String topology and closed geodesics

Hans-Bert Rademacher

(Universität Leipzig)

Seminar *Geometry, Topology and their Applications* Novosibirsk, 2021-04-19

Introduction, I

Joint work with *Nancy Hingston*

Setting: compact manifold M with a *Finsler metric f.*

Functional $F : \Lambda M \longrightarrow \mathbb{R}, F(c) = \left(\int_0^1 f^2 \left(c'(t) \right)^2 \, dt \right)^{1/2}$ defined on the *free loop space* $\Lambda = \Lambda M$.

A *closed (or periodic) geodesic* $c : S^1 = \mathbb{R}/\mathbb{Z} \longrightarrow M$ is a *critical point* of the functional F of length $L(c) = F(c)$.

$$\boxed{\text{homology of } \Lambda \quad \overset{\textit{Morse theory}}{\longleftrightarrow} \quad \text{critical Points of } F = \text{closed geodesics}}$$

UNIVERSITÄT LEIPZIG

Existence of one closed geodesics

The *index* $\mathrm{ind}(c)$ of a closed geodesic c is the *index* of the *hessian* $d^2F(c)$. If $\Lambda^{\leq a} := \{\sigma \in \Lambda\,;\, F(c) \leq a\}$ denotes the sublevel sets of F and if there is only one closed geodesic whose length lies in the interval $[a-\epsilon, a+\epsilon]$ (and if the closed geodesic is non-degenerate), then:

$$H_k\left(\Lambda^{\leq a+\epsilon}, \Lambda^{\leq a-\epsilon}; \mathbb{Q}\right) = \begin{cases} \mathbb{Q} & ; \quad k = \mathrm{ind}(c), k = \mathrm{ind}(c)+1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

The Morse-inequalities imply:

Theorem (Birkhoff 1927; Lusternik-Fet 1951)

On a simply-connected and compact manifold with a Finsler metric there exists a (nontrivial) closed geodesic.

UNIVERSITÄT LEIPZIG

Gromoll-Meyer theorem

It is a consequence from a result by Bott 1956 that

$$m\alpha_c - (n-1) \leq \operatorname{ind}(c^m) \leq m\alpha_c + (n-1)$$

Theorem (Gromoll-Meyer 1969)

Let the sequence $(b_k(\Lambda M))_{k\geq 1}$ of Betti numbers of the free loop space on a compact manifold M be unbounded.
Then for any Finsler metric there are infinitely many closed geodesics.

Rational homotopy theory shows that the assumption of the Theorem is satisfied, if the rational cohomology ring has at least two generators (Sullivan, Vigue-Poirrier 1976).

The assumption is not satisfied for spheres and complex resp. quaternionic projective spaces, then
$H^*(M;\mathbb{Q}) = T_{d,h+1}(u) =< 1, u, u^2, \ldots, u^h >, \deg u = d.$

UNIVERSITÄT LEIPZIG

Introduction, II

For a closed curve $c : S^1 \to M$ and $m \geq 1$ we denote by c^m the m-th cover, i.e. $c^m(t) = c(mt)$. Hence $L(c^m) = mL(c)$.

Principal Problem

$$\boxed{c \text{ closed geodesic} \quad \longleftrightarrow \quad c^m, m \geq 1 \text{ closed geodesics, too}}$$

Question:

Does there exist an operation on $H_*(\Lambda)$ corresponding to iteration?

Partial answer is given by *string topology*

UNIVERSITÄT LEIPZIG

String topology

String theory: Particles are made of vibrating bits of (closed) strings (very tiny)

Configuration spaces of string theory: free loop space ΛM

String topology: Algebraic and topological description of intersection theory on the free loop space (M.Chas and D.Sullivan 1999)

(co)homology products in string topology

String-topology defines products in the (co)homology of the free loop space $\Lambda = \Lambda M$ with $\dim M = n$:
Chas-Sullivan 1999

$$\bullet : H_j(\Lambda M) \otimes H_k(\Lambda M) \to H_{j+k-n}(\Lambda M) \tag{1}$$

Goresky-Hingston 2009

$$\circledast : H^j(\Lambda M, \Lambda^0 M) \otimes H^k(\Lambda M, \Lambda^0 M) \to H^{j+k+n-1}(\Lambda M, \Lambda^0 M) \tag{2}$$

These products generalize the *intersection product* in the homology of compact manifolds.

