On operators commuting with a noncommutative C*-algebra (Schrödinger operator in magnetic field as an example)

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Noncommutative torus

Let $\mathbb{S}$ denote the unit circle parametrized by $s \in [0, 1]$. Let $\theta \in [0, 1)$. Consider on $L^2(\mathbb{S})$ the operators U, V, $Uf(s) = f(s - \theta)$, $Vf(s) = e^{2\pi is} f(s)$, $s \in \mathbb{S}$. Then $UV = e^{2\pi i\theta} VU$. Let A_θ be the C*-algebra (i.e. the norm closed involutive algebra of operators on $l^2(\mathbb{S})$) generated by U and V. When $\theta = 0$, this gives the algebra generated by two commuting unitaries, i.e. the algebra of continuous functions on a 2-torus.

If θ is irrational then

- A_θ is simple;
- A_θ is 'zero-dimensional', i.e. invertible elements are dense in A_θ, and invertible selfadjoint elements are dense in the set of selfadjoints.

This C*-algebra is usually called a noncommutative torus, or an irrational rotation algebra.

Smooth structure on A_θ

A_θ is the completion of finite sums $a = \sum_{n,m\in\mathbb{Z}} a_{nm}U^nV^m$. It contains a 'smooth' subalgebra A_θ^∞ of sums of 'rapid decay'. The 2-torus $\mathbb{T}^2$ acts on A_θ: set $\alpha_{(s,t)}U = e^{2\pi is}U$, $\alpha_{(s,t)}V = e^{2\pi it}V$, $(s,t) \in \mathbb{T}^2$; then α extends to an endomorphism of A_θ, and, by simplicity, it is an isomorphism. Define $\phi_a : \mathbb{T}^2 \to A_\theta$ by $\phi_a(g) = \alpha_g(a)$, $a \in A_\theta$, $g \in \mathbb{T}^2$, and set $A_\theta^\infty = \{a \in A_\theta : \phi_a \text{ is } C^\infty\}$.

If M^∞ is a projective module of finite type over A_θ^∞ then $M = M^\infty \otimes_{A_\theta^\infty} A_\theta$ is a projective module of finite type over A_θ. If M is a projective module of finite type over A_θ then there exists a projective module M^∞ of finite type over A_θ^∞ such that $M = M^\infty \otimes_{A_\theta^\infty} A_\theta$.

A projective module over A_θ

Let $\mathcal{S}(\mathbb{R})$ be the space of Schwartz functions on $\mathbb{R}$. Define a right action of A_θ^∞ on $\mathcal{S}(\mathbb{R})$ by $(\varphi U)(t) = \varphi(t+\theta)$, $(\varphi V)(t) = e^{2\pi i t}\varphi(t)$, $\varphi \in \mathcal{S}(\mathbb{R})$.

$\mathcal{S}(\mathbb{R})$ is a projective module of finite type over A_θ^∞.

One may view $\mathcal{S}(\mathbb{R})$ as a vector bundle over a noncommutative torus. A. Connes developed gifferential geometry on this vector bundle, e.g. a connection given by $\nabla_1\varphi(t) = \frac{d\varphi(t)}{dt}$, $\nabla_2\varphi(t) = \frac{2\pi i t}{\theta}\varphi(t)$ has curvature θ.

Schrödinger operator with irrational magnetic flux

Let $\left(i\frac{\partial}{\partial x}+2\pi\theta y\right)^2-\frac{\partial^2}{\partial y^2}+V(x,y)$,
$V(x+1,y)=V(x,y+1)=V(x,y)$, be a perturbed Schrödinger operator with irrational magnetic flux.
After Fourier transform $x\to\xi$ and change of variables $t=-\frac{\xi}{2\pi}+\theta y$, $s=\frac{\xi}{2\pi}$, this operator takes the form

$$D=\theta^2\left(\left(\frac{2\pi t}{\theta}\right)^2-\frac{\partial^2}{\partial t^2}\right)+\sum_{k,l}v_{kl}T_t^kT_s^{-k}e^{2\pi ilt/\theta}e^{2\pi ils/\theta},$$

where T_t and T_s are unit translations in t and in s, and v_{kl} are Fourier coefficients of V.

