Grothendieck's dessins d'enfants in a web of dualities

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Overview

Witten Conjecture/Kontsevich Theorem and mirror symmetry are two driving forces in the study of Gromov-Witten type theories.

The former is concerned with integrable hierarchy (KdV hierarchy) or infinite-dimensional symmetry (Virasoro constraints).

The latter is concerned with the duality of two different theories.

Overview

So one has the following paradigm:

First, one constructs a “theory”.

Secondly, one shows the partition function of this theory is the tau-function of some integrable hierarch, or it satisfies some constraints forming an infinite-dimensional Lie algebra.

Thirdly, one shows the theory is equivalent to some other theory.

Overview

There have now been many famous examples in the literature about the connections between QFT and integrable hierarchies, and there are many examples of duality of different theories.

Inspired by such connections I propose to study the dualities of quantum field theories as applications of integrable hierarchies.

Overview

Arguably the simplest kind of integrable hierarchies is the KP hierarchy, and the simplest kind of infinite-dimensional constraints is the Virasoro constraints.

So I will first focus on them.

Formal QFT as physicist's algebraic topology

By a formal QFT we mean the following data:

1. A Hilbert space $\mathcal{H}$ with a basis $\{\mathcal{O}_m\}_{m\geq 0}$ (called the space of observables).

2. For each $n \geq 1$, a symmetric linear map $\langle - \rangle : \mathcal{H}^{\otimes n} \to \mathbb{C}$

$$\mathcal{O}_{m_1} \otimes \cdots \otimes \mathcal{O}_{m_n} \mapsto \langle \mathcal{O}_{m_1}, \cdots, \mathcal{O}_{m_n} \rangle$$

(called the n-point correlators).

QFT as physicist's algebraic topology

One constructs the free energy of the QFT by:

$$F := \sum_{n=1}^{\infty} \frac{t_{m_1} \cdots t_{m_n}}{n!} \langle \mathcal{O}_{m_1}, \cdots, \mathcal{O}_{m_n} \rangle$$

The partition function of the QFT is defined by:

$$Z = e^F.$$

For many QFTs that interest mathematicians, their partition functions are tau-functions of some suitable integrable hierarchies.

Formal QFT with Λ as space of observables

Let Λ be the space of symmetric functions.

A formal QFT with Λ as space of observables is given by a linear map $\langle - \rangle : \Lambda \to \mathbb{C}$.

Introduce formal variables $g_1, g_2, \dots$ (the coupling constants). For a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l)$, and let $g_\lambda = g_{\lambda_1} \cdots g_{\lambda_l}$. As usual, when λ is the empty partition $\emptyset$, $p_\emptyset = 1$ and $g_\emptyset = 1$. Define the partition function by

$$Z := \sum_{\lambda \in \mathcal{P}} \frac{1}{z_\lambda} \langle p_\lambda \rangle g_\lambda = \langle \exp \sum_{n=1}^{\infty} \frac{1}{n} g_n p_n \rangle,$$

where the summation is taken over the set $\mathcal{P}$ of all partitions.

Formal QFT with Λ as space of observables

The partition function is then of the following form:

$$Z := \sum_{\lambda \in \mathcal{P}} c_\lambda \cdot s_\lambda,$$

where $c_\lambda = \langle s_\lambda \rangle$, $p_n = g_n$. This is called the Schur expansion of the the partition function.

We are interested in formal QFT of this form such that Z is a tau-function of the KP hierarchy with $T_n = \frac{g_n}{n}$.

There are many examples: Hermitian one-matrix models, Witten-Kontsevich tau-function, Witten r-spin tau-function, Hurwitz numbers, Marino-Vafa formula, ...

Schur expansion of the tau-function of the KP hierarchy

Sato's school: A solution of the KP hierarchy is encoded in a tau-function.

The space of solutions of the KP is a homogeneous space of $\widehat{GL(\infty)}$.

By Boson-Fermion correspondence, this space corresponds to the orbit of the vacuum by the action of $\widehat{GL(\infty)}$ on the fermionic Fock space.

Affine coordinates for elements in the Sata Grassmannian

Let $z^{1/2}\mathbb{C}[[z,z^{-1}]]$ be the space of formal series in z of half integral powers.

An element V in the big cell Gr_0 of Sato's Grassmannian is specified by a sequence of series

$$\Psi_n(z) = z^{n+1/2} + \sum_{m=0}^{\infty} A_{m,n} z^{-m-1/2}, \;\; n \geq 0,$$

where the coefficients $A_{m,n}$ are called the *affine coordinates* of V.

This sequence is called the *normalized basis* of V.

A formula for n-point function of KP tau-function

Given an element V with normalized basis $\{\Psi_n\}_{n\geq 0}$, the tau-function Z_V can be obtained as follows. One first gets the Plücker coordinates of V as follows:

$$Z_V := \Psi_0(z) \wedge \Psi_1(z) \wedge \cdots = \sum_{\mu} \det(A_{m_i,n_j})_{1\leq i,j\leq k} \cdot |\mu\rangle,$$

where the summation is taken over all partitions μ, expressed as

$$(m_1, \cdots, m_k | n_1, \cdots, n_k)$$

in Frobenius notation.

Here

$$|\mu\rangle = (-1)^{n_1+n_2+\cdots+n_k} \prod_{i=1}^{k} \psi_{-m_i-\frac{1}{2}} \psi^*_{-n_i-\frac{1}{2}} |0\rangle_F$$

in the fermionic Fock space, $|0\rangle_F$ is the fermionic vacuum:

$$|0\rangle_F = z^{\frac{1}{2}} \wedge z^{\frac{3}{2}} \wedge \cdots ,$$

ψ_r is the operator $z^r\wedge$, and ψ^*_r is the adjoint operator of ψ_{-r}.

