On Steady Navier-Stokes equations in 2D exterior domains

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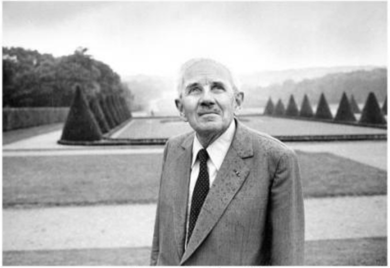
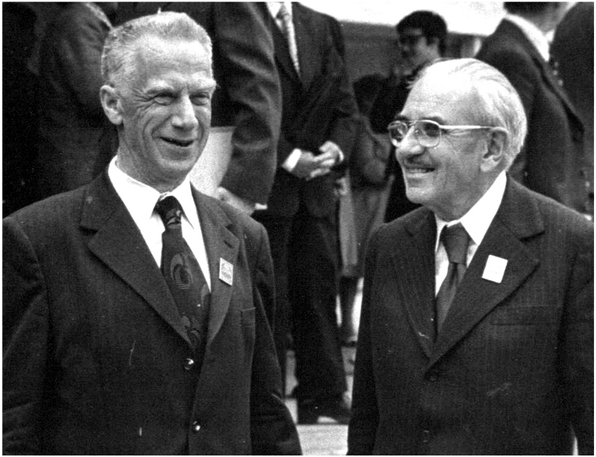
In the talk we concern the recent progress in the study of Steady Navier–Stokes System. The talk is based on some new results obtained in our joint papers with Konstantin Pileckas (University of Vilnius), Remigio Russo (Universita della Campania L.Vanvitelli), and Xiao Ren (Fudan University).

Sea waves



Jean Leray - Parc de Sceaux (1985)

Our results are related to the so-called Leray's problems in mathematical fluid mechanics, which remaine open for more than 80 years (starting from the publication of the famous paper of Jean Leray 1933).



Two outstanding mathematicians of XX century: J. Leray and S.L. Sobolev.

Consider the Navier–Stokes problem

$$\begin{cases} -\nu\Delta\mathbf{u} + \big(\mathbf{u}\cdot\nabla\big)\mathbf{u} + \nabla p & = & \mathbf{0} & \text{in} \quad \Omega, \\ \operatorname{div}\mathbf{u} & = & 0 & \text{in} \quad \Omega, \\ \mathbf{u} & = & \mathbf{a} & \text{on} \quad \partial\Omega, \\ \lim\limits_{|x|\to+\infty} \mathbf{u}(x) & = & \mathbf{u}_\infty \end{cases} \qquad (NS)$$

in the exterior domain $\Omega = \mathbb{R}^3 \setminus \big(\bigcup\limits_{j=1}^{N} \bar{\Omega}_j\big),$ (*)

Leray Theorem 1933. Assume that $\Omega \subset \mathbb{R}^3$ is an exterior axially symmetric domain of type $(*)$ with C^2-smooth boundary $\partial\Omega$, $\mathbf{u}_\infty \in \mathbb{R}^3$, and suppose that the boundary data $\mathbf{a} \in W^{3/2,2}(\partial\Omega)$ satisfies the condition of zero fluxes

$$F_j = \int\limits_{\partial\Omega_j} \mathbf{a}\cdot\mathbf{n}\,dS = 0 \quad j = 1, \ldots, N.$$

Then (NS) admits at least one solution $\mathbf{u}$ satisfying $\int_\Omega |\nabla\mathbf{u}|^2\,dx < +\infty.$

Consider the Navier–Stokes problem

$$\left\{\begin{array}{rcll} -\nu\Delta\mathbf{u} + \big(\mathbf{u}\cdot\nabla\big)\mathbf{u} + \nabla p & = & \mathbf{f} & \text{in} \quad \Omega, \\ \operatorname{div}\,\mathbf{u} & = & 0 & \text{in} \quad \Omega, \\ \mathbf{u} & = & \mathbf{a} & \text{on} \quad \partial\Omega, \\ \lim\limits_{|x|\to+\infty} \mathbf{u}(x) & = & \mathbf{u}_\infty & \end{array}\right. \qquad (NS)$$

in the exterior domain $\Omega = \mathbb{R}^3 \setminus \big(\bigcup\limits_{j=1}^{N} \bar{\Omega}_j\big),$ (*)

Theorem [KPR2018]. Assume that $\Omega \subset \mathbb{R}^3$ is an exterior axially symmetric domain of type $(*)$ with C^2-smooth boundary $\partial\Omega$, and $\mathbf{u}_\infty \in \mathbb{R}^3$, $\mathbf{f} \in W^{1,2}(\Omega)$, $\mathbf{a} \in W^{3/2,2}(\partial\Omega)$ are axially symmetric. Then (NS) admits at least one axially symmetric solution $\mathbf{u}$ satisfying $\int_\Omega |\nabla\mathbf{u}|^2\,dx < +\infty$.