Critical values of homology classes, Part I

sublevel sets: $\Lambda^{\leq a} := \{c \in \Lambda; F(c) \leq a\}$

Let $\mathrm{cr}(X), \mathrm{cr}(x)$ be *critical value* of the homology class $X \in H_k(\Lambda M)$, (resp. cohomology class $x \in H^*(\Lambda)$:)

$$\mathrm{cr}(X) = \inf \left\{ a > 0\, ;\, X \in \mathrm{Image}\left(H_*\left(\Lambda^{\leq a}\right) \longrightarrow H_*\left(\Lambda\right)\right)\right\}$$

$$\mathrm{cr}(x) = \sup \left\{ a > 0\, ;\, x \in \mathrm{Kernel}\left(H^*\left(\Lambda\right) \longrightarrow H_*\left(\Lambda^{\leq a}\right)\right)\right\}$$

Theorem (Gromov 1978)

For a compact and simply-connected manifold (M, g) there is a constant $\alpha = \alpha(M, g) > 0$ such that

$$\mathrm{cr}(X) \leq \frac{\deg(X)}{\alpha}\ ;\ \forall X \in H_*(\Lambda M)$$

Critical valus of homology classes, Part II

The loop products $\bullet$ and $\circledast$ satisfy the following

basic inequalities (GORESKY, HINGSTON 2009)

$$\mathrm{cr}(X \bullet Y) \leq \mathrm{cr}(X) + \mathrm{cr}(Y) \text{ for all } X, Y \in H_*(\Lambda)$$

$$\mathrm{cr}(x \circledast y) \geq \mathrm{cr}(x) + \mathrm{cr}(y) \quad \text{for all } x, y \in H^*(\Lambda, \Lambda^0).$$

With these products and these inequalities we obtain the following improvement of Gromov's estimate for spheres:

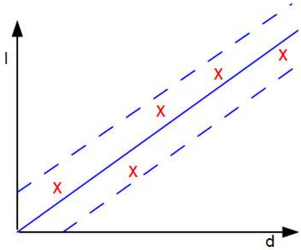
Resonance theorem

Theorem (Hingston-R.)

A Finsler metric on $S^n, n \geq 3$ determines a *global mean frequency* $\overline{\alpha} > 0$ *such that the following holds: There is positive* β *such that*

$$-\beta \leq \mathrm{cr}(X) - \frac{\deg(X)}{\overline{\alpha}} \leq \beta$$

as X ranges over all nontrivial homology or cohomology classes on Λ.



The countably infinite set of points $(\deg(X), \mathrm{cr}(X))$ in the (d, l)-plane lies in bounded distance from the line $l = d/\overline{\alpha}$.

Eigenfrequency of a closed geodesic

For a closed geodesic $c : S^1 \to M$ denote by $\widetilde{c} : \mathbb{R} \to M$ the corresponding covering, then

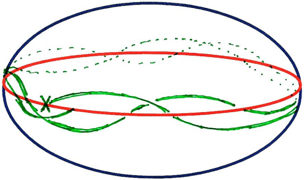
The *eigen frequency* of a closed geodesics of length $L(c)$ is given by

$$\overline{\alpha}_c = \lim_{m\to\infty} \frac{\#\{\text{ conjugate points } \widetilde{c}(t)\,;\ t \le m\,L(c)\}}{m}$$

Then

$$\overline{\alpha}_c = \frac{\alpha_c}{L(c)} = \frac{1}{L(c)} lim_{m\to\infty} \frac{\mathrm{ind}(c^m)}{m}$$

i.e. the average index α_c is the *mean value of conjugate points per period.*



Resonant closed geodesics

As an application one obtains:

Theorem (Hingston-R.)

