An overview of rationalizations of groups and spaces

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Spaces	Groups
Rationalization of nilpotent spaces	Malcev completion
Bousfield-Kan $\mathbb{Q}$-completion	$\mathbb{Q}$-completion
Homology rationalization	$H\mathbb{Q}$-localization
Ω-rationalization	Baumslag rationalization
π_1-fiberwise rationalization	—

1. Rationalization of nilpotent groups.
2. Rationalization**s** of all groups.
3. Rationalization of nilpotent spaces.
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Rationalization of nilpotent groups:
Malcev completion.

Rationalization of abelian groups

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- Rationalization for abelian groups:

$$A \longrightarrow A \otimes \mathbb{Q}$$

$$a \mapsto a \otimes 1$$

Malcev completion

- Lower central series of G is

$$\gamma_{n+1}(G) = [\gamma_n(G), G], \qquad G = \gamma_1(G) \supseteq \gamma_2(G) \supseteq \dots$$

G is **nilpotent** if $\gamma_n(G) = 1$ for some n.

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- For nilpotent groups $- \otimes \mathbb{Q}$ is a very nice functor: it takes exact sequences to exact sequences; and central extensions to central extensions; commutes with filtered colimits...

Rationalizations of all groups:

Baumslag rationalization,

$\mathbb{Q}$-completion,

$H\mathbb{Q}$-localization

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- For any central extension
$$1 \to A \to E \to G \to 1$$
the sequence
$$A \otimes \mathbb{Q} \to \mathsf{Bau}(E) \to \mathsf{Bau}(G) \to 1$$
is exact.

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- **Theorem**('21) If F_2 is a free group of rank 2, the cokernel

$$\mathsf{Coker}(F_2 \to \mathsf{Bau}(F_2))$$

contains a free non-abelian group.

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- Open question: $H_3(\widehat{F}_{\mathbb{Q}}, \mathbb{Q}) \stackrel{?}{=} 0$.

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- **Theorem**(-,Mikhailov'17) For a free group F of rank ≥ 2 the groups $\widehat{F}_{\mathbb{Q}}$ and $\ell_{H\mathbb{Q}}(F)$ are not isomorphic.
- If N is nilpotent,

$$N \otimes \mathbb{Q} \cong \mathsf{Bau}(N) \cong \ell_{H\mathbb{Q}}(N) \cong \widehat{N}_{\mathbb{Q}}$$

Rationalization of nilpotent spaces

Nilpotent spaces

- A space X is **nilpotent** if
  1. $\pi_1(X)$ is nilpotent;
  2. $\pi_n(X)$ is nilpotent as a $\mathbb{Z}[\pi_1(X)]$-module.

- X is nilpotent iff $\pi_1(X) \ltimes \pi_n(X)$ is nilpotent for any n.

Rationalization for nilpotent spaces

X, Y are nilpotent.

$$f : X \to Y \text{ is } \textbf{rational homotopy equivalence}$$

$$\Updownarrow$$

$$\pi_*(X) \otimes \mathbb{Q} \cong \pi_*(Y) \otimes \mathbb{Q}.$$

$$\Updownarrow$$

$$H_*(X, \mathbb{Q}) \cong H_*(Y, \mathbb{Q})$$

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- $X_{\mathbb{Q}}$ is uniquely defined (up to homotopy) and

$$\pi_*(X_{\mathbb{Q}}) \cong \pi_*(X) \otimes \mathbb{Q}, \qquad H_*(X_{\mathbb{Q}},\mathbb{Z}) \cong H_*(X,\mathbb{Q})$$

Rationalizations of all spaces:

Bousfield-Kan $\mathbb{Q}$-completion (1972)
Bousfield homology rationalization (1975)
Casacuberta-Peschke Ω-rationalization (1993)
Gómez-Tato–Halperin–Tanré's π_1-fiberwise rationalization (2001)

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- Ho(connected spaces) $\simeq$ Ho(simplicial groups)

- Bousfield-Kan $\mathbb{Q}$-completion is the "derived functor" of $\mathbb{Q}$-completion of groups

