Frobenius Manifolds and Bihamiltonian Integrable Hierarchies

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Based on the work

1. Si-Qi Liu, Zhe Wang, Youjin Zhang, Super tau-covers of bihamiltonian integrable hierarchies, J. Geom. Phys., **170** (2021), 104351.
2. Si-Qi Liu, Zhe Wang, Youjin Zhang, Variational bihamiltonian cohomologies and integrable hierarchies I: foundations. eprint arXiv: 2106.13038.
3. Si-Qi Liu, Zhe Wang, Youjin Zhang, Variational bihamiltonian cohomologies and integrable hierarchies II: Virasoro symmetries. eprint arXiv: 2109.01845.
4. Si-Qi Liu, Zhe Wang, Youjin Zhang, Linearization of Virasoro symmetries associated with semisimple Frobenius manifolds. eprint arXiv: 2109.01846.

Outline of the talk

1. Introduction

We consider a class of evolutionary PDEs that are closely related to 2d topological field theory (TFT), they have the form

$$\frac{\partial w^\alpha}{\partial t} = K^\alpha_\gamma(w) w^\gamma_x + \varepsilon \left(A^\alpha_\gamma(w) w^\gamma_{xx} + B^\alpha_{\gamma\xi}(w) w^\gamma_x w^\xi_x \right) + \varepsilon^2 \left(P^\alpha_\gamma(w)\, w^\gamma_{xxx} + Q^\alpha_{\gamma\xi}(w)\, w^\gamma_{xx} w^\xi_x + R^\alpha_{\gamma\xi\zeta}(w)\, w^\gamma_x w^\xi_x w^\zeta_x \right) + \dots$$

for the unknown functions $w^1(x,t), \dots, w^n(x,t)$, and have bihamiltonian structures

$$\frac{\partial w^\alpha}{\partial t} = \{w^\alpha(x), H_a\}_a = \mathcal{P}^{\alpha\beta}_a \frac{\delta H_a}{\delta w^\beta}, \quad \alpha = 1, \dots, n;\ \ a = 1, 2.$$

Here the Hamiltonians and the compatible Hamiltonian operators have the form

$$\mathcal{P}^{\alpha\beta}_a = g^{\alpha\beta}_a(w)\partial_x + \Gamma^{\alpha\beta}_{a;\gamma}(w) w^\gamma_x + \sum_{m\ge 1} \varepsilon^m \mathcal{A}^{\alpha\beta}_{a;m}(w; w_x, \dots; \partial_x), \quad a = 1, 2;$$

$$H_a = \int \left[h_a(w) + \varepsilon f_{a;\beta}(w) w^\beta_x + \varepsilon^2 \left(p_{a;\beta}(w) w^\beta_{xx} + q_{a;\beta\gamma}(w) w^\beta_x w^\gamma_x \right) + \mathcal{O}(\varepsilon^3) \right] dx.$$

1. Introduction

Each evolutionary PDEs belongs to a certain hierarchy of Hamiltonian systems

$$\frac{\partial w^\alpha}{\partial t^{\beta,p}} = \{w^\alpha(x), H_{\beta,p}\}_1, \quad \alpha, \beta = 1, \dots, n, \; p \geq 0,$$

here the Hamiltonians $H_{\beta,p}$ are related by the bihamiltonian recursion relations, and are in involution w.r.t. both Poisson brackets. Thus each flow of the hierarchy possesses an infinite number of conserved quantities. So we usually call such a hierarchy of Hamiltonian systems a hierarchy of bihamiltonian integrable systems, or a bihamiltonian integrable hierarchy.

The KdV hierarchy and the Toda lattice hierarchy are two prototypical examples of bihamiltonian integrable hierarchies.