Let f be a Finsler metric with reversibility $\lambda = \max\{f(-v); f(v) = 1\}$ and of positive flag curvature $\lambda^2/(1+\lambda)^2 < K \leq 1$ on an odd-dimensional sphere S^n with global mean frequency $\overline{\alpha} = \overline{\alpha}(M, f) > 0$. Then one of the following holds:

- *There are two resonant closed geodesics c_1, c_2 with eigenfrequency $\overline{\alpha}_1 = \overline{\alpha}_2 = \overline{\alpha}$.*
- *There is a sequence c_k of infinitely many closed geodesics c_k with $\overline{\alpha}_k = \overline{\alpha}_{c_k} \neq \overline{\alpha}$ and*

$$\overline{\alpha} = \lim_{k \to \infty} \overline{\alpha}_k$$

In particular: If there are only finitely many closed geodesics, then there are *two resonant* closed geodesics.

UNIVERSITÄT LEIPZIG

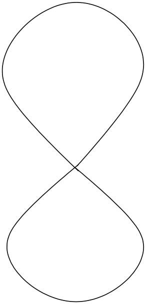
The Chas-Sullivan Loop Product, Part I

Let

$$\mathcal{F} = \{(\alpha, \beta) \in \Lambda \times \Lambda\, ;\, \alpha(0) = \beta(0)\}$$

be the *figure 8-space* of the compact manifold M.
We can view $\mathcal{F}$ as:

- The embedding $e : \mathcal{F} \longrightarrow \Lambda \times \Lambda$ is an *embedding of codimension* n
- $\mathcal{F} \longrightarrow \Lambda$ is a subset of Λ



The Chas-Sullivan Loop Product, Part II

Then we obtain the *Chas Sullivan product* as the following composition

$$H_k(\Lambda) \otimes H_l(\Lambda) \longrightarrow H_{k+l}(\Lambda \times \Lambda) \longrightarrow H_{k+l}(\Lambda \times \Lambda, \Lambda \times \Lambda - \mathcal{F}) \xrightarrow{\tau}$$
$$\xrightarrow{\tau} H_{k+l-n}(\mathcal{F}) \longrightarrow H_{k+l-n}(\Lambda)$$

The homomorphism τ is induced by the *Thom-isomorphism* of the *normal bundle* of the embedding $\mathcal{F} \longrightarrow \Lambda \times \Lambda$.

The last homorphism is induced by the inclusion $\mathcal{F} \longrightarrow \Lambda$.

short notation:

$$\Lambda \times \Lambda \longleftarrow \mathcal{F} \longrightarrow \Lambda$$

Morse theory and spectral sequence

Assume for simplicity: f is *bumpy,* i.e. all closed geodesics are non-degenerate. Then there are *no periodic Jacobi fields.*
Let $0 = l_0 < l_1 < l_2 < \ldots$ be the sequence of critical values.

$$\textit{filtration } \left\{\Lambda^{\leq l_j}\right\}_{j \geq 0} \longrightarrow \textit{spectral sequence} \longrightarrow H_*(\Lambda)$$

The filtration induces a spectral sequence converging to $H_*(\Lambda)$.

Each *page* is *bigraded* by the index set $\{j\}$ of the sequence $\{l_j\}$ and the non-negative integers d.

