Characterizing envelopes of cones

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joint work with

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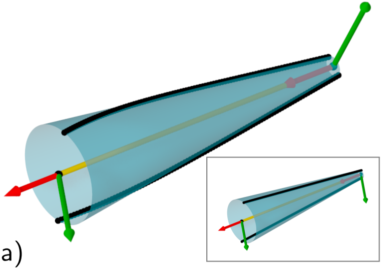
Overview

1. Motivation
2. Main idea
3. Statements

1
Motivation

5-Axis flank computer numerically controlled machining



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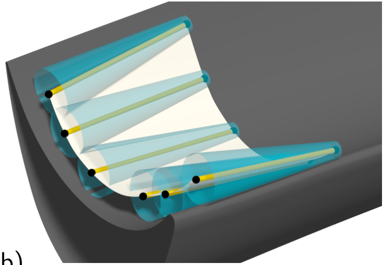
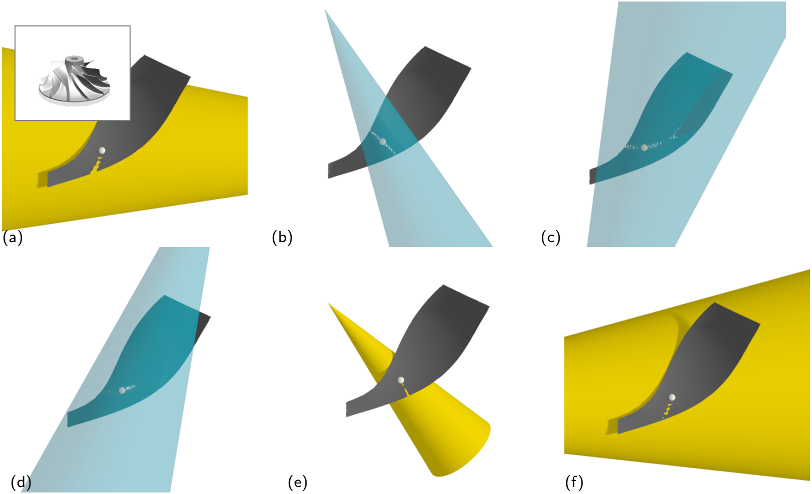


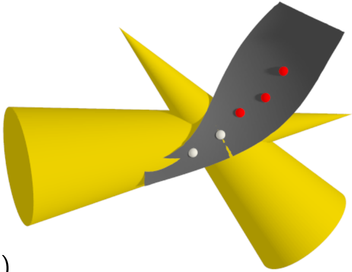
(b)

Engineering Problem. Approximate a given a surface by a one millable by a moving conical tool and reconstruct the motion.

Industrial benchmark



Industrial benchmark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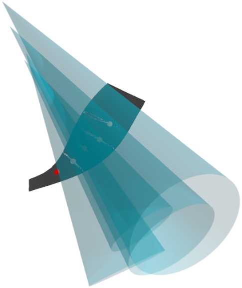
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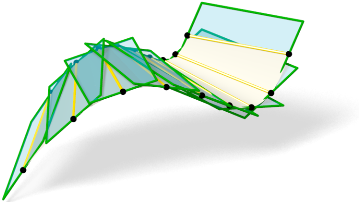
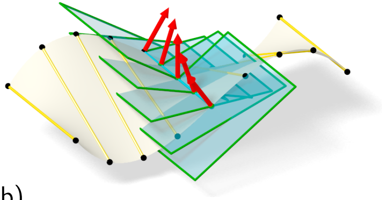
Mathematical Problem. Given a surface, decide if it is an envelope of a one-parametric family of cones, and reconstruct the family of cones.

Known limit cases: developable and ruled surfaces



(a)



(b)

Characterization of ruled surfaces

Theorem (folklore)

For a C^3 function $f\colon D \to \mathbb{R}$ defined in an open disk $D \subset \mathbb{R}^2$ the following conditions are equivalent:

- *Through a generic point of the graph of f there passes a line segment completely contained in the graph.*
- *For each $(x, y) \in D$ we have $f_{xx}f_{yy} - {f_{xy}}^2 \le 0$ and*

$$
\begin{aligned}
&{f_{yy}}^3{f_{xxx}}^2 + 6f_{yy}f_{xxx}f_{yyy}f_{xy}f_{xx} - 6{f_{yy}}^2f_{xxx}f_{xyy}f_{xx}\\
&\quad - 6f_{yyy}f_{xy}{f_{xx}}^2f_{xyy} + 9f_{yy}{f_{xyy}}^2{f_{xx}}^2 - 6f_{xy}{f_{yy}}^2f_{xxy}f_{xxx}\\
&+ 12{f_{xy}}^2f_{xxy}f_{yyy}f_{xx} - 18f_{xy}f_{yy}f_{xxy}f_{xyy}f_{xx} + 12f_{yy}f_{xyy}{f_{xy}}^2f_{xxx}\\
&- 8f_{yyy}{f_{xy}}^3f_{xxx} + 9f_{xx}{f_{yy}}^2{f_{xxy}}^2 - 6f_{yy}f_{xxy}f_{yyy}{f_{xx}}^2 + {f_{yyy}}^2{f_{xx}}^3 = 0.
\end{aligned}
$$

Proof of direct implication

3 times differentiate $z+wt = f(x+ut, y+vt)$ wrt t:

$$\begin{cases} f_{xx}u^2 + 2f_{xy}uv + f_{yy}v^2 = 0, \\ f_{xxx}u^3 + 3f_{xxy}u^2v + 3f_{xyy}uv^2 + f_{yyy}v^3 = 0. \end{cases}$$

(Contact order 3 between the line and the surface)

Discriminant nonnegative, resultant vanishing.

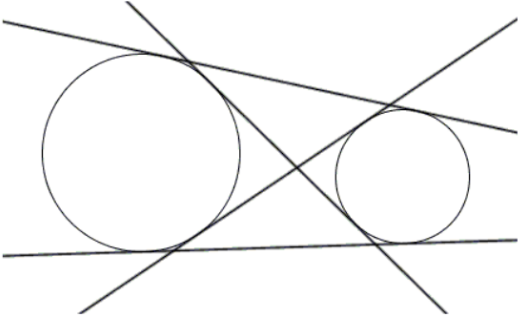
Theorem Sketch (made precise below, Bo–Bartoň–Pottmann–S'20)

Under technical conditions, if at each point a surface has contact order 4 *(in the space of planes) with a cone, then the surface is an envelope of a one-parametric family of cones.*

2 Main idea

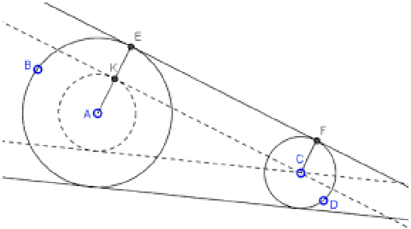
Antique geometry problem.

Construct a common tangent to 2 given circles using a compass and a straightedge.



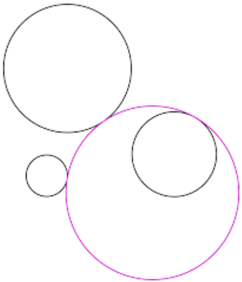
from cut-the-knot.org

Antique geometry problem.
Construct a common tangent to 2 given circles using a compass and a straightedge.
Solution: transform one circle to a point by an *offset*



from cut-the-knot.org

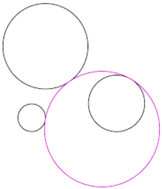
Apollonius problem. Construct a common tangent circle to 3 given circles using a compass and a straightedge.



from wikipedia.org

Apollonius problem. Construct a common tangent circle to 3 given circles using a compass and a straightedge.

Solution: transform one circle to a point by an *offset*, move the point to infinity by an *inversion*, apply the previous problem.

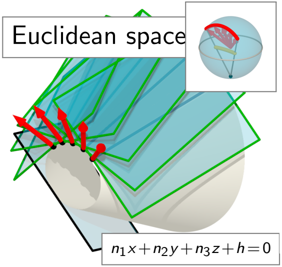


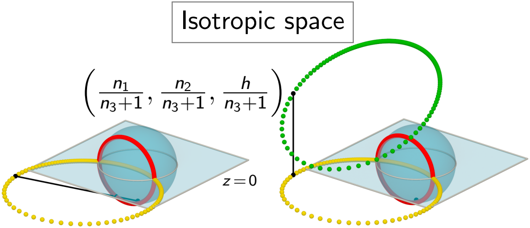
from wikipedia.org

Definition. A *Laguerre transformation* is a transformation of the set of oriented hyperplanes in $\mathbb{R}^n$ taking oriented tangent hyperplanes to an oriented sphere (possibly of radius 0) to oriented tangent hyperplanes to an oriented sphere (possibly of radius 0).

Examples. Offsets, similarities.