1. Introduction

Example (The KdV hierarchy, $n = 1$)

$$
\begin{aligned}
w_{t^0} &= w_x, \\
w_{t^1} &= w w_x + \frac{\varepsilon^2}{12} w_{xxx} \quad \text{(KdV)}, \\
w_{t^2} &= \frac{1}{2} w^2 w_x + \frac{\varepsilon^2}{12}\left(2 w_x w_{xx} + w w_{xxx}\right) + \frac{\varepsilon^4}{240} w^{(5)}, \ldots.
\end{aligned}
$$

It has a bihamiltonian structure

$$
\frac{\partial w}{\partial t^p} = \mathcal{P}_1 \frac{\delta H_p}{\delta w} = (p + \frac{1}{2})^{-1} \mathcal{P}_2 \frac{\delta H_{p-1}}{\delta w}, \quad p \geq 0.
$$

Here the Hamiltonian operators and the Hamiltonians are given by

$$
\begin{aligned}
&\mathcal{P}_1 = \partial_x, \quad \mathcal{P}_2 = w(x)\partial_x + \frac{1}{2} w_x + \frac{\varepsilon^2}{8} \partial_x^3, \\
&H_{-1} = \int w(x)\, dx, \quad \{w(x), H_{k-1}\}_2 = (k + \frac{1}{2})\{w(x), H_k\}_1, \quad k \geq 0.
\end{aligned}
$$

1. Introduction

Example (The Toda lattice hierarchy, $n = 2$)

We consider the Toda lattice equation for $q_n(T)$:

$$\ddot{q}_n = e^{q_{n-1}-q_n} - e^{q_n-q_{n+1}}, \quad -\infty < n < \infty.$$

By performing an interpolation we arrive at the following continuous version of the Toda lattice equation for $q(x,t) = q_n(T)$:

$$\varepsilon^2 q_{tt} = e^{q(x-\varepsilon)-q(x)} - e^{q(x)-q(x+\varepsilon)}.$$

Here $q(x,t) = q_n(T)$ with $x = n\varepsilon, t = T\varepsilon$, and we omit the t-dependence of q in the equation. Let

$$w^1(x,t) = -\varepsilon q_t(x,t), \quad w^2(x,t) = q(x-\varepsilon,t) - q(x,t),$$

then we can rewrite the Toda lattice equation in the form

$$\varepsilon \frac{\partial w^1}{\partial t} = e^{w^2(x+\varepsilon)} - e^{w^2(x)}, \quad \varepsilon \frac{\partial w^2}{\partial t} = w^1(x) - w^1(x-\varepsilon).$$

1. Introduction

Example (The Toda lattice hierarchy, $n = 2$)

The above-mentioned Toda lattice equation is given by the flow $\frac{\partial}{\partial t^{2,0}}$ *of the following bihamiltonian integrable hierarchy (called the extended Toda hierarchy)*

$$\frac{\partial w^\alpha}{\partial t^{\beta,q}} = \{w^\alpha(x), H_{\beta,q}\}_1, \quad \alpha, \beta = 1, 2;\ q \geq 0,$$

Here the bihamiltonian structure is given by the Hamiltonian operators

$$\mathcal{P}_1 = \frac{1}{\varepsilon}\begin{pmatrix} 0 & e^{\varepsilon\partial_x} - 1 \\ 1 - e^{-\varepsilon\partial_x} & 0 \end{pmatrix},$$
$$\mathcal{P}_2 = \frac{1}{\varepsilon}\begin{pmatrix} e^{\varepsilon\partial_x} e^{w^2} - e^{w^2} e^{-\varepsilon\partial_x} & w^1\left(e^{\varepsilon\partial_x} - 1\right) \\ \left(1 - e^{-\varepsilon\partial_x}\right) w^1 & e^{\varepsilon\partial_x} - e^{-\varepsilon\partial_x} \end{pmatrix},$$

and the Hamiltonians satisfy the bihamiltonian recursion relations

$$\{w^\alpha(x), H_{\beta,q-1}\}_2 = (q + \mu_\beta + \frac{1}{2})\{w^\alpha(x), H_{\beta,q}\}_1 + 2\delta_2^\gamma \delta_{\beta,1}\{w^\alpha(x), H_{\gamma,q-1}\}_1.$$

1. Introduction

The close relationship of the above-mention class of integrable hierarchies with 2d topological field theory was revealed by Witten in the beginning of the 90's of the last century. As it was conjectured by Witten and later proved by Kontsevich, that the partition function of the 2d topological gravity is a particular tau function of the KdV hierarchy. Witten also conjectured that for a general 2d TFT there should exist a hierarchy of Hamiltonian integrable systems which is satisfied by some special two-point correaltion functions of the 2d TFT.