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\text{(connected spaces)} \arrow[r, "\mathbb{Q}_\infty"] \arrow[d, "\simeq"] & \text{(connected spaces)} \arrow[d, "\simeq"] \\
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- $f : X \to Y$ is $\mathbb{Q}$-homology equivalence $\Longleftrightarrow$ $\mathbb{Q}_\infty X \simeq \mathbb{Q}_\infty Y$ is homotopy equivalence.
- $\mathbb{Q}_\infty$ is not idempotent:

$$\mathbb{Q}_\infty(\mathbb{Q}_\infty(S^1 \vee S^1)) \not\simeq \mathbb{Q}_\infty(S^1 \vee S^1).$$

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Ω-rationalization of spaces

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$$X \text{ is } \Omega\textbf{-rational}$$
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$$(-)^n : \Omega X \to \Omega X \text{ is homotopy equivalence for } n \geq 1$$
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- $\pi_1(L_{\Omega\mathbb{Q}}(X)) = \mathsf{Bau}(\pi_1(X))$.

Ω-rationalization and simplicial groups

Theorem (Bastardas-Casacuberta'2001). The diagram of homotopy categories

$$\begin{array}{ccc}
\text{(connected spaces)} & \xrightarrow{L_{\Omega\mathbb{Q}}} & \text{(connected spaces)} \\
\downarrow \simeq & & \downarrow \simeq \\
\text{(simplicial groups)} & \xrightarrow{\mathsf{LBau}} & \text{(simplicial groups)}
\end{array}$$

is commutative.

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- $\pi_n(L_{\mathbb{Q}}^{\pi_1}(X)) = \pi_n(X) \otimes \mathbb{Q}$ for $n \geq 2$.
- easy to compute, there is a version of minimal Sullivan models but for nilpotent spaces $L_{\mathbb{Q}}^{\pi_1}$ is “wrong”.

Comparison

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There are natural transformations:

$$L_{\mathbb{Q}}^{\pi_1} \longrightarrow L_{\Omega\mathbb{Q}} \longrightarrow L_{H\mathbb{Q}} \longrightarrow \mathbb{Q}_\infty$$

$\pi_1(L_{\mathbb{Q}}^{\pi_1}(X)) = \pi_1(X)$
$\pi_1(L_{\Omega\mathbb{Q}}(X)) = \mathsf{Bau}(\pi_1(X))$
$\pi_1(L_{H\mathbb{Q}}(X)) = \ell_{H\mathbb{Q}}(\pi_1(X))$
If $\pi_1(X)$ is finitely presented, $\pi_1(\mathbb{Q}_\infty X) \cong \widehat{\pi_1(X)}_{\mathbb{Q}}$

$$L_{\mathbb{Q}}^{\pi_1}(S^1 \vee S^1) = S^1 \vee S^1$$
$$L_{\Omega\mathbb{Q}}(S^1 \vee S^1) = K(\mathsf{Bau}(F), 1)$$
$$L_{H\mathbb{Q}}(S^1 \vee S^1) =?$$
$$\mathbb{Q}_\infty(S^1 \vee S^1) = K(\widehat{F}_{\mathbb{Q}}, 1)$$

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- $\{L_{\Omega\mathbb{Q}}$-equiv.$\}$=?

Ω-rational equivalences

- For any group G we set

$$\mathbf{R}(G) = \mathbb{Z}[\mathsf{Bau}(G)][\Sigma^{-1}],$$

where

$$\Sigma = \{1 + g + g^2 + \cdots + g^{n-1} \mid g \in \mathsf{Bau}(G), n \geq 1\}.$$

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- **Theorem.** (Casacuberta-Peschke'93)
  A map $f : X \to Y$ is Ω-rational equivalence iff
  (1) $\mathsf{Bau}(\pi_1(X)) \cong \mathsf{Bau}(\pi_1(Y))$
  (2) $H^*(Y, \mathcal{A}) \cong H^*(X, \mathcal{A})$ for any local system of coefficients $\mathcal{A}$ over $\mathbf{R}[\pi_1(Y)]$.
  (or
  (2') $H_*(X, \mathbf{R}[\pi_1(X)]) \cong H_*(Y, \mathbf{R}[\pi_1(Y)])$)

Integral homology of Ω-rationalization

$$H_*(L_{\Omega\mathbb{Q}}(X), \mathbb{Z}) \cong H_*(X, \mathbb{Q}) \oplus T_*(X),$$

where $T_*(X)$ is a torsion group.

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