Summary: The Schur expansion of the tau-function of the KP hierarchy is determined by the affine coordinates of the normalized basis of the element in the Sato grassmannian.

A formula for n-point function of KP tau-function

Theorem. (Z.) The n-point function associated with a tau-function of the KP hierarchy is given by the following formula:

$$G^{(n)}(\xi_1, \dots, \xi_n) = (-1)^{n-1} \sum_{n\text{-cycles}} \prod_{i=1}^{n} \widehat{A}(\xi_{\sigma(i)}, \xi_{\sigma(i+1)}) - \frac{\delta_{n,2}}{(\xi_1 - \xi_2)^2},$$

where

$$\widehat{A}(\xi_i, \xi_j) = \begin{cases} i_{\xi_i,\xi_j} \frac{1}{\xi_i - \xi_j} + A(\xi_i, \xi_j), & i < j, \\ A(\xi_i, \xi_i), & i = j, \\ i_{\xi_j,\xi_i} \frac{1}{\xi_i - \xi_j} + A(\xi_i, \xi_j), & i > j, \end{cases}$$

and

$$A(\xi, \eta) = \sum_{m,n \geq 0} A_{m,n} \xi^{-n-1} \eta^{-m-1}.$$

Genus zero one-point function and two-point function

Genus zero one-point function and two-point function are particularly interesting because of their application to Eynard-Orantin topological recursion.

Genus zero one-point function $\longrightarrow$ spectral curve

Genus zero two-point function $\longrightarrow$ Bergmann kernel

These are the main subjects of what I call the emergent geometry of KP hierarchy.

Grothendieck's Dessins d'enfants (Children's Drawings)

Belyi's Theorem. A smooth complex algebraic curve C is defined over $\bar{\mathbb{Q}}$ if and only if there exists a holomorphic branched cover $f : C \to \mathbb{P}^1$ that is ramified only over $0, 1, \infty$.

Grothendieck's Correspondence. There is a one-to-one correspondence between the isomorphism classes of Belyi pairs and connected bicolored ribbon graphs (called Grothendieck's dessin denfant), given by $f^{-1}([0,1])$.

Set $k = |f^{-1}(0)|$, $l = |f^{-1}(1)|$ and $m = |f^{-1}(\infty)|$, $g = g(C)$, $d = \deg(f)$. Then by Riemann-Hurwitz's formula:

$$2g - 2 = d - (k + l + m).$$

Counting Grothendieck's Dessins d'enfants

Assume that the poles of f are labeled. Denote their orders by $\mu = (\mu_1, \ldots, \mu_m)$.

Denote the set of all dessins of type (k, l, μ) by $\mathcal{D}_{k,l;\mu}$.

Define the dessin free energy by counting the dessins:

$$F_{Des}(s, u, v, p_1, p_2, \ldots) = \sum_{k,l,m \geq 1} \frac{1}{m!} \sum_{\mu} \sum_{\Gamma \in \mathcal{D}_{k,l;\mu}} \frac{1}{|\mathrm{Aut}(\Gamma)|} \cdot u^k v^l p_{\mu_1} \cdots p_{\mu_m}.$$

The dessin partition function is defined by:

$$Z_{Des} = e^{F_{Des}}.$$

Counting Grothendieck's Dessins d'enfants and KP hierarchy

Kazarian-Zograf: The dessin partition function satisfies the Virasoro constraints for $n \geq 0$:

$$L_n Z_{Dessins} = 0,$$

$$\begin{aligned} L_n &= -\frac{n+1}{s}\frac{\partial}{\partial p_{n+1}} + (u+v)n\frac{\partial}{\partial p_n} + \sum_{j=1}^{\infty} p_j(n+j)\frac{\partial}{\partial p_{n+j}} \\ &+ \sum_{i+j=n} ij\frac{\partial^2}{\partial p_i \partial p_j} + \delta_{n,0}uv, \end{aligned}$$

$$[L_m, L_n] = (m-n)L_{m+n}.$$

Cut-and-join representation of dessin partition function

Kazarian-Zograf: The dessin partition function can be obtained from the bosonic vacuum in the following way:

$$Z_{Des}(u,v,s) = e^{s((u+v)\Lambda_1 + M_1 + uvp_1)}1,$$

where Λ_1 and M_1 are differential operators defined as follows:

$$\Lambda_1 = \sum_{i=2}^{\infty}(i-1)p_i \frac{\partial}{\partial p_{i-1}},$$

$$M_1 = \sum_{i=2}\sum_{j=1}^{i-1}\left((i-1)p_j p_{i-j}\frac{\partial}{\partial p_{i-1}} + j(i-j)p_{i+1}\frac{\partial^2}{\partial p_j \partial p_{i-j}}\right).$$

As a corollary, Z_{Des} is a 3-parameter family of tau-functions of the KP hierarchy.

Explicit formula of dessin partition function

Theorem (Z.) The dessin partition function is explicitly given by:

$$Z_{Des} = \sum_{\mu \in \mathcal{P}} s_\mu \cdot \prod_{e \in \mu} \frac{s \cdot (u + c(e))(v + c(e))}{h(e)},$$

where e denotes the box in the partition a Young diagram μ, $c(e)$ and $h(e)$ denotes its content and hook length respectively. It follows from:

Theorem (Z.) In the fermionic picture,

$$Z_{Des} = \exp\Big(\sum_{m,n \geq 0} A_{m,n} \psi_{-m-\frac{1}{2}} \psi^*_{-n-\frac{1}{2}} \Big) |0\rangle,$$

as follows:

$$A_{m,n} = \frac{(-1)^n s^{m+n+1} uv}{(m+n+1)m!n!} \prod_{j=1}^{m} (u+j)(v+j) \cdot \prod_{i=1}^{n} (u-i)(v-i).$$