In 1993, Dubrovin introduced the notion of Frobenius manifold, which is a coordinate free formulation of the WDVV associativity equations of 2d TFT. For any Frobenius manifold, he constructed a bihamiltonian integrable hierarchy of hydrodynamic type, known as the Principal Hierarchy of the Frobenius manifold, and showed that the genus zero free energy of a 2d TFT is given by the logarithm of the tau function of a particular solution of the Principal Hierarchy of the corresponding Frobenius manifold.

1. Introduction

It is expected that the partition function of a 2d TFT is given by the tau function of a particular solution of a bihamiltonian integrable hierarchy, which is a certain deformation of the Principal Hierarchy. At the genus one approximation, this is confirmed by Dubrovin & Z. in 1998 under the assumption of semisimplicity of the associated Frobenius manifold.

In 2001, Dubrovin and & Z. proposed a project to classify bihamiltonian integrable hierarchies which possess the following additional properties:

1. Existence of tau function.
2. The integrable hierarchy possesses an infinite set of Virasoro symmetries which act linearly on the tau function.

1. Introduction

We showed in our 2001 preprint that the leading terms of the bihamiltonian integrable hierarchy which possesses tau function yields a Frobenius manifold structure or a degenerate one. The main challenge is to construct bihamiltonian integrable hierarchies starting from a Frobenius manifold which satisfy the above-mentioned additional properties.

Under the assumption of semisimplicity of the Frobenius manifold, we construct an integrable hierarchy via a quasi-Miura type transformation which is applied to the Principal Hierarchy of the Frobenius manifold. This quasi-Miura transformation is given by the unique solution of the so-called loop equation of the Frobenius manifold. We call this integrable hierarchy the topological deformation of the Principal Hierarchy, which is also called the Dubrovin-Zhang hierarchy in the literature.

1. Introduction

Since the quasi-Miura transformation is represented in terms of rational functions of the x-derivatives of the unknown functions, it was left to prove that the resulting integrable hierarchy has the above-mentioned form, i.e., it can be represented in terms of differential polynomials, and the bihamiltonian structure obtained via the application of the quasi-Miura transformation to that of the Principal Hierarchy also possesses the polynomiality property.

In 2012, Buryak, Posthuma, Shadrin proved the polynomiality of the first Poisson bracket and the integrable hierarchy. A proof of the polynomiality of the second Poisson bracket is given in the following recent preprint:

Si-Qi Liu, Zhe Wang, Youjin Zhang, Linearization of Virasoro symmetries associated with semisimple Frobenius manifolds. eprint arXiv: 2109.01846.

2. The Principal Hierarchy of a Frobenius manifold

Frobenius manifold (Dubrovin 1993):
Encodes the properties of the primary free energy $F = F(v^1, \dots, v^n)$ of a 2d topological field theory:

$$\frac{\partial^3 F}{\partial v^1 \partial v^\alpha \partial v^\beta} = \eta_{\alpha\beta} = \text{constant}, \quad (\eta_{\alpha\beta}): \text{ nondegenerate},$$

$$c^\gamma_{\alpha\beta} = \eta^{\gamma\xi} \frac{\partial^3 F}{\partial v^\xi \partial v^\alpha \partial v^\beta}: \quad \begin{array}{l}\text{structure constants of}\\ \text{an associative algebra}\end{array}$$

$$\partial_E F = (3-d) F + \text{quadratic terms in } v^\alpha,$$

$$\text{with the Euler vector } E = \sum_{\alpha=1}^n (d_\alpha v^\alpha + r_\alpha) \frac{\partial}{\partial v^\alpha}.$$

They are the WDVV equations of associativity.