But the case of steady NS in exterior plane domain surprisingly turns out to be much more difficult than in 3d case.

Let

$$\Omega = \mathbb{R}^2 \setminus \bigcup_{i=1}^{N} \overline{\Omega}_i,$$

where Ω_i are N pairwise disjoint smooth domains. The boundary value problem for the Navier–Stokes equations in Ω is

$$\begin{aligned} \nu\Delta\mathbf{u} - \mathbf{u}\cdot\nabla\mathbf{u} - \nabla p = \mathbf{0} \qquad & \text{in } \Omega, \\ \operatorname{div}\mathbf{u} = 0 \qquad & \text{in } \Omega, \\ \mathbf{u} = \mathbf{a} \qquad & \text{on } \partial\Omega, \end{aligned} \tag{NS}$$

with the condition at infinity

$$\lim_{x\to\infty} \mathbf{u}(x) = \mathbf{u}_\infty, \tag{ass}$$

where $\mathbf{a}$ and $\mathbf{u}_\infty$ are an assigned field on $\partial\Omega$ and a constant vector respectively.

$$
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\end{aligned}
\tag{NS}
$$

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\lim_{x\to\infty}\mathbf{u}(x) = \mathbf{u}_\infty. \tag{ass}
$$

From a pioneering paper by J. Leray [1933] it is now customary to look for a solution to (NS) with finite Dirichlet integral

$$
\int_\Omega |\nabla\mathbf{u}|^2 < +\infty, \tag{1}
$$

known also as *D–solution*. As is well known, it is analytic in Ω.

The problem is particularly meaningful in view of the famous *Stokes paradox* which asserts that the Stokes system

$$\begin{aligned} \nu\Delta\mathbf{u} - \nabla p = \mathbf{0} \qquad & \text{in } \Omega, \\ \operatorname{div}\mathbf{u} = 0 \qquad & \text{in } \Omega, \\ \mathbf{u} = \mathbf{0} \qquad & \text{on } \partial\Omega, \\ \lim_{x\to\infty}\mathbf{u}(x) = \mathbf{u}_\infty, & \end{aligned} \tag{Stokes}$$

obtained by linearizing (NS) has **no** solutions if $\mathbf{u}_\infty \neq \mathbf{0}$.

Stokes himself gave the following explanation: *The pressure of the cylinder on the fluid continuously tends to increase the quantity of fluid which it carries with it, while the friction of the fluid at a distance from the cylinder continually tends to diminish it. In the case of a sphere, these two causes eventually counteract each other, and the motion becomes uniform. But in the case of a cylinder, the increase in the quantity of the fluid carried continually gains on the decrease due to the friction of the surrounding fluid, and the quantity carried increases indefinitely as the cylinder moves on.*

The celebrated J.Leray's paper [JMPA1933] can be considered as a landmark point in the study of the *nonlinear* NS-problem.

JOURNAL

DE

MATHÉMATIQUES

PURES ET APPLIQUÉES.

Étude de diverses équations intégrales non linéaires et de quelques problèmes que pose l'Hydrodynamique;

PAR JEAN LERAY.

CHAPITRE I.

THÉORÈMES D'EXISTENCE NON LOCAUX.

1. SOMMAIRE. — Les théorèmes d'existence fournis par la Méthode des approximations successives sont en général locaux. Dans des cas appropriés on peut, en répétant l'application de cette méthode, atteindre de proche en proche des résultats non locaux : il faut que le rayon de convergence ne tombe pas à zéro [*cf.* les travaux relatifs à l'équation $\Delta u = e^u$; les Mémoires de M. S. Bernstein sur les équations aux dérivées partielles du second ordre et du type elliptique]. D'autre part M. E. Schmidt a montré que les solutions d'une équation intégrale, considérées comme fonctions des paramètres, peuvent admettre des singularités de nature algébrique. Dans ce cas le procédé ci-dessus

Journ. de Math., tome XII. — Fasc. I, 1933. 1

The existence of a D–solution to (NS) have been established by J. Leray [1933] by the so–called method of *invading domains* if the fluxes through every $\partial\Omega_i$ vanish

$$\mathcal{F}_i = \int\limits_{\partial\Omega_i} \mathbf{a} \cdot \mathbf{n} \, dS = 0, \quad i = 1 \ldots, N. \tag{VF}$$