Quantum spectral curve of dessin partition function

Quantum spectral curve of dessin partition function is given by studying:

$$\psi_{0,des} = \psi(s,t,u,v) = e^{F_{Dessins}(s,u,v,p_1,p_2,\ldots)}|_{p_i=t^i}.$$

Kazarian-Zogrof:

$$t^2\frac{d^2\psi}{dt^2} + \left((u+v+1)t - \frac{1}{s}\right)\frac{d\psi_{0,des}}{dt} + uv\psi_{0,des} = 0.$$

Z.: $\psi_{0,des} = 1 + \sum_{n=1}^{\infty} \frac{s^n t^n}{n!} \prod_{j=0}^{n-1}(u+j)(v+j)$ and it satisfies the Picard-Fuchs equation:

$$\frac{\partial}{\partial t}\psi_{0,des} = s(t\frac{\partial}{\partial t} + u)(t\frac{\partial}{\partial t} + v)\psi_{0,des}.$$

Kac-Schwarz operator for dessin partition function

Theorem (Z.) The operator

$$\mathcal{D} \;=\; z^2\frac{d}{dz} + z - (u+v-2)sz\frac{d}{dz} + sz^2\frac{d^2}{dz^2}$$

is a Kac-Schwarz operator for Z_{des}:

$$\mathcal{D}\phi_{k,Des}(z) = (k+1)\phi_{k+1,Des}(z) - ks(u+v-k-1)\phi_{k,Des}(z).$$

I.e., one can use $\{\mathcal{D}^k\phi_{0,des}\}_{k\geq 0}$ as a basis for the element corresponding to Z_{des} in Sato grassmannian.

Counting Grothendieck's dessins as a universal object

With the above explicit results we can specialize Z_{des} to partition functions of other models, hence putting Grothendieck's dessins in a web of dualities.

These models include: The partition function of Hermitian one-matrix models=partition function of clean dessins = partition function of fat-graphs (ribbon graphs), modified partition function of Hermitian one-matrix model with even couplings, generalized BGW models, the partition function of the Laguerre unitary ensemble (with Di Yang).

Furthermore, in joint work with Di Yang we study the dessin partition function from the point of view of some other integrable hierarchies.

Modified partition function of Hermitian one-matrix model with even couplings

Thoerem (Dubrovin, Liu, Yang, Zhang) A suitably modified partition function $\tilde{Z}_{even}$ of Hermitian one-matrix model with even couplings satisfies the Virasoro constraints:

$$\frac{1}{2}\frac{\partial}{\partial s_2}\tilde{Z}_{even} = \sum_{k\geq 1} ks_{2k}\frac{\partial}{\partial s_{2k}}\tilde{Z}_{even} + \left(\frac{t^2}{4\epsilon^2} - \frac{1}{16}\right)\tilde{Z}_{even},$$

$$\frac{1}{2}\frac{\partial}{\partial s_{2n+2}}\tilde{Z}_{even} = \sum_{k=1}^{n-1}\frac{\partial^2}{\partial s_{2k}\partial s_{2n-2k}}\tilde{Z}_{even} + t\frac{\partial}{\partial s_{2n}}\tilde{Z}_{even}$$
$$+\sum_{k\geq 1} ks_{2k}\frac{\partial}{\partial s_{2k+2n}}\tilde{Z}_{even},$$

$n \geq 1$, where $t = N\epsilon$, and $\Lambda = e^{\epsilon\partial_t}$.

Cut-and-join representation of modified partition function of Hermitian one-matrix model with even couplings

Theorem (Z.) Let $s_{2k} = \frac{p_k}{k}$, $\qquad k \geq 1$ and

$$
\begin{aligned}
W &= 2p_1\left(\frac{t^2\epsilon^{-2}}{4} - \frac{1}{16}\right) + 2\epsilon^2 \sum_{n=2}^{\infty} p_{n+1} \sum_{k=1}^{n-1} k(n-k) \frac{\partial^2}{\partial p_k \partial p_{n-k}} \\
&+ 2t \sum_{n=1}^{\infty} n p_{n+1} \frac{\partial}{\partial p_n} + 2 \sum_{n=0}^{\infty} p_{n+1} \sum_{k\geq 1} (k+n) p_k \frac{\partial}{\partial p_{k+n}},
\end{aligned}
$$

then

$$\tilde{Z}_{even} = e^W 1.$$

Modified partition function as specialization of dessin partition function

Theorem (Z.) $\tilde{Z}_{even}|_{\epsilon=1} = Z_{Des}|_{u=\frac{t}{2}+\frac{1}{4}, v=\frac{t}{2}-\frac{1}{4}, s=2}$.

Theorem (Z.) $\tilde{Z}_{even}$ is a one-parameter family of KP tau-functions. In the fermionic picture,

$$\tilde{Z}_{even}|_{\epsilon=1} = e^{\tilde{A}_{even}}|0\rangle,$$

$$\tilde{A}_{even} = \sum_{m,n\geq 0} A_{m,n}\psi_{-m-1/2}\psi^*_{-n-1/2},$$

$$\begin{aligned} A_{m,n} &= \frac{(-1)^n}{2^{3(m+n+1)}(m+n+1)\cdot m!n!} \\ &\quad \cdot \prod_{i=0}^{2m}(2t+2i+1)\cdot \prod_{j=0}^{2n}(2t-2j-1). \end{aligned}$$

Generalized BGW model

Originally the BGW model is described by a unitary matrix model

$$Z_{BGW} = \int [dU] e^{\frac{1}{\hbar} \mathrm{tr}(A^\dagger U + A U^\dagger)},$$

but it can be also described by a generalized Kontsevich model:

$$Z_{BGW,N}(M) \frac{1}{V_N} \int_{H_{N\times N}} dX e^{\mathrm{tr}(MX - N \log X + \frac{1}{X})}$$