2. The Principal Hierarchy of a Frobenius manifold

The deformed flat connection of a Frobenius manifold M^n:

$$\tilde{\nabla}_a b = \nabla_a b + z\, a \cdot b$$

Extend it to $M \times \mathbb{C}^*$ by

$$\tilde{\nabla}_{\frac{d}{dz}} b = \partial_z b + E \cdot b - \frac{1}{z} \mu\, b$$

with $\mu = \frac{2-d}{2} - \nabla E$.

The deformed flat coordinates $\tilde{v}_1(v; z), \dots, \tilde{v}_n(v; z)$ satisfying

$$\tilde{\nabla} d\tilde{v}_\alpha(v; z) = 0, \quad \alpha = 1, \dots, n.$$

2. The Principal Hierarchy of a Frobenius manifold

The functions $\theta_{\beta,q}(v)$

The deformed flat coordinates have the form

$$(\tilde{v}_1(v;z),\dots,\tilde{v}_n(v;z)) = (\theta_1(v;z),\dots,\theta_n(v;z))z^{\mu}z^{R}$$

Here $\theta_1(v;z),\dots,\theta_n(v;z)$ are analytic at $z=0$ with Taylor expansions

$$\theta_\alpha(v;z) = \sum_{p\geq 0}\theta_{\alpha,p}(v)z^p$$

satisfying the normalization conditions

$$\theta_\alpha(v;0) = \eta_{\alpha\beta}v^\beta, \quad \alpha = 1,\dots,n$$
$$\langle\nabla\theta_\alpha(v;-z),\nabla\theta_\beta(v;z)\rangle = \eta_{\alpha\beta}.$$

$\mu, R:$ monodromy data at $z=0$.

2. The Principal Hierarchy of a Frobenius manifold

The Principal Hierarchy: a bihamiltonian integrable hierarchy of hydrodynamic type

$$\frac{\partial v^\alpha}{\partial t^{\beta,q}} = \eta^{\alpha\gamma}\frac{\partial}{\partial x}\left(\frac{\delta H_{\beta,q}}{\delta v^\gamma}\right), \quad \alpha,\beta=1,\dots,n,\ \ q\ge 0,$$

where

$$H_{\beta,q} = \int \theta_{\beta,q+1}(v(x))\,dx.$$

It has a bihamiltonian structure given by the compatible Hamiltonian operators

$$\mathcal{P}_1^{\alpha\beta} = \eta^{\alpha\beta}\partial_x, \quad \mathcal{P}_1^{\alpha\beta} = g^{\alpha\beta}\partial_x + \Gamma^{\alpha\beta}_\gamma(v)v^\gamma_x.$$

Here g is the intersection form of the Frobenius manifold, in the flat coordinates $v^1,\dots,v^n$ it is given by

$$g^{\alpha\beta} = E^\gamma c^{\alpha\beta}_\gamma,$$

and $\Gamma^{\alpha\beta}_\gamma$ are the contravariant components of its Levi-Civita connection of η.

2. The Principal Hierarchy of a Frobenius manifold

The tau function

Define functions $\Omega_{\alpha,p;\beta,q}(v)$ as follows:

$$\sum \Omega_{\alpha,p;\beta,q}(v)\, z^p w^q = \frac{< \nabla\theta_\alpha(z), \nabla\theta_\beta(w) > -\eta_{\alpha\beta}}{z+w},$$

These functions satisfy the relations

$$\frac{\partial\theta_{\alpha,p}}{\partial t^{\beta,q}} = \frac{\partial\theta_{\beta,p}}{\partial t^{\alpha,q}} = \partial_x \Omega_{\alpha,p;\beta,q}(v).$$