Isotropic model of Laguerre geometry

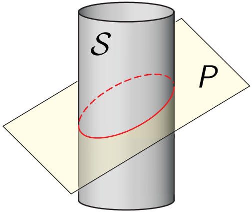


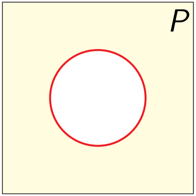


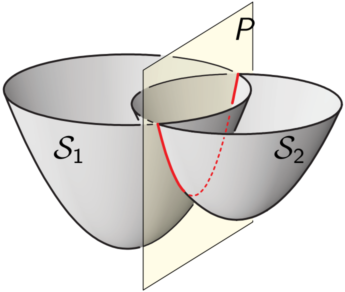
Example

Euclidean space	Isotropic space
oriented sphere of center (m_1, m_2, m_3), radius R, and inwards oriented normals	rotational paraboloid/plane $z = \frac{R+m_3}{2}(x^2+y^2) - m_1 x - m_2 y + \frac{R-m_3}{2}$.

$$\|(x, y, z)\|^2 = x^2 + y^2$$







i-circle of elliptic type
(top view is a circle)

i-circle of parabolic type

Isotropic model of Laguerre geometry

Euclidean space	Isotropic space
oriented plane	point
oriented sphere	i-sphere of parabolic type non-isotropic plane
cone (=oriented cone of revolution)	i-circle of elliptic type i-circle of parabolic type non-isotropic line

Proposition

For a cone C with the opening angle θ such that all the oriented unit normals are distinct from $(0, 0, -1)$ the set C^i is a conic satisfying the following condition:
(Θ) the top view of the conic is the stereographic projection of a circle of intrinsic radius $\pi/2 - \theta$ in the unit sphere (not passing through the projection center $(0, 0, -1)$).

Proposition

Let Φ be an oriented surface in $\mathbb{R}^3$ with nowhere vanishing Gaussian curvature and the oriented unit normals distinct from $(0, 0, -1)$. Then the following two conditions are equivalent:

- *through each point of Φ there passes an oriented cone which is tangent to Φ along a continuous curve containing the point (not a ruling because the Gaussian curvature of Φ does not vanish), has the opening angle θ, and has no oriented unit normals of the form $(0, 0, -1)$;*
- *through each point of Φ^i there passes an arc of a conic contained in Φ^i and satisfying condition (Θ).*

Assumption

Condition (*) Φ is an oriented surface in $\mathbb{R}^3$ with nowhere vanishing Gaussian curvature such that all the oriented unit normals are distinct from $(0, 0, -1)$, and Φ^i is the graph of a C^4 function $f : D \to \mathbb{R}$ in a disk $D \subset \mathbb{R}^2$.

Problem. Characterize functions in 2 variables whose graphs are covered by conics satisfying condition (Θ) and reconstruct the conics.

Surfaces containing two isotropic circles through each point

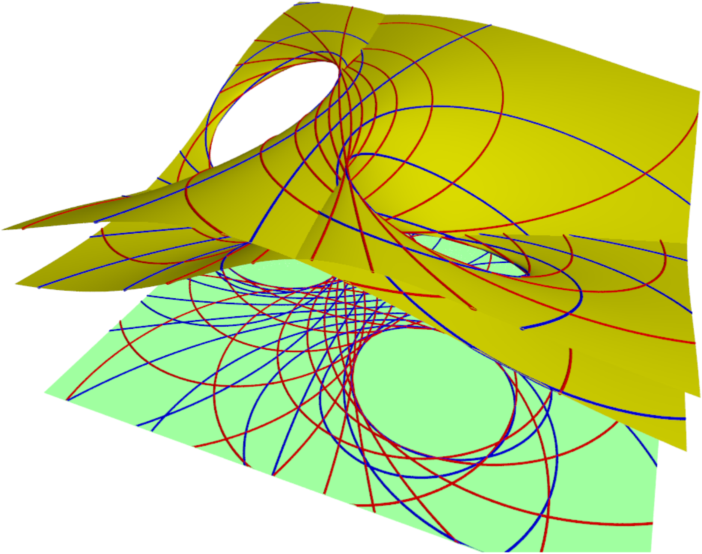
Theorem (Morozov'21)

Assume that through each point of an analytic surface in $\mathbb{R}^3$ one can draw two transversal arcs of isotropic circles fully contained in the surface. Assume (**). Then the surface has a parametrization

$$\Phi(u,v)=\left(\frac{P_0P_1-P_2P_3}{P_0^2+P_3^2},\frac{P_1P_3+P_0P_2}{P_0^2+P_3^2},\frac{Z}{P_0^2+P_3^2}\right)$$

for some $P_0,P_1,P_2,P_3,Z\in\mathbb{R}[u,v]$ such that P_0,P_1,P_2,P_3 have degree at most 1 in u and v, and Z has degree at most 2 in u and v.