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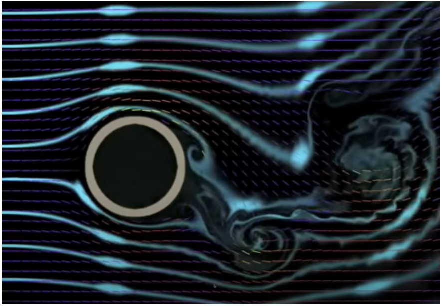
In particular, this method works for the physically most important “Flow Past an Obstacle” problem, i.e., when $\mathbf{a} = \mathbf{0}$.



Flow Past an Obstacle problem

Leray's *invading domains*-method essentially consists in showing that the solutions $(\mathbf{u}_k, p_k)$ to the sequence of problems

$$\begin{aligned} \nu\Delta\mathbf{u} - \mathbf{u}\cdot\nabla\mathbf{u} - \nabla p = \mathbf{0} \quad & \text{in } \Omega_k, \\ \operatorname{div}\mathbf{u} = 0 \quad & \text{in } \Omega_k, \\ \mathbf{u} = \mathbf{0} \quad & \text{on } \partial\Omega, \\ \mathbf{u} = \mathbf{u}_\infty \quad & \text{on } \partial B_{R_k}, \end{aligned} \tag{NS_k}$$

for $R_{k+1} > R_k \to \infty$, satisfies the estimate

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$$\int_{\Omega_k} |\nabla\mathbf{u}_k|^2 \le c \tag{est}$$

for some positive constant c independent of k.

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for $R_{k+1} > R_k \to \infty$, satisfies the estimate

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for some positive constant c independent of k. Then of course

$$\mathbf{u}_k \rightharpoonup \mathbf{u}_L,$$

where $\mathbf{u}_L$ is a solution to (NS_{1-3}), but the method is unable to give any information about the asymptotic behavior of $\mathbf{u}_L(x) \underset{|x|\to\infty}{\to} ???$, e.g., $\mathbf{u}$ could be even unbounded (apriory).

This achievement of Leray immediately raises two crucial questions:
(1) Is the constructed solution $\mathbf{u}_L$ nontrivial, i.e., can we exclude the identity $\mathbf{u}_L \equiv \mathbf{0}$?

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(2) If $\mathbf{w}_L$ is nontrivial, what can we say about its behavior at infinity? Namely, can we guarantee the desired convergence

$$\mathbf{u}_L(z) \to \mathbf{u}_\infty \text{ as } |z| \to \infty \quad ? \tag{2}$$

The result of Leray was rediscovered by H. Fujita [1961] by means of a different technique. Nevertheless, due to a lack of a uniqueness theorem, the two solutions could have different behavior at infinity.

H. Fujita (1961)

The first existence theorem for (NS) satisfying the condition at infinity

$$\mathbf{u}(x) \to \mathbf{u}_\infty \qquad \text{as } x \to \infty \qquad\qquad (Cond\_Inf)$$

is due to D.R. Smith and R. Finn [1967]: if $\mathbf{0} \neq \mathbf{u}_\infty$ is sufficiently small, then there is a D–solution to (NS) which converges uniformly to $\mathbf{u}_\infty$. This result is particularly meaningful in view of above mentioned *Stokes paradox* which asserts that the corresponding linear Stokes system has no solutions.

R. Finn and V.Pukhnachev

The “symmetric” question, whether the system

$$
\begin{aligned}
\nu\Delta\mathbf{u} - \mathbf{u}\cdot\nabla\mathbf{u} - \nabla p &= \mathbf{0} && \text{in } \Omega,\\
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admits a solution constant on $\partial\Omega$ and zero at infinity has no answer yet, also for small data!

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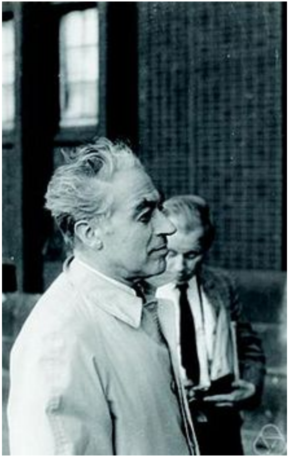
admits a solution constant on $\partial\Omega$ and zero at infinity has no answer yet, also for small data!
The existence of such solutions (for non-constant boundary data) was proved by [PileckasRusso2012] under symmetry conditions with respect to *both* coordinate axis.