These relations ensure the existence of a function $\tau^{[0]}(t)$ for any solution

$$v(t) = (v^1(t), \ldots, v^n(t))$$

of the Principal hierarchy satisfying

$$\Omega_{\alpha,p;\beta,q}(v(t)) = \frac{\partial^2 \log \tau^{[0]}(t)}{\partial t^{\alpha,p} \partial t^{\beta,q}}, \quad \alpha, \beta = 1, \ldots, n, \;\; p, q \geq 0.$$

2. The Principal Hierarchy of a Frobenius manifold

The Virasoro symmetries

The first symmetry is the Galilean symmetry:

$$v^\alpha \mapsto v^\alpha + \epsilon \left(\sum_{\beta,q} t^{\beta,q} \frac{\partial v^\alpha}{\partial t^{\beta,q-1}} + \delta_1^\alpha \right) + \mathcal{O}(\epsilon^2),$$
$$\tau \mapsto \tau + \epsilon L_{-1}\tau + \mathcal{O}(\epsilon^2),$$

where the operator L_{-1} defined by

$$L_{-1} = \sum_{q\geq 1} t^{\beta,q} \frac{\partial}{\partial t^{\beta,q-1}} + \frac{1}{2}\,\eta_{\alpha\beta}\, t^{\alpha,0} t^{\beta,0}.$$

2. The Principal Hierarchy of a Frobenius manifold

The Virasoro symmetries

In general, the action of the Virasoro symmetries on the tau function can be represented in the form

$$\tau \mapsto \tau + \epsilon\left(a_m^{\alpha,p;\beta,q}\frac{1}{\tau}\frac{\partial\tau}{\partial t^{\alpha,p}}\frac{\partial\tau}{\partial t^{\beta,q}}\right.$$
$$\left. + b_{m;\alpha,p}^{\beta,q} t^{\alpha,p}\frac{\partial\tau}{\partial t^{\beta,q}} + c_{m;\alpha,p\beta,q} t^{\alpha,p} t^{\beta,q}\tau + c\delta_{m,0}\tau\right) + \mathcal{O}(\epsilon^2).$$

with the Virasoro operators (for $m \geq -1$)

$$L_m = \varepsilon^2 a_m^{\alpha,p;\beta,q}\frac{\partial^2}{\partial t^{\alpha,p}\partial t^{\beta,q}} + b_{m;\alpha,p}^{\beta,q} t^{\alpha,p}\frac{\partial}{\partial t^{\beta,q}}$$
$$+ \varepsilon^{-2} c_{m;\alpha,p,\beta,q} t^{\alpha,p} t^{\beta,q} + \kappa_0\,\delta_{m,0}, \quad m \geq -1.$$

3. The topological deformation of the Principal Hierarchy

Linearization of the Virasoro symmetries

We consider a certain integrable deformation of the Principal Hierarchy such that the deformed hierarchy is given by a quasi-Miura transformation

$$v^\alpha \to w^\alpha = v^\alpha + \varepsilon^2 \eta^{\alpha\gamma} \frac{\partial^2}{\partial x \partial t^{\gamma,0}} \sum_{g\geq 1} \varepsilon^{2g} F_g(v, v_x, \dots, v^{(m_g)}).$$

Then the deformed hierarchy has a tau function of the form

$$\tau = e^{\varepsilon^{-2}\mathcal{F}_0 + \sum_{g\geq 1} \varepsilon^{2g-2} F_g(v, v_x, \dots, v^{(m_g)})|_{v=v(t)}}.$$

We require that the actions of the infinitesimal Virasoro symmetries on the tau function are linear, i.e.

$$\tau \mapsto \tau + \epsilon\, L_m \tau + \mathcal{O}(\epsilon^2), \quad m \geq -1.$$