Virasoro constraints for generalized BGW model

The generalized BGW tau-function satisfies the Virasoro constraints (Alexandrov):

$$\hbar\widehat{\mathcal{L}}_m^{(N)}\tau_N(t,\hbar) = \frac{\partial}{\partial t_{2m+1}}\tau_N(t,\hbar), m \geq 0,$$

where

$$\begin{aligned}\widehat{\mathcal{L}}_m^{(N)} &= \frac{1}{2}\sum_{k=0}^{\infty}(2k+1)t_{2k+1}\frac{\partial}{\partial t_{2k+2m+1}} \\ &+ \frac{1}{4}\sum_{a+b=m-1}\frac{\partial^2}{\partial t_{2a+1}\partial t_{2b+1}} + (\frac{1}{16} - \frac{N^2}{4})\delta_{m,0}.\end{aligned}$$

Cut-and-join representation for generalized BGW model

It follows that $Z_{BGW}^{(N)}(t)$ has the following cut-and-join representation:

$$Z_{BGW}^{(N)}(t) = e^{\hbar W_{BGW}^{(N)}}(1)$$

where the operator $W_{BGW}^{(N)}$ is defined by:

$$\begin{aligned} W_{BGW}^{(N)} &= \frac{1}{2}\sum_{k,m=0}^{\infty}(2k+1)(2m+1)t_{2k+1}t_{2m+1}\frac{\partial}{\partial t_{2k+2m+1}} \\ &+ \frac{1}{4}\sum_{k,m=0}^{\infty}(2k+2m+3)t_{2k+2m+3}\frac{\partial^2}{\partial t_{2k+1}\partial t_{2m+1}} \\ &+ \left(\frac{1}{16}-\frac{N^2}{4}\right)t_1. \end{aligned}$$

Relating generalized BGW to counting dessins of odd valence

So comparing $\hbar W_{BGW}^{(N)}$ with $W_{Dessins}$, we get:

The cut-and-join operator for the generalized BGW tau-function is a special case of the cut-and-join operator for the dessin tau-function:

$$\hbar W_{BGW}^{(N)} = W_{Dessins}\Big|_{p_{2n}=0, p_{2n-1}=\sqrt{2}(2n-1)t_{2n-1}, s=\frac{\hbar}{2\sqrt{2}}, u=-v=\sqrt{\frac{N^2}{2}-\frac{1}{8}}}.$$

Affine coordinates for the τ-function of generalized BGW model

Theorem. (Z.) We have the following explicit expression for the normalized basis hence the affine coordinates for the generalized BGW tau-function:

$$\Psi^{(N)}_{m-1} = z^{m-1} + \sum_{n=1}^{\infty} z^{-n} \sum_{k=0}^{m-1} \frac{(-1)^{m+n-1-k}\hbar^{m+n-1}}{4^{m+n-1}k!(m+n-1-k)!} \cdot [N-m-1/2]_1^{2k} [N-n-1/2]^0_{2k-2m+3}.$$

Here we have used the following notation: For integers $m < n$,

$$[x]^n_m = \prod_{j=m}^{n} (x+j).$$

Quantum spectral curve for generalized BGW theory

This has been discussed in Alexandrov. It is related to the modified Bessel equation by taking $x = z^2$.

Here we write down the equation of hypergeometric type satisfied by the principal specialization. We have

$$\Phi_1^{(N)}(z) = 1 + \sum_{k=1}^{\infty} \frac{\hbar^k}{z^k} \frac{\prod_{l=1}^{k}((2l-1)^2 - 4N^2)}{16^k k!}.$$

Quantum spectral curve for generalized BGW theory

Hence the principal specialization of the generalized BGW theory is:

$$\psi^{(N)}(t) = 1 + \sum_{k=1}^{\infty} \hbar^k t^k \frac{\prod_{l=1}^{k}((2l-1)^2 - 4N^2)}{16^k k!}.$$

It satisfies the following equation of hypergeometric type:

$$\frac{d}{dt}\psi = \frac{\hbar}{16}(2t\frac{d}{dt} + 1 + 2N)(2t\frac{d}{dt} + 1 - 2N)\psi.$$

Genus zero one-point function of dessin partition function

I obtain the following explicit formula for one-point function:

$$\begin{aligned} G_{0,1}(x) &= \frac{1}{2s}\left(1-\frac{s(u+v)}{x}-\sqrt{\left(1-\frac{s(u+v)}{x}\right)^2-\frac{4s^2uv}{x^2}}\right) \\ &= \frac{1}{2s}\left(1-\frac{s(u+v)}{x}-\sqrt{1-\frac{2s(u+v)}{x}+\frac{s^2(u-v)^2}{x^2}}\right) \\ &= \sum_{n=1}^{\infty}\frac{s^n uv}{x^{n+1}}\sum_{k=1}^{n}\frac{\binom{n}{k}}{n}\frac{\binom{n}{k-1}}{n}u^{n-k}v^{k-1}. \end{aligned}$$

The coefficients are Narayana numbers

$$N_{n,k} = \frac{1}{k}\binom{n-1}{k-1}\binom{n}{k-1} = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}.$$

It is well-known that the Narayana numbers provide a version of q-analogues of the Catalan numbers. By taking $q = 1$ in the Narayana polynomials, one gets the Catalan numbers:

$$\sum_{k=1}^{n} \frac{1}{n}\binom{n}{k}\binom{n}{k-1} = \frac{1}{n+1}\binom{2n}{n}.$$

Naryana numbers have numerous combinatorial meanings: The number of noncrossing set partitions of $\{1, \ldots, n\}$ into k blocks, the h-vectors of the associahedra as well as its dual. They appear in the study of cluster algebras of type A.