The problem of the asymptotic behavior of an arbitrary D–solution $(\mathbf{u}, p)$ to (NS) at infinity was tackled by D. Gilbarg & H. Weinberger and C. Amick.



David Gilbarg

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$$p(x) - p_0 = o(1) \qquad \text{as } r = |x| \to \infty,$$

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$$p(x) - p_0 = o(1) \qquad \text{as } r = |x| \to \infty,$$

i.e., pressure has a limit at infinity (one can choose, say, $p \to 0$), and

$$\begin{aligned} \mathbf{u}(x) &= o(\log^{1/2} r), \\ \omega &= o(r^{-3/4} \log^{1/8} r), \\ \nabla \mathbf{u}(x) &= o(r^{-3/4} \log^{9/8} r), \end{aligned} \tag{3}$$

where

$$\omega = \partial_1 u_2 - \partial_2 u_1$$

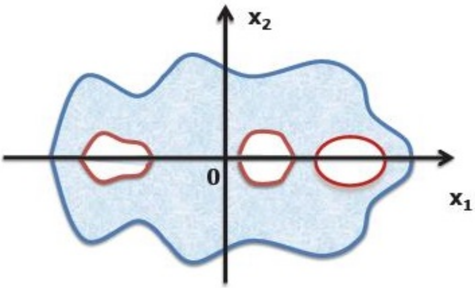
is the corresponding vorticity.

Amick proved that if

$$\mathbf{u} \equiv \mathbf{0} \quad \text{on } \partial\Omega,$$

then $\mathbf{u}$ is bounded.

Also, he proved that the solutions obtained by the Leray method are nontrivial in case of symmetric domains with respect to the x_2–axis (say), $i.e$, $(x_1, x_2) \in \Omega \Rightarrow (x_1, -x_2) \in \Omega$.



Amick symmetry assumption (1984)

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Furthermore, Amick proved that under these symmetry assumptions any D-solution to the flow around the obstacle problem converges uniformly to some constant vector $\mathbf{u}_0 \in \mathbb{R}^2$.

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$$\mathbf{u}(x) \rightrightarrows \mathbf{u}_0 \quad \text{as} \ \ |x| \to \infty.$$

Under Amick's symmetry assumptions with respect to one axis, the existence theorem for *arbitrary fluxes* was proved in the recent paper [KPR2014].

Main results

Main results

Theorem 1 [KPR2017]. *Let* $\mathbf{u}$ *be a solution to the N–S system*

$$\begin{cases} -\nu\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \mathbf{0} & \text{in } \Omega, \\ \operatorname{div}\mathbf{u} = 0 & \text{in } \Omega \end{cases} \tag{NS}$$

in the exterior domain $\Omega \subset \mathbb{R}^2$*. Suppose that* $\int\limits_{\Omega} |\nabla\mathbf{u}|^2 < \infty$ *(i.e.,* $\mathbf{u}$ *is a* D*-solution). Then* $\mathbf{u}$ *is uniformly bounded in the neighborhood of infinity, i.e.,*

$$\sup_{|x|>R_0} |\mathbf{u}(x)| < \infty, \tag{4}$$

where $B_{R_0} = \{x \in \mathbb{R}^2 : |x| < R_0\} \supset \partial\Omega$*.*

Theorem 2 [KPR2017]. *Let Ω be an exterior domain with smooth compact boundary. Suppose that $\mathbf{a} \in W^{1/2,2}(\partial\Omega)$ and the identity*

$$\sum_{i=1}^{N} \mathcal{F}_i = 0$$

holds, i.e., if the total flux is zero. Then there exists a D-solution $\mathbf{u}$ to the Navier–Stokes system (NS) with boundary condition

$$\mathbf{u} = \mathbf{a} \qquad \text{on } \partial\Omega.$$

Theorem 3 [KPR2018]. *Let* $\mathbf{u}$ *be a solution to the N–S system*

$$\begin{cases} -\nu\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \mathbf{0} & \text{in } \Omega, \\ \operatorname{div}\mathbf{u} = 0 & \text{in } \Omega \end{cases} \qquad (NS)$$

in the exterior smooth domain $\Omega \subset \mathbb{R}^2$*. Suppose that* $\int\limits_\Omega |\nabla\mathbf{u}|^2 < \infty$ *(i.e.,* $\mathbf{u}$ *is a* D*-solution). Then* $\mathbf{u}$ *has a uniform limit at infinity, i.e.,*

$$\mathbf{u}(x) \rightrightarrows \mathbf{u}_0 \quad \text{as } |x| \to \infty$$

for some $\mathbf{u}_0 \in \mathbb{R}^2$*.*

Theorem 4 [KPR2018]. *Leray solution of the flow around an obstacle problem (when $\mathbf{a} = 0, \mathbf{u}_\infty \neq 0$) is always nontrivial. These solutions converges to some constant vector $\mathbf{u}_0$ when* $|x| \to \infty$.