3. The topological deformation of the Principal Hierarchy

The loop equation

The condition of lineariztion of the Virasoro symmetries is equivalent to a system of equations, called the loop equation, for the functions $F_g, g \geq 1$. For example, when $g = 1$ we have

$$\sum_{i=1}^{n} \frac{\partial F_1}{\partial u_i} \frac{1}{u_i - \lambda} - \sum_{i=1}^{n} \frac{\partial F_1}{\partial u_i'} \frac{u_i'}{(u_i - \lambda)^2} + \sum \frac{\partial F_1}{\partial v_x^\gamma} \partial_1 p_\alpha G^{\alpha\beta} \partial_x \partial^\gamma p_\beta$$
$$= -\frac{1}{16} \sum_{i=1}^{n} \frac{1}{(\lambda - u_i)^2} - \frac{1}{2} \sum_{i<j} \frac{V_{ij}^2}{(\lambda - u_i)(\lambda - u_j)} + \frac{1}{4\,\lambda^2} \mathrm{tr}\left(\frac{1}{4} - \hat{\mu}^2\right) - \frac{\kappa_0}{\lambda^2}.$$

Here $u_1, \dots, u_n$ are the canonical coordinates of the semisimple Frobenius manifold M, which have the property

$$\frac{\partial}{\partial u_i} \cdot \frac{\partial}{\partial u_j} = \delta_{i,j} \frac{\partial}{\partial u_i}$$

3. The topological deformation of the Principal Hierarchy

The genus one free energy

$$F_1(v, v_x) = \frac{1}{24} \log \det \left(c_{\alpha\beta\gamma}(v) v_x^\gamma\right) + G(v)$$

(Witten-Dijkgraaf and Getzler)

The function $G(v)$ has the form (Dubrovin, Z.)

$$G(v) = \log \frac{\tau_I(v)}{J^{1/24}(v)}\,.$$

- For the 1-dimensional Frobenius manifold, the topological deformation of the Principal Hierarchy is the KdV hierarchy (Witten-Kontsevich).
- For the Frobenius manifold associated to the $\mathbb{P}^1$ topological sigma model, the topological deformation of the Principal Hierarchy is the Toda lattice hierarchy (Dubrovin & Z. 2004).

3. The topological deformation of the Principal Hierarchy

Example: the KdV hierarchy

The quasi-Miura transformation that relate the Principal Hierarchy of the 1-dimensional Frobenius manifold – the dispersionless KdV hierarchy

$$\frac{\partial v}{\partial t^p} = \frac{1}{p!} v^p v_x, \quad p \geq 0$$

and the KdV hierarchy

$$w_{t^0} = w_x, \quad w_{t^1} = w w_x + \frac{\varepsilon^2}{12} w_{xxx}, \ldots$$

is given by

$$w = v + \frac{\epsilon^2}{24} (\log v_x)_{xx} + \epsilon^4 \left(\frac{v_{xxxx}}{1152\, v_x^2} - \frac{7\, v_{xx} v_{xxx}}{1920\, v_x^3} + \frac{v_x x^3}{360\, v_x^4} \right)_{xx} + O(\epsilon^6).$$

3. The topological deformation of the Principal Hierarchy

Example: the KdV hierarchy

The above-mentioned quasi-Miura transformation also transforms the bihamiltonian structure

$$\{v(x), v(y)\}_1 = \delta'(x-y),$$
$$\{v(x), v(y)\}_2 = v(x)\delta'(x-y) + \frac{1}{2}v_x\delta(x-y)$$

to that of the KdV hierarchy

$$\{w(x), w(y)\}_1 = \delta'(x-y),$$
$$\{w(x), w(y)\}_2 = w(x)\delta'(x-y) + \frac{1}{2}w_x\delta(x-y) + \frac{\varepsilon^2}{8}\delta'''(x-y).$$

4. Classification of bihamiltonian integrable hierarchies

Classification of deformations of bihamiltonian structure of hydrodynamic type

$$
\begin{aligned}
\{u^i(x), u^j(y)\}_a = & g_a^{ij}(u(x))\delta'(x-y) + \Gamma^{ij}_{k;a}(u(x))\, u_x^k\, \delta(x-y) \\
& + \sum_{m\geq 1} \sum_{l=0}^{m+1} \varepsilon^m A^{ij}_{m,l;a}(u, u_x, \dots, u^{(m+1-l)})\delta^{(l)}(x-y), \\
& i,j = 1,\dots,n, \ \ a = 1,2
\end{aligned}
$$

under Miura type transformations

$$
u^i \mapsto u^i + \sum_{k\geq 1} \varepsilon^k F^i_k(u; u_x, \dots, u^{(k)}), \quad i = 1,\dots,n.
$$

Polynomiality:

The coefficients $A^{ij}_{m,l;a}$ and F^i_k are homogeneous polynomials of $u^i_x, u^i_{xx}, \dots$.