UNIVERSITÄT LEIPZIG

The spectral sequence, part I

E_1-page:

$$E_1^{d,l_j} := H_d\left(\Lambda^{\le l_j}, \Lambda^{\le l_{j-1}}; \mathbb{Q}\right);$$

$$E_1^{d,l_j} = \bigoplus_{c;\mathrm{ind}(c)=d, L(c)=l_j} \mathbb{Q} \oplus \bigoplus_{c;\mathrm{ind}(c)=d-1, L(c)=l_j} \mathbb{Q}$$

$$E_1 = \bigoplus_{d,j} E_1^{d,l_j}$$

defines a *first quadrant spectral sequence* in (l,d)-plane indexed by (l_j,d). E_k is obtained by taking homology w.r.t. $D_k, k \ge 1$ of degree $(-k,-1)$:

$$D_k : H_d\left(\Lambda^{\le l_j}, \Lambda^{\le l_{j-1}}\right) \longrightarrow H_{d-1}\left(\Lambda^{\le l_{j-k}}, \Lambda^{\le l_{j-k-1}}\right)$$

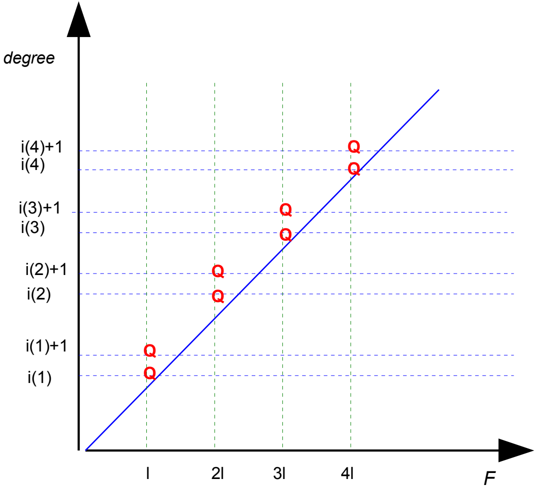
The products $\bullet, \circledast$ on $E_1, \ldots, E_k$ are *compatible* with the filtration on $H_*(\Lambda)$.

UNIVERSITÄT LEIPZIG

The spectral sequence, part II

Contribution of one non-degenerate closed geodesic lies in a bounded distance of a line of slope $\overline{\alpha}_c$

$i(m) = \mathrm{ind}(c^m), l = L(c)$



The spectral sequence, Part III

The *homology version* of the resonance theorem states: In the E_∞ page of the spectral sequence all non-trivial entries lie within a bounded distance of the line

$$d = \overline{\alpha} \cdot l$$

In particular, all sufficiently high iterates of c must *die* (are *killed*) in the spectral sequence, unless $\overline{\alpha}_c = \overline{\alpha}$.

If $N < \infty$ there is *at least one closed geodesic* c with $\overline{\alpha}_c = \overline{\alpha}$.

And in *high degrees* these are the only closed geodesics whose iterates appear in E_∞.

Examples, standard sphere, part I

The Resonance theorem is obviously true if the following cases:

1. *Hypothetical example* $N = 1$.
2. *standard metric on S^n :* c great circle, $L(c) = 2\pi, K \equiv 1, \alpha_c = 2(n-1), \overline{\alpha}_c = (n-1)/\pi$. All closed geodesics are *resonant.* Sequence of critical values: $2\pi m, m \geq 1$. Let $B = BS^n = T^1S^n \subset \Lambda S^n$ be the set of great circles, which we can identify with the unit tangent bundle T^1S^n.
   Then the functional $F : \Lambda S^n \longrightarrow \mathbb{R}$ is a Morse-Bott function, the critical set is the disjoint union of the set $B^m = \{c^m; c \in B\}$ of m-fold covered great circles.

   The manifolds $B^m, m \geq 1$ are non-degenerate critical submanifolds and
   $$i(m) = \mathrm{ind}(c^m) = (2m-1)(n-1).$$

Examples, standard sphere, part II

Take rational coefficients for (co)homology.
The Thom isomorphism implies:

$$E_1^{d,m} = H_d\left(\Lambda^{\leq 2\pi m}, \Lambda^{\leq 2\pi(m-1)}\right) \cong H_{d-(2m-1)(n-1)}\left(T^1 S^n\right)$$

$$H_d\left(\Lambda S^n, \Lambda^0 S^n\right) \cong \bigoplus_{m\geq 1} E_1^{d,m}$$

Since $\dim(T^1 S^n) = 2n-1$ there is a class

$$\theta \in H_{3n-2}\left(\Lambda S^n, \Lambda^0 S^n\right)$$

corresponding to the top dimensional class of the set AS^n of circles.