Genus zero two-point function of dessin partition function

I also obtain the following explicit formula for two-point function:

$$
\begin{aligned}
& G_{0,2}(x_1, x_2) \\
= \quad & \frac{1 - \frac{s(u+v)}{x_1} - \frac{s(u+v)}{x_2} + \frac{s^2(u-v)^2}{x_1 x_2}}{2(x_1 - x_2)^2 \sqrt{(1 - \frac{2s(u+v)}{x_1} + \frac{s^2(u-v)^2}{x_1^2})(1 - \frac{2s(u+v)}{x_2} + \frac{s^2(u-v)^2}{x_2^2})}} \\
- \quad & \frac{1}{2(x_1 - x_2)^2}.
\end{aligned}
$$

Emergence of spectral curve and its special deformation

Consider the following series:

$$y = \frac{1}{2}\sum_{n=1}^{\infty}(p_n - \frac{\delta_{n,1}}{s})x^{n-1} + \frac{u+v}{2x} + \sum_{n=1}^{\infty}\frac{n}{x^{n+1}}\frac{\partial F_0(t)}{\partial p_n}.$$

One has by Viraroso constraints:

$$\begin{aligned} y^2 \;&=\; \frac{1}{4}\Big(\sum_{n=1}^{\infty}(p_n - \frac{\delta_{n,1}}{s})x^{n-1}\Big)^2 + \frac{(u-v)^2}{4x^2} + \frac{u+v}{2}\sum_{n=1}^{\infty}(p_n - \frac{\delta_{n,1}}{s})x^{n-2} \\ &+\; \sum_{k\geq 1}\sum_{j\geq k+1}(p_j - \frac{\delta_{j,1}}{s})kx^{j-k-2}\frac{\partial F_0(t)}{\partial p_k}. \end{aligned}$$

When $p_n = 0$ for all p_n:

$$y^2 = \frac{1}{4s^2} - \frac{u+v}{2sx} + \frac{(u-v)^2}{4x^2}.$$

Eynard-Orantin topological recursion of dessin partition function as emergent geometry

We refer to the above two plane algebraic curve as the dessin spectral curve and its special deformation respectively. One can also consider the multilinear differential forms on the spectral curve:

$$W_{g,n}(p_1,\ldots,p_n)=\widehat{G}_{g,n}(y_1,\ldots,y_n)dx_1\cdots dx_n,$$

where $\widehat{G}_{g,n}(y_1,\ldots,y_n)=G_{g,n}(y_1,\ldots,y_n)$ except for the following two exceptional cases:

$$\widehat{G}_{0,1}(y_1)=-\frac{1}{2s}+\frac{(u+v)}{2x_1}+G_{0,1}(y_1),$$
$$\widehat{G}_{0,2}(y_1,y_2)=\frac{1}{(x_1-x_2)^2}+G_{0,2}(y_1,y_2).$$

Eynard-Orantin topological recursion of dessin partition function as emergent geometry

Theorem. (Kazarian-Zograf) The dessin partition function satisfies a version of topological recursions from which one can derive the Eynard-Orantin topological recursion.

Theorem. (Z.) The following are equivalent: (a) $W_{g,n}$ satisfies the EO topological recursion on the spectral curve

$$y^2 = \frac{1}{4s^2} - \frac{u+v}{2sx} + \frac{(u-v)^2}{4x^2}$$

(b) Z_{Des} satisfies the Virasoro constraints above.

Dessin partition function and Seiberg-Witten curve

By comparing the dessin spectral curve

$$y^2 = \frac{1}{4s^2} - \frac{u+v}{2sx} + \frac{(u-v)^2}{4x^2}$$

with the Seiberg-Witten curve

$$Y^2 = (X^2 - U)^2 - 4\Lambda^2$$

we find they are related to each other by:

$$Y = 2xy, \qquad X^2 = \frac{x}{s}, \qquad U = u + v, \qquad \Lambda^2 = uv.$$

Seiberg-Witten curve and associahedra

$$
\begin{aligned}
Y &= \sqrt{(X^2-U)^2-4\Lambda^2} \\
&= X^2\Bigg[1-\frac{u}{X^2}-2\Bigg(\frac{\Lambda^2}{X^4}+\frac{u\Lambda^2}{X^6}+\frac{(u^2\Lambda^2+\Lambda^4)}{X^8}+\frac{(u^3\Lambda^2+3u\Lambda^4)}{X^{10}} \\
&+ \frac{(u^4\Lambda^2+6u^2\Lambda^4+2\Lambda^6)}{X^{12}}+\frac{(u^5\Lambda^2+10u^3\Lambda^4+10u\Lambda^6)}{X^{14}} \\
&+ \frac{(u^6\Lambda^2+15u^4\Lambda^4+30u^2\Lambda^6+5\Lambda^8)}{X^{16}}\cdots\Bigg)\Bigg]
\end{aligned}
$$

The coefficients $1, 1, (1, 1), (1, 3)$, $(1, 6, 2)$, $(1, 10, 10)$,$(1, 15, 30, 5)$, $(1, 21, 70, 35)$, $\ldots$ are A055151: $T(n,k) = \frac{1}{k+1}\binom{n}{k,k,n-2k}$ = number of Motzkin paths of length n with k up steps.

They are the γ-vectors of the n-dimensional (type A) associahedra.