An example and top view



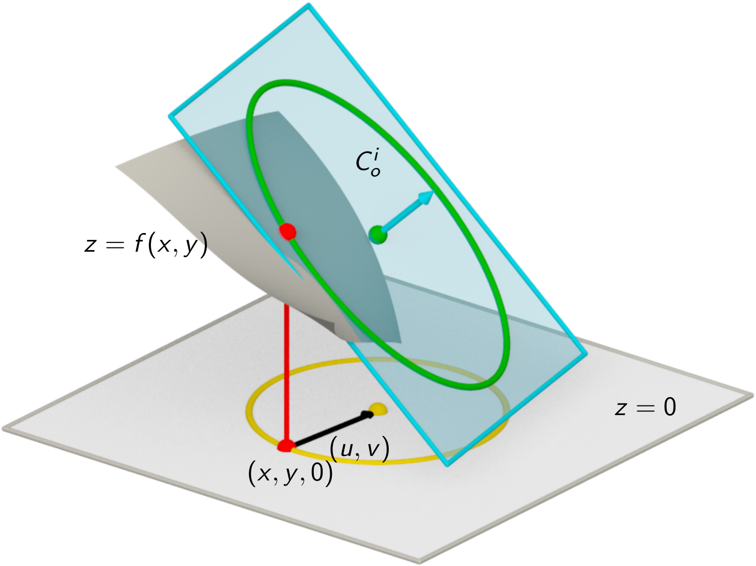
by Morozov

Technical assumptions (**)

- the two arcs analytically depend on the point;
- the two arcs lie neither in the same isotropic sphere nor in the same plane;
- through each point in some dense subset of the surface one can draw only finitely many (not nested) arcs of isotropic circles and line segments contained in the surface.

3
Statements

What if just one isotropic circle through each point?



(a)

Conic parametrization

Proposition

Each conic satisfying condition (Θ) *can be parametrized as*

$$\begin{cases} x(t) & = x + v\sin t + u(1-\cos t), \\ y(t) & = y - u\sin t + v(1-\cos t), \\ z(t) & = z + a\sin t + b(1-\cos t), \end{cases} \tag{1}$$

where $a, b, u, v, x, y, z \in \mathbb{R}$ *satisfy*

$$\left(x^2 + y^2 + 1 + 2xu + 2yv\right)^2 - 4\tan^2\theta\left(u^2 + v^2\right) = 0. \tag{2}$$

Definition. Let $(x(t), y(t), z(t))$, where t runs through an interval I, be a smooth curve such that $(\dot{x}(t), \dot{y}(t)) \neq 0$ for each $t \in I$. The curve *has contact order* n with the graph of a C^n function f at $t = 0$, if $\frac{z(t) - f(x(t), y(t))}{t^n} \to 0$ as $t \to 0$.

Proposition

If conic (1) has contact order 2 with the graph of f ("osculation"), if and only if

$$\begin{cases} z = f(x, y), \\ a = f_x v - f_y u, \\ b = f_x u + f_y v + f_{xx} v^2 - 2 f_{xy} uv + f_{yy} u^2. \end{cases}$$

Proposition

If conic (1) has contact order 3 ("hyperosculation"), then

$$f_{xxx}v^3 - 3f_{xxy}v^2u + 3f_{xyy}vu^2 - f_{yyy}u^3 + 3(f_{xx} - f_{yy})uv + 3f_{xy}(v^2 - u^2) = 0. \quad (3)$$

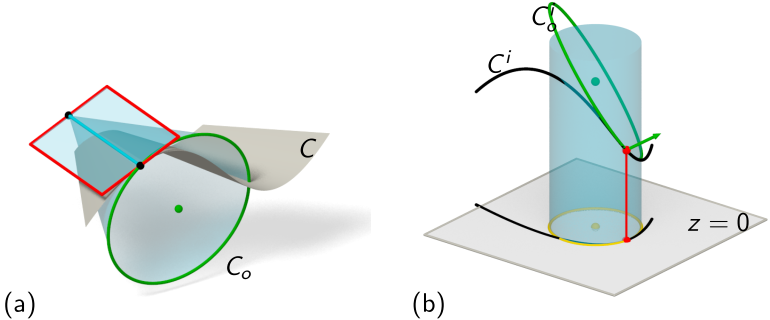
If the contact order is 4, then

$$f_{xxxx}v^4 - 4f_{xxxy}v^3u + 6f_{xxyy}v^2u^2 - 4f_{xyyy}vu^3 + f_{yyyy}u^4 + 6uv^2f_{xxx} + 6v(v^2 - 2u^2)f_{xxy} + 6u(u^2 - 2v^2)f_{xyy} + 6u^2vf_{yyy} + 3(u^2 - v^2)(f_{xx} - f_{yy}) + 12uvf_{xy} = 0. \quad (4)$$

