-manifold is parallel with respect to the characteristic connection, $\nabla^c T^c = 0$. [Kirichenko, 1977]

Non homogeneous nearly Kähler manifolds

Were widely believed to exist, but their explicit construction was an open problem for many years!

Thm. There exists a non-homogeneous nearly Kähler structure on S^6 and on $S^3 \times S^3$. [Foscolo-Haskins, 2017]

They admit an isometric action of a compact Lie group such that generic orbits of the action are of codimension one.

The Lie group considered in this case is $\mathrm{SU}(2) \times \mathrm{SU}(2)$ and the generic orbits are $S^2 \times S^3$, which is motivated by results of Podesta and Spiro (2012) characterizing all possible groups and orbits for cohomogeneity one nearly Kähler.

Local homogeneous non-homogeneous examples of nearly Kähler manifolds are constructed by Cortés and Vásquez (2015).

– Why are nearly Kähler manifolds spin?

The Riemannian Dirac operator

(M^n, g): compact Riemannian spin mnfd, Σ: spin bdle

Classical Riemannian Dirac operator D^g:

$$\text{Dfn}: \qquad D^g : \ \Gamma(\Sigma) \longrightarrow \Gamma(\Sigma), \quad D^g\psi \ := \ \sum_{i=1}^n e_i \cdot \nabla^g_{e_i}\psi$$

Properties:

- D^g is elliptic differential operator of first order, essentially self-adjoint on $L^2(\Sigma)$, pure point spectrum

- Of equal fundamental importance than the Laplacian

- In dimension 4: $\text{index}(D^g) = \sigma(M^4)/8$ [Atiyah-Singer, $\sim$ 1963]

- Schrödinger (1932), Lichnerowicz (1962): $\boxed{(D^g)^2 = \Delta + \frac{1}{4}\text{Scal}^g}$

$\sim$ "'root of the Laplacian"' for $\text{Scal}^g = 0$

Spinors and Riemannian eigenvalue estimates

SL formula $\Rightarrow$ EV of $(D^g)^2$: $\quad \lambda \ \geq \ \frac{1}{4}\mathrm{Scal}^g_{\min}$

- optimal only for spinors with $\langle \Delta\psi, \psi\rangle = \|\nabla^g\psi\|^2 = 0$, i. e. parallel spinors

Thm. (M, g) has parallel spinors iff $\mathrm{Hol}_0(M) = \mathrm{SU}(n), \mathrm{Sp}(n), G_2, \mathrm{Spin}(7)$, and then $\mathrm{Ric}^g = 0$. [Wang, 1989]

Thm. Optimal EV estimate: $\lambda \geq \dfrac{n}{4(n-1)}\mathrm{Scal}^g_{\min}$ [Friedrich, 1980]

- "=" iff $\exists$ a Killing spinor (KS) ψ: $\nabla^g_X\psi = \mathrm{const}\cdot X\cdot\psi \quad \forall X$

Link to special geometries:

Thm. $\exists$ KS $\Leftrightarrow n = 5 : (M, g)$ is Sasaki-Einstein mnfd

$\Leftrightarrow n = 6 : (M, g)$ nearly Kähler mnfd

$\Leftrightarrow n = 7 : (M, g)$ nearly parallel G_2 mnfd

(similarly for other n) [Friedrich, Grunewald, Kath, 1985-90]

Killing spinors and submanifolds

Thm. Suppose (M,g) is Sasaki-Einstein $(n=5)$, nearly Kähler $(n=6)$, or nearly parallel G_2 (n=7). Then the metric cone

$$(\bar{M},\bar{g}) := (M\times\mathbb{R}^+, \tfrac{1}{4}r^2g^2+dr^2)$$

has a ∇^g-parallel spinor; in particular, it is Ricci-flat of Riemannian holonomy $\mathrm{SU}(3)$, G_2, resp. $\mathrm{Spin}(7)$. [Bryant 1987 ⇝ B-Salamon 1989, Bär 1993 (+ Wang '89)]

Observe: Construction relies on existence of a Killing spinor

Thm. Let (M,g) be a spin manifold with a ∇^g-parallel spinor ψ, $N\subset M$ a codimension one hypersurface. Then $\varphi := \psi\big|_N$ is a *generalized Killing spinor* on N, i. e. $\nabla^g_X\varphi = A(X)\cdot\varphi$ for a symmetric endomorphism A (Weingarten map). [Friedrich 1998, Bär-Gauduchon-Moroianu 2005]

Observe: Generalizes the Weierstraß representation of minimal surfaces, based on ideas of Eisenhardt (1909)

Beyond nearly Kähler – Spin structures and topology in dimension 6

[A-Fr-Chiossi-Höll, 2014]

Observation:

Any 8-dimensional real vector bundle over a 6-dimensional manifold admits a section of length one

• In $n = 6$, the spin representation is real and $2^3 = 8$-dimensional, call it $\Delta := \mathbb{R}^8$.