Theorem 4 [KPR2018]. *Leray solution of the flow around an obstacle problem (when* $\mathbf{a} = 0, \mathbf{u}_\infty \neq 0$ *) is always nontrivial. These solutions converges to some constant vector* $\mathbf{u}_0$ *when* $|x| \to \infty$*.*

Open question: $\mathbf{u}_0 = \mathbf{u}_\infty$?

Finn and Smith, 1967: “We remark that for solutions in three dimensions, the uniqueness of a small solution has been proved in the class of all solutions which exhibit the same qualitative asymptotic properties at infinity. We have been unable to obtain a comparable result in the present case.”
G.P. Galdi, 2011: “Maybe uniqueness of generalized solutions is a more complicated question than existence itself. Indeed it represents a formidable problem that, for its resolution, requires in my opinion the contribution of completely new ideas and methods... Nevertheless, what is certainly true is that, if the Reynolds number is ‘sufficiently’ large and the flux through the wall is not zero, uniqueness of generalized solutions is lost, as can be seen by means of simple examples.”

J. Guillod and P. Wittwer, Proc. Amer. Math. Soc., 2018: “The question of the uniqueness of weak solutions for small data is even more open in two-dimensional exterior domains... For two-dimensional exterior domains with nonempty boundary, we would apriory also expect the existence of infinitely many weak solutions parameterized by some parameter”.

Theorem 5 [K,XiaoRen2020]. *There exists a positive constant λ_0 depending only on the geometry of $\partial\Omega$, such that, for $0 < |\mathbf{u}_\infty| < \lambda_0$ and for arbitrary D-solution $\mathbf{u}$ to the exterior problem*

$$\begin{aligned} \nu\Delta\mathbf{u} - \mathbf{u}\cdot\nabla\mathbf{u} - \nabla p = \mathbf{0} \qquad & \text{in } \Omega, \\ \operatorname{div}\mathbf{u} = 0 \qquad & \text{in } \Omega, \\ \mathbf{u} = \mathbf{0} \qquad & \text{on } \partial\Omega, \\ \lim_{x\to\infty}\mathbf{u}(x) = \mathbf{u}_\infty, & \end{aligned} \tag{NS}$$

the identity $\mathbf{u}(z) \equiv \mathbf{u}_{FS}(z)$ holds.

Here $\mathbf{u}_{FS}(z)$ is the classical Finn-Smith solution constructed in 1967.

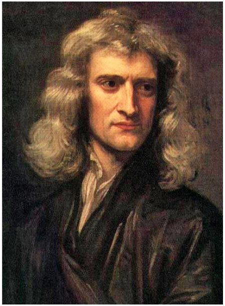
Theorem 6 [K,XiaoRen2021]. *There exists a positive constant λ_0 depending only on the geometry of $\partial\Omega$, such that for $0 < |\mathbf{u}_\infty| < \lambda_0$ the corresponding Leray solution $\mathbf{u} = \mathbf{u}_L$ (obtained by the invading domain method) has the prescribed limit at infinity, i.e., it satisfies all the required conditions:*

$$\begin{aligned} \nu\Delta\mathbf{u} - \mathbf{u}\cdot\nabla\mathbf{u} - \nabla p &= \mathbf{0} \qquad \text{in } \Omega, \\ \operatorname{div}\mathbf{u} &= 0 \qquad \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} \qquad \text{on } \partial\Omega, \\ \lim_{x\to\infty}\mathbf{u}(x) &= \mathbf{u}_\infty. \end{aligned} \tag{NS}$$

Thus, despite the recent advances, there remain many unsolved and attractive problems for steady NS system:
the Leray problem in a bounded three-dimensional domain in the general case (without assumptions of axial symmetry),
the similar problem in exterior three-dimensional domains, and,
of course, the Leray problem in exterior two-dimensional domains.

There are also other very interesting problems, for example, existence of a solution in a pipe with asymptotic as the Poiseuille flow, Liouville type theorems for steady NS system in the whole three-dimensional space, etc.

太谢谢您了！！！



Isaac Newton (1689, by Godfrey Kneller)