UNIVERSITÄT LEIPZIG

Examples, standard sphere, part III

The Chas-Sullivan product $\bullet$ corresponds to the intersection product restricted to the energy sublevels:

$$\begin{tikzcd} H_i\left(\Lambda^{\le 2\pi m},\Lambda^{<2\pi m}\right)\otimes H_j\left(\Lambda^{\le 2\pi},\Lambda^{<2\pi}\right) \arrow[r, "\bullet"] \arrow[d] & H_{i+j-n}\left(\Lambda^{\le 2\pi(m+1)},\Lambda^{<2\pi(m+1)}\right) \arrow[d] \\ H_{i-i(m)}\left(T^1S^n\right)\otimes H_{j-i(1)}\left(T^1S^n\right) \arrow[r, "I"] & H_{i-i(m)+j-i(1)-(2n-1)}\left(T^1S^n\right) \end{tikzcd}$$

Here it is important that:

$$i(m+1) = i(m) + i(1) + n - 1.$$

Examples, standard sphere, part IV

Multiplication

$$\theta \bullet : H_* \left(\Lambda S^n, \Lambda^0 S^n\right) \longrightarrow H_{*+(2n-2)} \left(\Lambda S^n, \Lambda^0 S^n\right)$$

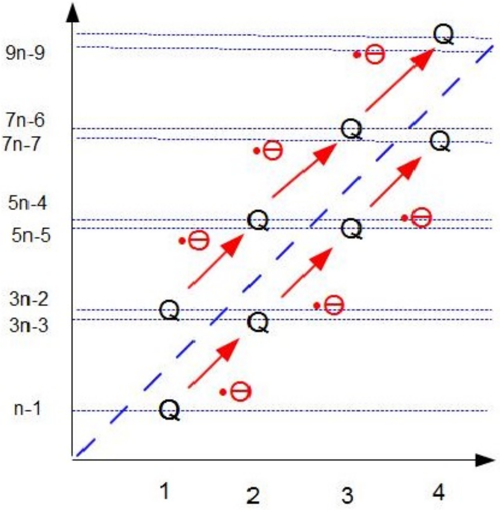
is an isomorphism resembling an *iteration map*

$$\theta \bullet : H_* \left(\Lambda^{\leq 2\pi m} S^n, \Lambda^0 S^n\right) \longrightarrow H_{*+(2n-2)} \left(\Lambda^{\leq 2\pi(m+1)} S^n, \Lambda^0 S^n\right)$$

resp.

$$\theta \bullet : H_* \left(\Lambda^{\leq 2\pi m} S^n, \Lambda^{\leq 2\pi(m-1)} S^n\right) \longrightarrow H_{*+(2n-2)} \left(\Lambda^{\leq 2\pi(m+1)} S^n, \Lambda^{\leq 2\pi m} S^n\right)$$

Examples, standard sphere, even dimension n, Part I



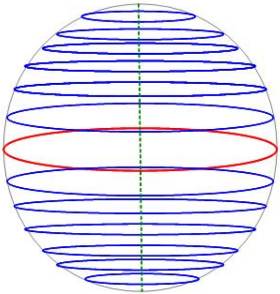
The algebra $(H_*(\Lambda S^n;\mathbb{Q}), \bullet)$; is finitely generated.

