El's Nonlocal Kinetic Equation and its Hydrodynamic Reductions

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The Talk is based on joint works with my friends and colleagues:
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11 October 2021

Geometry, Topology and Their Applications

El's Nonlocal Kinetic Equation

El's integro-differential kinetic equation for dense soliton gas (2003)

$$f_t + (sf)_x = 0,$$

$$s(\eta) = S(\eta) + \int_0^\infty G(\mu,\eta) f(\mu)[s(\mu) - s(\eta)]\, d\mu,$$

where $f(\eta) = f(\eta, x, t)$ is a distribution function and $s(\eta) = s(\eta, x, t)$ is the associated transport velocity. Here the variable η is the spectral parameter in the Lax pair; the function $S(\eta)$ (free soliton velocity) and the kernel $G(\mu,\eta)$ (phase shift due to pairwise soliton collisions) are independent of x and t. The kernel $G(\mu,\eta)$ is assumed to be symmetric: $G(\mu,\eta) = G(\eta,\mu)$. This system describes the evolution of a dense soliton gas and represents a broad generalisation of Zakharov's kinetic equation for rarefied soliton gas. In this case

$$S(\eta) = 4\eta^2, \qquad G(\mu,\eta) = \frac{1}{\eta\mu}\log\left|\frac{\eta-\mu}{\eta+\mu}\right|,$$

the above system was derived by G. El as a thermodynamic limit of the KdV Whitham equations

Hydrodynamic Reductions. Dirac Delta-Functional Ansatz

Under a delta-functional ansatz (an iso-spectral case, 2010, G.A. El, A.M. Kamchatnov, MVP, S.A. Zykov),

$$f(\eta, x, t) = \sum_{i=1}^{N} u^i(x, t)\ \delta(\eta - \eta^i),$$

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reduces to a $N \times N$ quasilinear system for $u^i(x, t)$,

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where v^i can be recovered from the linear system (here $\xi^i = -S(\eta^i)$)

$$v^i = \xi^i + \sum_{m \neq i} \epsilon^{mi} u^m (v^m - v^i), \qquad \epsilon^{ki} = G(\eta^k, \eta^i),\ k \neq i.$$

Integrability of Diagonalisable Hydrodynamic Type Systems

The algorithm for integrability of diagonalisable hydrodynamic type systems

$$r_t^i = \mu^i(\mathbf{r}) r_x^i.$$

1. Diagonal metric coefficients $g_{kk}(\mathbf{r}) = H_k^2$ are determined by (here $\partial_k \equiv \partial / \partial r^k$)

$$\partial_k \ln H_i = \frac{\partial_k \mu^i}{\mu^k - \mu^i}, \quad i \neq k.$$

2. Integrability condition

$$\partial_j \frac{\partial_k \mu^i}{\mu^k - \mu^i} = \partial_k \frac{\partial_j \mu^i}{\mu^j - \mu^i}, \quad i \neq j \neq k.$$

3. The linear system

$$\partial_k \zeta^i = \frac{\partial_k \mu^i}{\mu^k - \mu^i}(\zeta^k - \zeta^i), \quad i \neq k.$$

4. The Tsarev Generalised Hodograph Method

$$x + \mu^i(\mathbf{r}) t = \zeta^i(\mathbf{r}).$$

Parametrisation

Now we introduce the new variables r^i by the formula

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In the dependent variables r^i, the quasilinear system

$$u^i_t = (u^i v^i)_x,$$

reduces to a diagonal form

$$r^i_t = v^i r^i_x,$$

where velocities v^i can be expressed in terms of Riemann invariants r^k as follows. Let us introduce the $N \times N$ matrix $\hat{\epsilon}$ with diagonal entries $r^1, \dots, r^N$ (so that $\epsilon^{ii} = r^i$) and off-diagonal entries $\epsilon^{ik} = G(\eta^i, \eta^k),\ k \neq i$. Note that this matrix is symmetric due to the symmetry of the kernel G.