The Seiberg-Witten differential

The Seiberg-Witten differential

$$\lambda = \frac{XdX}{Y} \cdot \frac{d}{dX}(X^2 - U) = \frac{2X^2 dX}{Y} = \frac{dx}{2s^{3/2}x^{1/2}y},$$

$$\begin{aligned}
\lambda &= \frac{2X^2 dX}{((X^2-u)^2 - 4\Lambda^2)^{1/2}} = \frac{2dX}{(1-u/X^2)^2 - 4\Lambda^2/X^4)^{1/2}} \\
&= 2dX\left(1 + \frac{u}{X^2} + \frac{(u^2+2\Lambda^2)}{X^4} + \frac{(u^3+6u\Lambda^2)}{X^6}\right. \\
&+ \left.\frac{(u^4+12u^2\Lambda^2+6\Lambda^4)}{X^8} + \frac{(u^5+20u^3\Lambda^2+30u\Lambda^4)}{X^{10}} + \cdots\right)
\end{aligned}$$

Combinatorial meaning of the coefficients

The coefficients are

$$T(n,k) = \binom{n}{k,k,2n-k}.$$

The rows of this triangle are the γ-vectors of the n-dimensional type-B associahedra (cyclohedra introduced by Bott and Taubes). Recall

$$\begin{aligned} G_{0,1}(x) &= \frac{1}{2s}\left(1 - \frac{s(u+v)}{x} - \sqrt{\left(1 - \frac{s(u+v)}{x}\right)^2 - \frac{4s^2uv}{x^2}}\right) \\ &= \sum_{n=1}^{\infty} \frac{s^n uv}{x^{n+1}} \sum_{k=1}^{n} \frac{\binom{n}{k}}{n} \frac{\binom{n}{k-1}}{n} u^{n-k} v^{k-1}. \end{aligned}$$

The coefficients are Naryana numbers: the h-vectors of the type-A associahedra .

f-vector, h-vector and γ-vector

For a d-dimensional polytope P, the face number $f_i(P)$ is the number of i-dimensional faces of P. The vector $(f_0(P), \ldots, f_d(P))$ is called the f-vector, and the polynomial $f_P(t) = \sum_{i=0}^{d} f_i(P)t^i$ is called the f-polynomial of P.

The h-vector $(h_0(P), \ldots, h_d(P))$ and
the h-polynomial $h_P(t) = \sum_{i=0}^{d} h_i(P)t^i$ are defined by

$$f_P(t) = h_P(t+1), \quad f_j(P) = \sum_i \binom{j}{i} h_i(P), j = 0, \ldots, d.$$

The γ-vector $(\gamma_0, \gamma_1, \ldots, \gamma_{[d/2]})$ and
the γ-polynomial $\gamma_P(t) := \sum_{i=0}^{[d/2]} \gamma_i t^i$ are defined by

$$h_P(t) = \sum_{i=0}^{[d/2]} \gamma_i t^i (1+t)^{d-2i} = (1+t)^d \gamma_P\left(\frac{t}{(1+t)^2}\right).$$

Dessins and Laguerre Unitary Ensemble (LUE)

The following are joint work with Di Yang.

In the above we relate dessin countings to two different matrix mdoels:

$$\tilde{Z}_{even}|_{\epsilon=1} = Z_{Dessins}|_{u=\frac{t}{2}+\frac{1}{4}, v=\frac{t}{2}-\frac{1}{4}, s=2},$$

$$Z_{BGW}^{(N)} = Z_{Dessins}\Big|_{p_{2n}=0, p_{2n-1}=\sqrt{2}(2n-1)t_{2n-1}, s=\frac{1}{2\sqrt{2}}, u=-v=\sqrt{\frac{N^2}{2}-\frac{1}{8}}}.$$

recently we have found:

$$Z_{Laguerre} = Z_{Dessin}|_{u=x, v=x+a}.$$

Laguerre Unitary Ensemble (LUE)

Let $\mathcal{H}_n^+$ denote the space of positive hermitian matrices of size n.

Define *the normalized LUE partition function of size* n as follows:

$$Z_n^{\mathsf{LUE1}}(\mathbf{s};\alpha;\epsilon) = \frac{G(\alpha+1)\epsilon^{-n^2-\alpha n}}{\pi^{\frac{n(n-1)}{2}} G(n+\alpha+1)} \int_{\mathcal{H}_n^+} (\det M)^{\alpha} e^{-\frac{1}{\epsilon}\mathrm{tr} V(M;\mathbf{s})} dM,$$

where $\mathbf{s} = (s_1, s_2, s_3, \cdots)$ is an infinite vector of indeterminates, α is a parameter, $G(z)$ denotes the Barnes G-function,

$$V(M;\mathbf{s}) = M - \sum_{i\geq 1} s_i M^i,$$

$$dM = \prod_{1\leq i\leq n} dM_{ii} \prod_{1\leq i<j\leq n} d\mathrm{Re} M_{ij} d\mathrm{Im} M_{ij}.$$

Cut-join representation for LUE

Define the *normalized LUE partition function* Z^{one} as

$$Z^{\mathsf{one}}(x,\mathbf{s};a;\epsilon)=Z^{\mathsf{LUE1}}_{x/\epsilon}\Big(\mathbf{s};a/\epsilon;\epsilon\Big).$$

It has the following cut-and-join representation:

$$Z^{\mathsf{one}}(x,\mathbf{s};a;\epsilon)=e^{W^{\mathsf{one}}}(1),$$

$$W^{\mathsf{one}}:=2\Big(x+\frac{a}{2}\Big)\Lambda_1^{\mathsf{one}}+M_1^{\mathsf{one}}+\frac{x(x+a)}{\epsilon^2}s_1,$$

with

$$\Lambda_1^{\mathsf{one}}=\sum_{i\geq 2}\frac{i(i-2)}{i-1}s_i\frac{\partial}{\partial s_{i-1}},$$

$$M_1^{\mathsf{one}}=\sum_{i\geq 2}\sum_{j=1}^{i-1}\left(j(i-j)s_js_{i-j}\frac{\partial}{\partial s_{i-1}}+\epsilon^2(i+1)s_{i+1}\frac{\partial^2}{\partial s_j\partial s_{i-j}}\right).$$