Let $(\varphi_j(t))_{j\in\mathbb{N}}$ be the eigenfunctions of $\Delta = \left(\frac{2\pi t}{\theta}\right)^2 - \frac{\partial^2}{\partial t^2}$. They form an orthonormal basis in $L^2(\mathbb{R}) \supset \mathcal{S}(\mathbb{R})$, and lie in $\mathcal{S}(\mathbb{R})$. Hence the set S_0 of finite sums $\sum_j \varphi_j(t)m_j(s)$, where $m_j \in \mathcal{S}(\mathbb{R})$, is dense in $\mathcal{S}(\mathbb{R}^2)$. Define the structure of a right A_θ^∞-module on S_0 by $(\sum_j \varphi_j m_j \cdot a)(t,s) = \sum_j \varphi_j(t)(m_j \cdot a)(s)$, $a \in A_\theta^\infty$.

As $\mathcal{S}(\mathbb{R})$ is a projective module of finite type over A_θ^∞, there is a well-defined A_θ-valued inner product $\langle\cdot,\cdot\rangle$ on $M = \mathcal{S}(\mathbb{R}) \otimes_{A_\theta^\infty} A_\theta$. Set $\langle\sum_j \varphi_j m_j, \sum_k \varphi_k n_k\rangle = \sum_j \langle m_j, n_j\rangle$. This gives an inner product on S_0. Completion of S_0 w.r.t. the norm $\|a\|^2 = \|\langle a,a\rangle\|$ gives a Hilbert C*-module $l_2(M)$ over A_θ.

Hilbert C*-modules

Let A be a unital C*-algebra, M a right A-module equipped with the sesquilinear map $\langle\cdot,\cdot\rangle : M \times M \to A$ such that

- $\langle xa, y\rangle = a^*\langle x, y\rangle$; $\langle x, ya\rangle = \langle x, y\rangle a$, $x, y \in M$, $a \in A$;
- $\langle y, x\rangle = \langle x, y\rangle^*$, $x, y \in M$;
- $\langle x, x\rangle$ is positive in A.

Then M is called a pre-Hilbert C*-module over A. It has a norm $\|x\|^2 = \|\langle x, x\rangle\|$. It is a Hilbert C*-module if it is complete w.r.t. this norm.

Theorem

The operator D is a selfadjoint unbounded (densely defined) operator on the Hilbert C*-module $l_2(M)$ of the form ‘positive plus bounded’.

Corollary

The operator $(C + D)^{-1}$ exists for sufficiently great $C > 0$, and is compact.

Diagonalization

Let $A = C(X)$ be a commutative C*-algebra. Let $T \in M_n(A)$ be a selfadjoint matrix with entries from A. Then it can be diagonalized pointwise, with diagonal entries from A. The coordinates of eigenvectors need not be continuous, but we may allow them to be from a greater W*-algebra $L^\infty(X)$. The same works for compact selfadjoint operators with matrix entries from A.

For general noncommutative C*-algebras this is no longer true, but this holds for a certain class including the noncommutative torus. Namely, for a positive compact selfadjoint operator T over A_θ there exist $\lambda_j \in A_\theta$, $j \in \mathbb{N}$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ such that $diag(\lambda_1, \lambda_2, \cdots) = U^* TU$ for some unitary U with matrix entries in a greater W*-algebra (where A_θ is a weakly dense subalgebra).

The same works for the C*-algebra $A'_\theta = End_{A_\theta}(M)$.

When $V = 0$, λ_j are scalars (the eigenvalues of the operator $\Delta = \left(\frac{2\pi t}{\theta}\right)^2 - \frac{\partial^2}{\partial t^2}$). When V is small, λ_j are close to scalars. When V is great, these λ_j have no physical meaning. The situation is similar to diagonalization of usual compact operators considered as operators with matrix entries in the algebra $A = M_n(\mathbb{C})$. We diagonalize a positive selfadjoint compact operator as usual, obtain the usual eigenvalues μ_j, $j \in \mathbb{N}$, ordered by decreasing, and then group them into n-tuples to get $\lambda_i = diag(\mu_{(n-1)i+1}, \cdots, \mu_{ni}) \in M_n(\mathbb{C})$, then the diagonalized operator is given by $diag(\lambda_1, \lambda_2, \cdots)$. For example, if the usual eigenvalues μ_j analytically depend on a parameter then this is damaged when we group the eigenvalues into n-tuples using the ordering (which may change). The theory should be more analytic than smooth.