Examples, standard sphere, part V

There is a cohomology class

$$\omega \in H^{n-1}\left(\Lambda S^n, \Lambda^0 S^n; \mathbb{Z}\right)$$

which is dual to the homology class of dimension $(n-1)$ which can be represented by *circles* parametrized by a *disc of dimension* $(n-1)$ lying over a given *great circle*:



Examples, standard sphere, part VI

The Goresky-Hingston product $\circledast$ corresponds to the cup product $\cup$ restricted to the energy sublevels:

$$
\begin{tikzcd}
H^i\left(\Lambda^{\leq 2\pi m}, \Lambda^{<2\pi m}\right) \otimes H^j\left(\Lambda^{\leq 2\pi}, \Lambda^{<2\pi}\right) \arrow[r, "\circledast"] \arrow[d] & H^{i+j+n-1}\left(\Lambda^{\leq 2\pi(m+1)}, \Lambda^{<2\pi(m+}\right. \arrow[d] \\
H^{i-i(m)}\left(T^1 S^n\right) \otimes H^{j-i(1)}\left(T^1 S^n\right) \arrow[r, "\cup"] & H^{i+j-i(m+1)-i(1)}\left(T^1 S^n\right)
\end{tikzcd}
$$

Examples, standard sphere, part VII

Multiplication

$$\omega \circledast : H^{*}\left(\Lambda S^{n}, \Lambda^{0} S^{n}\right) \longrightarrow H^{*+(2n-2)}\left(\Lambda S^{n}, \Lambda^{0} S^{n}\right)$$

is an isomorphism resembling an *iteration map*

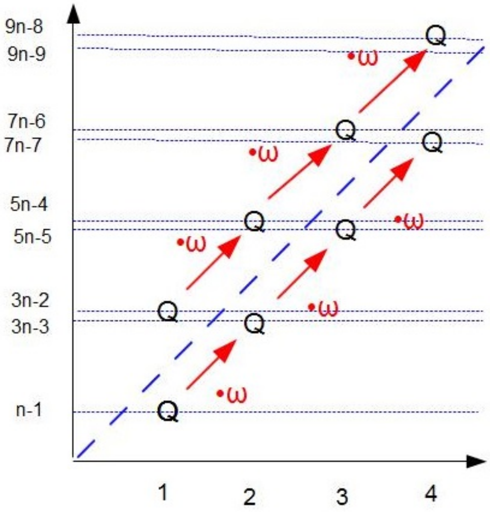
$$\omega \circledast : H^{*}\left(\Lambda^{\leq 2\pi m} S^{n}, \Lambda^{0} S^{n}\right) \longrightarrow H^{*+(2n-2)}\left(\Lambda^{\leq 2\pi(m+1)} S^{n}, \Lambda^{0} S^{n}\right)$$

resp.

$$\omega \circledast : H^{*}\left(\Lambda^{\leq 2\pi m} S^{n}, \Lambda^{\leq 2\pi(m-1)} S^{n}\right) \longrightarrow H^{*+(2n-2)}\left(\Lambda^{\leq 2\pi(m+1)} S^{n}, \Lambda^{\leq 2\pi m} S^{n}\right)$$

UNIVERSITÄT LEIPZIG

Examples, standard sphere, even dimension n, Part II



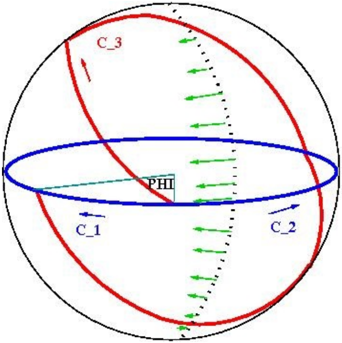
The algebra $\left(H^{*}\left(\Lambda S^{n};\mathbb{Q}\right), \circledast\right)$ is finitely generated.

Katok example

Family of *non-reversible Finsler metrics* on the sphere S^n with n (if n is even) resp. $n+1$ (if n is odd) closed geodesics. The *flag curvature* is constant $K \equiv 1$, hence

$$\overline{\alpha}_c = \frac{n-1}{\pi} = \overline{\alpha}$$

Hence all closed geodesics are *resonant*.



$n = 2$

closed geodesics c_1, c_2 which differ only by orientation (and length)

In all other cases the result is surprising!