Virasoro constraints for LUE

The cut-and-join representation follows from the Virasoro constraints:

$$L_m^{\text{one}}\Big(Z^{\text{one}}(x,\mathrm{s};a;\epsilon)\Big)=0,\quad m\geq 0,$$

where

$$\begin{aligned} L_m^{\text{one}} &= \epsilon^2\sum_{k=1}^{m-1}\frac{\partial^2}{\partial s_k\partial s_{m-k}}+\sum_{k\geq 1}k\tilde{s}_k\frac{\partial}{\partial s_{k+m}} \\ &+ 2\Big(x+\frac{a}{2}\Big)\delta_{m\geq 1}\frac{\partial}{\partial s_m}+\frac{x(x+a)}{\epsilon^2}\delta_{m,0}. \end{aligned}$$

Dessins and LUE

Using the cut-and-join representations, one can identify the dessin partition function with the normalized LUE partition function:

$$Z_{dessins}(x, x+a, \mathbf{p}; \epsilon) = Z^{\text{one}}\Big(x, \mathbf{s}; a; \epsilon\Big),$$
$$Z_{dessins}\Big(u, v, \mathbf{p}; \epsilon\Big) = Z^{\text{one}}\Big(u, \mathbf{s}; v-u; \epsilon\Big).$$

From

$$Z_{dessins}(v, u, \mathbf{p}; \epsilon) = Z_{dessins}(u, v, \mathbf{p}; \epsilon),$$

one gets the following duality for LUE partition function:

$$n \mapsto n + \alpha, \quad \alpha \mapsto -\alpha.$$

Dessins and monotone Hurwitz numbers

By a recent result due to Cunden–Dahlqvist–O'Connell, the correlators of the LUE are related to strictly monotone Hurwitz numbers and weakly monotone Hurtwitz numbers.

So we get:

$$N_{k,l}(\mu) = \frac{\prod_{i=1}^{\infty} n_i!}{|\mu|!} \sum_{\substack{\nu \in \mathcal{P}_{|\mu|} \\ \ell(\nu)=l}} h_g(\mu,\nu).$$

$N_{k,l}(\mu)$: number of dessins

$h_g(\mu,\nu)$: strictly monotone Hurwitz numbers.

Corrected dessins partition function

Define the *LUE partition function of size* n by:

$$\begin{aligned} Z_n^{\mathsf{LUE}}(s;\alpha;\epsilon) &= (-1)^{\frac{1}{24}}\epsilon^{-\frac{1}{12}+n(n+\alpha)}(2\pi)^{-n} \\ &\quad \cdot \frac{G(n+1)G(n+\alpha+1)}{G(\alpha+1)} Z_n^{\mathsf{LUE1}}(s;\alpha;\epsilon) \\ &= \frac{G(n+1)\epsilon^{-\frac{1}{12}}}{\pi^{\frac{n(n+1)}{2}}2^n} \int_{\mathcal{H}_n^+} (\det M)^{\alpha} e^{-\frac{1}{\epsilon}\operatorname{tr} V(M;\mathrm{s})} dM. \end{aligned}$$

By the identification of dessin partition function with the normalized LUE partition, this is a corrected dessin partition function.

Corrected dessins partition function and Toda lattice hierarchy

Theorem (Yang-Z.) The LUE partition function $Z(x, \mathbf{s}; a; \epsilon)$ is a particular tau-function for the Toda lattice hierarchy. In particular, the power series $V(x, \mathbf{s}; \epsilon)$ and $W(x, \mathbf{s}; \epsilon)$ defined by

$$V(x, \mathbf{s}; \epsilon) = \epsilon(\Lambda - 1)\frac{\partial \log Z(x, \mathbf{s}; a; \epsilon)}{\partial s_1},$$
$$W(x, \mathbf{s}; \epsilon) = \frac{Z(x + \epsilon, \mathbf{s}; a; \epsilon)Z(x - \epsilon, \mathbf{s}; a; \epsilon)}{Z(x, \mathbf{s}; a; \epsilon)^2}$$

satisfy the Toda lattice hierarchy with $t_j = s_{j+1}$, $j \geq 0$. Moreover, the solution $(V(x, \mathbf{s}; \epsilon), W(x, \mathbf{s}; \epsilon))$ is uniquely specified by the following initial data:

$$V(x, \mathbf{0}; \epsilon) = 2x + a + \epsilon, \quad W(x, \mathbf{0}; \epsilon) = x(x + a).$$

Toda lattice hierarchy

Let

$$L = \Lambda + V(x) + W(x)\Lambda^{-1}$$

be a linear difference operator, called the *Lax operator*, where $\Lambda : f(x) \mapsto f(x+\epsilon)$ is the shift operator. The *Toda lattice hierarchy* is a system of evolutionary differential-difference equations, which can be defined by

$$\frac{\partial L}{\partial t_j} = \frac{1}{\epsilon}\left[(L^{j+1})_+, L\right], \quad j \geq 0.$$

Here, for a difference operator P written in the form $P = \sum_{i\in\mathbb{Z}} P_i\Lambda^i$, P_+ is defined as $\sum_{i\geq 0} P_i\Lambda^i$.