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Tsarev's Generalised Hodograph Method

Denote β_{ik} the matrix elements of $\hat{\beta}$ (indices i and k are allowed to coincide). Then we obtain the following formulae for u^i, v^i:

$$u^i = \sum_{m=1}^{N} \beta_{mi}, \;\; v^i = \frac{1}{u^i} \sum_{m=1}^{N} \xi^m \beta_{mi}.$$

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$$x + \xi_i t = P_i(r^i) - r^i P_i'(r^i) - \sum_{m \neq i} \epsilon^{mi} P_m'(r^m), \quad i = 1, 2, ..., N,$$

where $P_i(r^i)$, $i = 1, \ldots, N$, are arbitrary functions.

Tsarev's Generalised Hodograph Method

Under the re-parametrization

$$P_k''(\xi) = -\frac{\phi_k(\xi)}{f(\xi)}$$

the generalized hodograph solution

$$x + \xi_i t = P_i(r^i) - r^i P_i'(r^i) - \sum_{m \neq i} \epsilon^{mi} P_m'(r^m), \quad i = 1, 2, ..., N,$$

becomes

$$x + \xi_i t = \int^{r^i} \frac{\xi \phi_i(\xi)}{f(\xi)} d\xi + \sum_{m \neq i} \epsilon^{mi} \int^{r^m} \frac{\phi_m(\xi)}{f(\xi)} d\xi \, .$$

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Now we consider the particular choice of $f(\xi)$ defined as $f(\xi) = \sqrt{R_K(\xi)}$, where

$$R_K(\xi) = \prod_{m=1}^{K} (\xi - E_m),$$

and $E_1 < E_2 < \cdots < E_K$ are real constants ($K = 2g + 1$ and $K = 2g + 2$ for odd and even cases, respectively, where g is a genus of this hyperelliptic curve); and $\phi_k(\xi)$ being arbitrary polynomials in ξ of degrees less than g.

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$$x + \xi_i t = \int^{r^i} \frac{\xi\phi_i(\xi)}{f(\xi)} d\xi + \sum_{m\neq i} \epsilon^{mi} \int^{r^m} \frac{\phi_m(\xi)}{f(\xi)} d\xi\,,$$

describes quasiperiodic solutions of the form

$$x + \xi_i t = \int^{r^i} \frac{\xi\phi_i(\xi)d\xi}{\sqrt{R_K(\xi)}} + \sum_{m\neq i} \epsilon^{mi} \int^{r^m} \frac{\phi_m(\xi)d\xi}{\sqrt{R_K(\xi)}}, \quad i = 1, 2, ..., N.$$

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reduces to a $2N \times 2N$ quasilinear system for $u^i(x, t)$ and $\eta^i(x, t)$,

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Block-Diagonal Hydrodynamic Type Systems

Introducing new field variables

$$r^i = -\frac{1}{u^i}\left(1 + \sum_{m\neq i} \epsilon^{mi} u^m\right),$$

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can be rewritten in a block-diagonal form

$$r^i_t = v^i r^i_x + p^i \eta^i_x, \quad \eta^i_t = v^i \eta^i_x,$$

where

$$u^i = \sum_{m=1}^{N} \beta_{mi}, \;\; v^i = \frac{1}{u^i} \sum_{m=1}^{N} \xi^m \beta_{mi}, \;\; p^i = \frac{1}{u^i}\left(\sum_{m \neq i} \epsilon^{mi}_{,\eta^i} (v^m - v^i) u^m + (\xi^i)'\right).$$

Block-Diagonal Hydrodynamic Type Systems

Now we study integrability aspects of quasilinear systems

$$u_t^i = V_k^i(\mathbf{u})u_x^k,$$

whose matrix V consists of N Jordan blocks of size 2×2:

$$\begin{aligned} r_t^i &= v^i r_x^i + p^i \eta_x^i, \\ \eta_t^i &= v^i \eta_x^i, \end{aligned}$$

$i = 1, \ldots, N$, where the coefficients $v^i(r, \eta)$ and $p^i(r, \eta)$ are functions of the N dependent variables $r = (r^1, \ldots, r^N)$ and N dependent variables $\eta = (\eta^1, \ldots, \eta^N)$.

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We are looking for commuting flows in the form (2021, E.V. Ferapontov, MVP)

$$r_y^i = w^i r_x^i + q^i \eta_x^i,$$
$$\eta_y^i = w^i \eta_x^i.$$

Then unknown expressions $w^i(\mathbf{r}, \boldsymbol{\eta})$, $q^i(\mathbf{r}, \boldsymbol{\eta})$ can be found from the compatibility conditions $(r_y^i)_t = (r_t^i)_y$, $(\eta_y^i)_t = (\eta_t^i)_y$, $i = 1, 2, ..., N$.