$\Rightarrow$ a 6-dim. oriented Riemannian manifold admits a spin structure iff it admits a reduction from $\mathrm{Spin}(6) \cong \mathrm{SU}(4)$ to $\mathrm{SU}(3)$

Use this section to give a uniform spinorial description of $\mathrm{SU}(3)$-manifolds!

Spin linear algebra in dimension 6

- Δ admits a $\mathrm{Spin}(6)$-invariant cplx structure j (because $\mathrm{Spin}(6) \cong \mathrm{SU}(4)$)
- any real spinor $0 \neq \phi \in \Delta$ decomposes Δ into three pieces,

$$\Delta = \mathbb{R} \cdot \phi \oplus \mathbb{R} \cdot j(\phi) \oplus \underbrace{\{X \cdot \phi \ : \ X \in \mathbb{R}^6\}}_{\cong \mathbb{R}^6, \text{ the base space}} \qquad (*)$$

- the following formula defines an orthogonal cplx str. on the last piece,

$$J_\phi(X) \cdot \phi \ := \ j(X \cdot \phi)$$

- the spinor defines a 3-form by $\psi_\phi(X, Y, Z) := -(X \cdot Y \cdot Z \cdot \phi, \phi)$.

Thm. The following is a 1-1 correspondence: (well-known)

- $\mathrm{SU}(3)$-structures on $\mathbb{R}^6 \longleftrightarrow$ real spinors of length one $(\mathrm{mod}\, \mathbb{Z}_2)$,

$$\mathrm{SO}(6)/\mathrm{SU}(3) = \{\mathrm{SU}(3)\text{-structures on } \mathbb{R}^6\} = \mathbb{P}(\Delta) = \mathbb{RP}^7.$$

Special almost Hermitian geometry

• $\mathrm{SU}(3)$ manifold (M^6, g, ϕ): Riemannian spin manifold (M^6, g) equipped with a global spinor ϕ of length one, j as before, J induced almost cplx str., Ω its kähler form, ψ_ϕ induced 3-form, $\psi_\phi^J := J \circ \psi_\phi$.

Decomposition $(*) \Rightarrow \exists_1$ 1-form η and endomorphism S s. t.

$$\boxed{\nabla^g_X \phi = \eta(X) j(\phi) + S(X) \cdot \phi}$$

η: "intrinsic 1-form", S: "intr. endomorphism" (indeed: $\Gamma = S \lrcorner \psi_\phi - \frac{2}{3}\eta \otimes \Omega$)

This equation summarizes all spinor eqs. previously known in dim.6!

Thm. $(\nabla^g_X \Omega)(Y, Z) = 2\psi^J_\phi(S(X), Y, Z)$

This generalizes the classical nearly Kähler condition $\nabla^g_X \Omega(X, Y) = 0 \, \forall X, Y$.

Thm. The classes of $\mathrm{SU}(3)$ str. are determined as follows:

class	description	dimension
χ_1	$S = \lambda \cdot J_\phi$, $\eta = 0$	1
$\chi_{\bar{1}}$	$S = \mu \cdot \mathrm{Id}$, $\eta = 0$	1
χ_2	$S \in \mathfrak{su}(3)$, $\eta = 0$	8
$\chi_{\bar{2}}$	$S \in \{A \in S_0^2(\mathbb{R}^6) \vert AJ_\phi = J_\phi A\}$, $\eta = 0$	8
χ_3	$S \in \{A \in S_0^2(\mathbb{R}^6) \vert AJ_\phi = -J_\phi A\}$, $\eta = 0$	12
χ_4	$S \in \{A \in \Lambda^2(\mathbb{R}^6) \vert AJ_\phi = -J_\phi A\}$, $\eta = 0$	6
χ_5	$S = 0$, $\eta \neq 0$	6

where $\lambda, \mu \in \mathbb{R}$.

. . . from here, a whole world of almost Hermitian mnfds is accessible to a spinorial description.

– THANK YOU FOR YOUR ATTENTION –