Initial values of Toda lattice hierarchy

The following is a table of the initial values for various tau-functions of the Toda lattice hierarchy:

	V	W
LUE/Dessins	$2x + a + \epsilon$	$x(x + a)$
JUE	$\frac{1}{2} + \frac{b^2 - a^2}{2(2x+a+b)(2x+a+b+2\epsilon)}$	$\frac{x(x+a)(x+b)(x+a+b)}{(2x+a+b-\epsilon)(2x+a+b)^2(2x+a+b}$
GUE	0	x
mEven GUE	$2x + \frac{\epsilon}{2}$	$x(x - \frac{\epsilon}{2})$
$\mathbb{P}^1$	$x + \frac{\epsilon}{2}$	1

Genus zero part of initial values of Toda lattice hierarchy

In genus zero we have

	v	$w = e^u$	Equations
LUE/Des	$2x + a$	$x(x + a)$	$w = \frac{1}{4}(v^2 - a^2)$
JUE	$\frac{1}{2} + \frac{b^2 - a^2}{2(2x+a+b)^2}$	$\frac{x(x+a)(x+b)(x+a+b)}{(2x+a+b)^4}$	$w = \frac{1}{4}v^2 - \frac{b^2(2v-1)}{4(b^2-a^2)}$
GUE	0	x	$v \equiv 0$
mEven GUE	$2x$	x^2	$w = \frac{1}{4}v^2$
$\mathbb{P}^1$	x	1	$w \equiv 1$

The last column gives the equations of the algebraic curves in the (v, w)-plane determined by the initial values in genus zero.

Dessins and extended Toda hierarchy of Dubrovin and Zhang

Let $M^{\mathbb{P}^1}$ be the Frobenius manifold associated with the $\mathbb{P}^1$ topological σ-model. The Frobenius potential of $M^{\mathbb{P}^1}$ is given by

$$F(\mathbf{v}) = \frac{1}{2}v^2 u + e^u$$

and the Euler vector field E on $M^{\mathbb{P}^1}$ is

$$E = v\partial_v + 2\partial_u.$$

Dubrovin and Zhang proposed a certain extension of the Toda lattice hierarchy and showed that Dubrovin–Zhang hierarchy for the Frobenius manifold of $\mathbb{P}^1$ is normal Miura equivalent to the extended Toda lattice hierarchy. In particular, the corresponding principal hierarchy coincides with the dispersionless extended Toda lattice hierarchy.

Dessins and extended Toda hierarchy of Dubrovin and Zhang

For the purpose of computing the dessins correlators, it suffices to look at the stationary flows.

In genus zero it suffices to find the power-series-in-$\mathbf{T}$ solution to the $\partial_{T^{2,p}}$-flows in the principal hierarchy with the following initial condition:

$$v(x, \mathbf{T} = \mathbf{0}; a) = 2x + a, \quad e^{u(x,\mathbf{T}=\mathbf{0};a)} = x(x + a).$$

Dessins and extended Toda hierarchy of Dubrovin and Zhang

Theorem. (Yang, Z.) The genus 0 corrected dessins free energy can be explicitly solved by the hodograph method. The genus g ($g \geq 1$) part of the corrected dessins free energy satisfies that

$$F_g(x, \mathbf{p}; a; \epsilon) = F_g^{M^{\mathbb{P}^1}} \left(\mathbf{v}(x, \mathbf{T}; a), \frac{\partial \mathbf{v}(x, \mathbf{T}; a)}{\partial x}, \ldots, \frac{\partial^{3g-2} \mathbf{v}(x, \mathbf{T}; a)}{\partial x^{3g-2}} \right).$$

Here, $\mathbf{p}$ and $\mathbf{T}$ are related by

$$T^{2,i+1} = (i+2)! \frac{p_i}{i}, \quad i \geq 0,$$

and $\mathbf{v}(x, \mathbf{T}; a) = (v(x, \mathbf{T}; a), u(x, \mathbf{T}; a))$ denotes the unique power-series-in-$\mathbf{T}$ solution to the principal hierarchy.

Generalized Penner model, $\mathbb{P}^1$ and nonlinear Schrödinger system

The LUE is just the generalized Penner model called by string theorists. For the Penner model,

$$F_{0,0}(z) = \frac{1}{2}z^2 \log(z) - \frac{3}{4}z^2.$$

The Legendre type transformation S_2 transforms the Frobenius manifold associated with Toda lattice with the potential

$$F_{Toda} = \frac{1}{2}v^2 u + e^u$$

to the Frobenius manifold associated with nonlinear Schrödinger system with the potential

$$F_{NLS}(\varphi, \rho) = \frac{1}{2}\varphi^2 \rho + \frac{1}{2}\rho^2 \log \rho - \frac{3}{4}\rho^2$$

where

$$\varphi = v, \rho = e^u.$$

By taking $\varphi = 0$ and $\rho = z$, we get

$$F_{0,0}(z) = F_{NLS}(0, z).$$

This establishes a connection between the generalized Penner model with the topological $\mathbb{P}^1$-sigma model.

Dessins, 2-dimensional Toda lattice hierarchy and Ablowitz-Ladik hierarchy

Recall dessins are branched covering of $\mathbb{P}^1$ ramified only at 0, 1 and ∞.

Like in the case of the double Hurwitz numbers, one can define a tau-function of the two-dimensional Toda lattice hierarchy from counting of dessins.

One-dimensional Toda lattice hierarchy and Ablowitz-Ladik hierarchy are both reductions of the 2D Toda lattice hierarchy.

Dessins are also related to Ablowitz-Ladik hierarchy (work in progress, Yang and Z.)

More work to do

$$
\begin{tikzcd}
& \text{Dessin Counting} \arrow[r] \arrow[d] & \text{Galois group } Gal(\bar{\mathbb{Q}}/\mathbb{Q}) \\
& \text{Matrix Models} \arrow[dl] \arrow[dr] & \\
\text{Seiberg-Witten} \arrow[d] & & \text{Local Calabi-Yau} \arrow[d] \\
\text{Periods, Toda} & & n\text{-component KP}
\end{tikzcd}
$$

Open intersection theory, Norbury's classes, ...

How much insight on Grothendieck's dessin counting from the dualities of QFTs can be transferred to the side of absolute Galois group $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$?

Dessins and Seiberg-Witten theory $\leftrightarrows$ cluster algebra?

Thank you very much for your attentions!