Block-Diagonal Hydrodynamic Type Systems

Indeed, the compatibility conditions

$$(r^i_y)_t = (r^i_t)_y, \quad (\eta^i_y)_t = (\eta^i_t)_y, \quad i = 1, 2, ..., N$$

lead to the set of equations

$$w^i_{r^i} = a_i q^i, \quad w^i_{\eta^i} = b_i q^i + q^i_{r^i},$$

where we denote

$$a_i = \frac{v^i_{r^i}}{p^i}, \quad b_i = \frac{v^i_{\eta^i} - p^i_{r^i}}{p^i}.$$

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$$w^i_{r^j} = a_{ij}(w^j - w^i), \quad w^i_{\eta^j} = b_{ij}(w^j - w^i) + a_{ij} q^j,$$

$$q^i_{r^j} = c_{ij}(w^j - w^i) - a_{ij} q^i, \quad q^i_{\eta^j} = d_{ij}(w^j - w^i) + c_{ij} q^j - b_{ij} q^i,$$

where we denote

$$a_{ij} = \frac{v^i_{r^j}}{v^j - v^i}, \quad b_{ij} = \frac{v^i_{\eta^j} - a_{ij} p^j}{v^j - v^i}, \quad c_{ij} = \frac{p^i_{r^j} + a_{ij} p^i}{v^j - v^i}, \quad d_{ij} = \frac{p^i_{\eta^j} + b_{ij} p^i - c_{ij} p^j}{v^j - v^i}.$$

Integrability Conditions I

The list of integrability conditions for every pair of distinct indices is

$$a_{i,r^j} = 0, \qquad a_{ij,r^i} = a_{ij}a_{ji} + a_i c_{ij};$$

$$a_{i,\eta^j} = 0, \qquad b_{ij,r^i} = b_{ij}a_{ji} + a_{ij}c_{ji} + a_i d_{ij};$$

$$b_{i,r^j} = 2a_{ij}a_{ji} + 2a_i c_{ij},$$

$$a_{ij,\eta^i} = a_{ij}b_{ji} - c_{ij}a_{ji} + b_i c_{ij} + c_{ij,r^i};$$

$$b_{i,\eta^j} = 2a_{ij}c_{ji} + 2b_{ij}a_{ji} + 2a_i d_{ij},$$

$$b_{ij,\eta^i} = b_{ij}b_{ji} + a_{ij}d_{ji} - d_{ij}a_{ji} - c_{ij}c_{ji} + b_i d_{ij} + d_{ij,r^i};$$

$$a_{ij,r^j} = b_j a_{ij} - a_j b_{ij} - a_{ij}^2, \qquad a_{ij,\eta^j} = b_{ij,r^j};$$

$$c_{ij,r^j} = b_j c_{ij} - a_j d_{ij} - 2a_{ij}c_{ij}, \qquad c_{ij,\eta^j} = d_{ij,r^j}.$$

Integrability Conditions II

The list of integrability conditions for every triad of distinct indices is

$$a_{ij,r^k} = a_{ij}a_{jk} + a_{ik}a_{kj} - a_{ij}a_{ik}.$$

$$a_{ij,\eta^k} = a_{ij}b_{jk} + a_{ik}c_{kj} + b_{ik}a_{kj} - a_{ij}b_{ik},$$
$$b_{ij,r^k} = b_{ij}a_{jk} + a_{ik}b_{kj} + a_{ij}c_{jk} - a_{ik}b_{ij}.$$

$$b_{ij,\eta^k} = a_{ij}d_{jk} + a_{ik}d_{kj} + b_{ij}b_{jk} + b_{ik}b_{kj} - b_{ij}b_{ik}.$$

$$c_{ij,r^k} = c_{ij}a_{jk} + c_{ik}a_{kj} - c_{ij}a_{ik} - c_{ik}a_{ij}.$$

$$c_{ij,\eta^k} = c_{ij}b_{jk} + c_{ik}c_{kj} + d_{ik}a_{kj} - a_{ij}d_{ik} - c_{ij}b_{ik},$$
$$d_{ij,r^k} = d_{ij}a_{jk} + c_{ij}c_{jk} + c_{ik}b_{kj} - a_{ik}d_{ij} - c_{ik}b_{ij}.$$

$$d_{ij,\eta^k} = c_{ij}d_{jk} + c_{ik}d_{kj} + d_{ij}b_{jk} + d_{ik}b_{kj} - b_{ij}d_{ik} - b_{ik}d_{ij}.$$