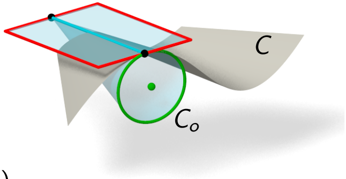
Corollary (Bo–Bartoň–Pottmann–S'20)

Let f be a C^4 function in a disk $D \subset \mathbb{R}^2$. If through each point of the surface $z = f(x, y)$ there passes an arc of a conic satisfying condition (Θ) and completely contained in the surface, then for each $(x, y) \in D$ three equations (2),(3),(4) have a common real solution (u, v).

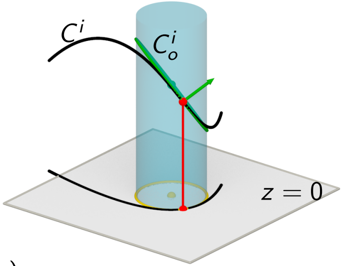
Osculating cone of a developable surface



Hyperosculation



(a)



(b)

One more assumption

Conic (1) is *multiple*, if (u, v) is a common real multiple root of (2) and (3), i.e.

$$f_{xxx}v^2\tilde{u} + f_{xxy}v(v\tilde{v} - 2u\tilde{u}) + f_{xyy}u(u\tilde{u} - 2v\tilde{v}) + f_{yyy}u^2\tilde{v} \\ + (f_{xx} - f_{yy})(u\tilde{u} - v\tilde{v}) + 2f_{xy}(u\tilde{v} + v\tilde{u}) = 0, \quad (5)$$

where

$$\tilde{u} = x(x^2 + y^2 + 1 + 2xu + 2yv) - 4u\tan^2\theta,$$
$$\tilde{v} = y(x^2 + y^2 + 1 + 2xu + 2yv) - 4v\tan^2\theta.$$

Theorem (Bo–Bartoň–Pottmann–S'20)

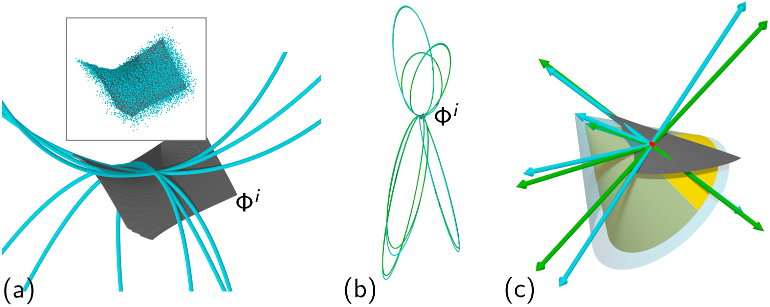
Let f be a C^4 function in a disk $D \subset \mathbb{R}^2$. Suppose that through each point (x, y, z) of the graph of f, there passes an arc of a nonmultiple conic $C_{x,y}$ having contact order 4 at (x, y, z) with the graph, continuously depending on (x, y), and such that the top view of $C_{x,y}$ is the stereographic projection of a circular arc of intrinsic radius $\frac{\pi}{2} - \theta$ (not passing through the projection center). Then an arc of a generic conic $C_{x,y}$ is contained in the graph.

Main Theorem

Theorem (Bo–Bartoň–Pottmann–S'20)

Assume (*).
If through each point of Φ *there passes a cone which is tangent to* Φ *along a curve (containing the point), has the opening angle* θ*, and has no tangent planes orthogonal to* $(0,0,-1)$*, then for each* $(x,y) \in D$ *three equations* (2), (3),(4) *have a common nonzero real solution* (u,v)*.*
Conversely, if for each $(x,y) \in D$ *three equations* (2), (3), (4) *have a common real solution* (u,v) *continuously depending on* (x,y) *and nowhere satisfying* (5)*, then through each point of* Φ *there passes a cone which is tangent to* Φ *along a continuous curve (containing the point) and has the opening angle* θ*.*

Implementation: $f(x, y) = y^2/(x^2 + y^2)$



References

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- M. Skopenkov, P. Bo, M. Bartoň, H. Pottmann, Characterizing envelopes of moving rotational cones and applications in CNC machining, Computer Aided Geom. Design 83 (2020), 101944. arXiv:2001.01444.

THANKS!