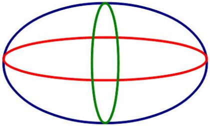
UNIVERSITÄT LEIPZIG

Ellipsoid

$$\frac{x_1^2}{a_1^2} + \ldots + \frac{x_{n+1}^2}{a_{n+1}^2} = 1, 1 \le a_1 < \ldots < a_{n+1}$$

- $m = \begin{pmatrix} n+1 \\ 2 \end{pmatrix}$ coordinate planes define simple closed geodesics $c_1, c_2, \ldots, c_m$
- $N = \infty$ (geodesic flow is integrable)
- $L(c_{m+1}) \to \infty$ for $a_{n+1} \to 1$.
  Expect: *generically* the eigenfrequencies $\overline{\alpha}_1, \ldots, \overline{\alpha}_m$ are pairwise distinct.

  Conclusion: Iterates of all but at most one of $c_1, \ldots, c_m$ do not contribute.



Resonance theorem, sketch of proof, part I

- Definition of the *global mean frequency* $\overline{\alpha}$ of (S^n, f) : The following (co)homology classes are *non-nilpotent* with respect to the product $\bullet$ resp. $\circledast$:

$$\theta \in H_{3n-2}\left(\Lambda S^n; \mathbb{Q}\right) \; ; \; \omega \in H^{n-1}\left(\Lambda S^n, \Lambda^0 S^n; \mathbb{Q}\right)$$

Then the following limit exists and defines the *global mean frequency:*

$$\begin{aligned} \frac{1}{\overline{\alpha}} &= \lim_{m\to\infty} \frac{\mathrm{cr}\left(\theta^{\bullet m}\right)}{\deg\left(\theta^{\bullet m}\right)} = \lim_{m\to\infty} \frac{\mathrm{cr}\left(\omega^{\circledast m}\right)}{\deg\left(\omega^{\circledast m}\right)} \\ &= \frac{1}{2(n-1)} \lim_{m\to\infty} \frac{\mathrm{cr}\left(\theta^{\bullet m}\right)}{m} = \frac{1}{2(n-1)} \lim_{m\to\infty} \frac{\mathrm{cr}\left(\omega^{\circledast m}\right)}{m} \end{aligned}$$

Resonance theorem, sketch of proof, part II

- $$n \geq 3 \Rightarrow \dim H_* \left(\Lambda; \mathbb{Q}\right) = \dim H^* \left(\Lambda; \mathbb{Q}\right) \leq 1$$

  Therefore: If X, x are (co)homology classes *dual* to each other:

  $$\mathrm{cr}(X) = \mathrm{cr}(x).$$

- Computation of the homomorphism

  $$\Delta = (S^1)_* : H_{(2m-1)(n-1)} \left(\Lambda S^n; \mathbb{Z}\right) \longrightarrow H_{(2m-1)(n-1)+1} \left(\Lambda S^n; \mathbb{Z}\right)$$

  defining a Batalin-Vilkovisky (BV)-algebra.

  It is important that *both products* $\bullet, \circledast$ are involved.

  The inequalities for the critical values go in different directions!

Recent related results

- Abbondandolo-Schwarz 2008, Cielibak-Hingston-Oancea 2020: The pair of pants product on the Floer homology $HF_*(T^1M)$ corresponds to $(H_*(\Lambda M), \bullet)$:
- Laudenbach 2011: Finite-dimensional approach to the Chas-Sullivan product $\bullet$
- Xiao, Long 2015: Chas Sullivan product on $\Lambda(\mathbb{R}P^{2m+1})$
- Jones, Mc Cleary 2016: The sequence $(b_k(\Lambda M; \mathbb{Z}_p))_{k\geq 1}$ for $\mathbb{Z}_p$-elliptic spaces compact and simply-connected manifolds M which are not monogenic.
- Maiti 2017: Morse theoretic description of the Goresky-Hingston product $\circledast$.
- Hingston-Wahl 2019: Homotopy invariance of the Goresky-Hingston product $\circledast$.
- Kupper 2020: Chas-Sullivan product on quotient spaces $\Lambda M/\mathbb{Z}_2$.

UNIVERSITÄT LEIPZIG