Commuting Flows

The block-diagonal system

$$r^i_t = v^i r^i_x + p^i \eta^i_x, \quad \eta^i_t = v^i \eta^i_x,$$

where

$$u^i = \sum_{m=1}^{N} \beta_{mi}, \;\; v^i = \frac{1}{u^i} \sum_{m=1}^{N} \xi^m \beta_{mi}, \;\; p^i = \frac{1}{u^i} \left(\sum_{m \neq i} \epsilon^{mi}_{,\eta^i} (v^m - v^i) u^m + (\xi^i)' \right),$$

possesses infinitely many commuting block-diagonal flows

$$r^i_y = w^i r^i_x + q^i \eta^i_x, \quad \eta^i_y = w^i \eta^i_x,$$

where

$$w^i = \frac{1}{u^i} \sum_{m=1}^{N} \varphi^m \beta_{mi}, \qquad q^i = \frac{1}{u^i} \left(\sum_{m \neq i} \epsilon^{mi}_{,\eta^i} (w^m - w^i) u^m - r^i \mu^i + \varphi^i_{,\eta^i} \right).$$

Here $\mu^i(\eta^i)$ are N arbitrary functions of one variable and the functions $\varphi^i(\eta^1, \ldots, \eta^N)$ satisfy the relations $\partial_{\eta^k} \varphi^i = \epsilon^{ki} \mu^k, \;\; k \neq i$. The general commuting flow depends on $2N$ arbitrary functions of one variable: N functions $\mu^i(\eta^i)$ plus extra N functions coming from φ^i.

Conservation Laws

Conservation laws $h_t = g_x$ provide an alternative way to derive integrability conditions for the block-diagonal system

$$r^i_t = v^i r^i_x + p^i \eta^i_x, \quad \eta^i_t = v^i \eta^i_x.$$

Their existence leads to a system of second-order linear PDEs

$$h_{r^i r^i} = b_i h_{r^i} - a_i h_{\eta^i}, \quad h_{r^i \eta^j} = a_{ji} h_{\eta^j} + c_{ji} h_{r^j} + b_{ij} h_{r^i},$$

$$h_{r^i r^j} = a_{ij} h_{r^i} + a_{ji} h_{r^j}, \quad h_{\eta^i \eta^j} = d_{ij} h_{r^i} + d_{ji} h_{r^j} + b_{ij} h_{\eta^i} + b_{ji} h_{\eta^j},$$

where $g_{r^i} = v^i h_{r^i}$, $g_{\eta^i} = p^i h_{r^i} + v^i h_{\eta^i}$.

The general conservation law has the form ($\sigma^i(\eta^i)$ are arbitrary functions)

$$\left(\sum_{m=1}^{N} u^m \psi^m(\eta) + \sum_{m=1}^{N} \sigma^m(\eta^m)\right)_t = \left(\sum_{m=1}^{N} u^m v^m \psi^m(\eta) + \sum_{m=1}^{N} \tau^m(\eta^m)\right)_x,$$

where $(\tau^i)' = (\sigma^i)' \xi^i$ and $\psi^i_{,\eta^k} = (\sigma^j)' \epsilon^{ik}$, $k \neq i$. This general conservation law depends on $2N$ arbitrary functions of one variable: N functions $\sigma^i(\eta^i)$, plus extra N functions coming from ψ^i.

Tsarev's Generalised Hodograph Method

If the hydrodynamic type system $u_t = V(u)u_x$ has a commuting flow $u_y = W(u)u_x$, where $V(u)$ and $W(u)$ are $N \times N$ matrices (the commutativity conditions $u_{ty} = u_{yt}$ impose differential constraints on V and W),

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$$W(u) = I\,x + V(u)\,t,$$

where I is the $N \times N$ identity matrix, defines an implicit solution $u(x,t)$. Note that, due to the commutativity conditions, only N out of the above N^2 relations will be functionally independent.

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$$r^i_t = v^i r^i_x + p^i \eta^i_x, \quad \eta^i_t = v^i \eta^i_x,$$
$$r^i_y = w^i r^i_x + q^i \eta^i_x, \quad \eta^i_y = w^i \eta^i_x,$$

the hodograph formula becomes

$$w^i(r,\eta) = x + v^i(r,\eta)\,t, \qquad q^i(r,\eta) = p^i(r,\eta)\,t,$$

which is a system of $2N$ implicit relations for the $2N$ dependent variables.

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$$u^i = \sum_{m=1}^{N} \beta_{mi}, \; v^i = \frac{1}{u^i} \sum_{m=1}^{N} \xi^m \beta_{mi}, \; p^i = \frac{1}{u^i} \left(\sum_{m \neq i} \epsilon^{mi}_{,\eta^i} (v^m - v^i) u^m + (\xi^i)' \right).$$

Then the general solution of the block-diagonal system

$$r^i_t = v^i r^i_x + p^i \eta^i_x,$$
$$\eta^i_t = v^i \eta^i_x,$$

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$$r^i = \frac{\varphi^i_{,\eta^i} - (\xi^i)' \, t}{\mu^i}, \qquad \varphi^i(\eta^1, \dots, \eta^N) = x + \xi^i(\eta^i)\, t;$$

where $\mu^i(\eta^i)$ are arbitrary functions of their arguments and the functions $\varphi^i(\eta^1, \dots, \eta^N)$ satisfy the relations $\varphi^i_{,\eta^k} = \epsilon^{ki}(\eta^i, \eta^k)\, \mu^k(\eta^k)$, $i \neq k$. The last N above equations define $\eta^i(x, t)$ as implicit functions of x and t; then the first N equations define $r^i(x, t)$ explicitly.

Appendix. A Multi-Component Case. A Nijenhuis tensor

Recall that, given an affinor V_k^i, its Haantjes tensor is defined by the formula

$$H_{jk}^i = N_{pr}^i V_j^p V_k^r - N_{jr}^p V_p^i V_k^r - N_{rk}^p V_p^i V_j^r + N_{jk}^p V_r^i V_p^r,$$

where

$$N_{jk}^i = V_j^p \partial_p V_k^i - V_k^p \partial_p V_j^i - V_p^i (\partial_j V_k^p - \partial_k V_j^p)$$

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where

$$N^i_{jk} = V^p_j \partial_p V^i_k - V^p_k \partial_p V^i_j - V^i_p (\partial_j V^p_k - \partial_k V^p_j)$$

is a Nijenhuis tensor.

In a generic case all characteristic velocities μ^k are *pairwise distinct*. If all components of a Nijenhuis tensor vanish, then corresponding hydrodynamic type system

$$u^i_t = V^i_k(\mathbf{u}) u^k_x$$

can be reduced to the totally decoupled form

$$\tilde{u}^i_t = \mu^i(\tilde{u}^i) \tilde{u}^i_x$$

by an appropriate invertible point transformation $\tilde{u}^k(\mathbf{u})$.

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$$H^i_{jk} = N^i_{pr} V^p_j V^r_k - N^p_{jr} V^i_p V^r_k - N^p_{rk} V^i_p V^r_j + N^p_{jk} V^i_r V^r_p,$$

while a Nijenhuis tensor is

$$N^i_{jk} = V^p_j \partial_p V^i_k - V^p_k \partial_p V^i_j - V^i_p (\partial_j V^p_k - \partial_k V^p_j).$$

Appendix. A Multi-Component Case. A Haantjes tensor

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In a generic case all characteristic velocities μ^k are *pairwise distinct*. If all components of a Haantjes tensor vanish, then corresponding hydrodynamic type system

$$u^i_t = V^i_k(\mathbf{u}) u^k_x$$

can be diagonalised, i.e. rewritten in the Riemann invariants

$$r^i_t = \mu^i(\mathbf{r}) r^i_x$$

by an appropriate invertible point transformation $r^k(\mathbf{u})$.

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