4. Classification of bihamiltonian integrable hierarchies

Central invariants

Definition

$$c_i(u) = \frac{1}{3(f^i(u))^2}\left(A^{ii}_{2,3;2} - u^i A^{ii}_{2,3,;1} + \sum_{k\neq i}\frac{(A^{ki}_{1,2;2} - u^i A^{ki}_{1,2;1})^2}{f^k(u)(u^k-u^i)}\right),$$

$$i = 1,\dots,n$$

here $u^1,\dots,u^n$ *are the canonical coordinates of the semisimple bihamiltonian structure of hydrodynamic type,* $u^1,\dots,u^n$*, in these coordinates,*

$$g_1^{ij}(u) = \delta_{ij} f^i(u),\quad g_2^{ij}(u) = \delta_{ij} u^i f^i(u),\quad i = 1,\dots,n.$$

4. Classification of bihamiltonian integrable hierarchies

Main properties of the central invariants

Theorem (Dubrovin, Liu, Z., 2006)

i) *Each function* $c_i(u)$ *depends only on* u^i, $i = 1, \ldots, n$.

ii) *The functions* $c_i(u)$, $i = 1, \cdots, n$ *are invariant under the action of the Miura group on the semisimple bihamiltonian structures.*

iii) *Two semisimple bihamiltonian structures with the same leading terms are equivalent if and only if they have the same set of central invariants.*

iv) *The space of equivalence classes of infintesimal deformations of a semisimple bihamiltonian structure of hydrodynamic type is parametrized by a set of smooth functions* $c_1(u^1), \ldots, c_n(u^n)$.

4. Classification of bihamiltonian integrable hierarchies

Theorem

For any given semisimple bihamiltonian structure of hydrodynamic type and a set of smooth functions $c_1(u^1), \dots, c_n(u^n)$, there exists a deformation of the bihamiltonian structure with the given set of functions as its central invariants.

Proof of the theorem:

- Liu, Z. 2013, for the bihamiltonian structure of the Principal Hierarchy of the 1-dimensional Frobenius manifold, i.e. the bihamiltonian structure of the dispersionless KdV hierarchy,
- Cartlet, Posthuma, Shadrin, 2018, for general semisimple bihamiltonian structure of hydrodynamic type,

by showing the triviality of the bihamiltonian cohomology $BH^3_{d\geq 5}(M; P_1, P_2)$, which is introduced by Dubrovin & Z. (2001), and characterizes obstructions of extending an infinitesimal deformations to a full deformation.

4. Classification of bihamiltonian integrable hierarchies

The associated deformations of the Principal Hierarchy

Theorem (Dubrovin, Liu, Z., 2018)

For any given semisimple Frobenius manifold, a deformation of the bihamiltonian structure of the Principal Hierarchy with constant central invariants yields a unique tau-symmetric deformation of the Principal Hierarchy which possesses Galilean symmetry.

Conjecture

For a semisimple Frobenius manifold of dimension n, the deformation of the Principal Hierarchy given by the deformation of the bihamiltonian structure of hydrodynamic type with central invariants

$$c_1 = \cdots = c_n = \frac{1}{24}$$

is equivalent to its topological deformation.

5. The Virasoro symmetries

Virasoro symmetries: bihamiltonian integrable hierarchies with constant central invariants

Theorem (Liu, Wang, Z. 2021)

For the Principal Hierarchy associated with a semisimple Frobenius manifold and for any of its tau-symmetric bihamiltonian deformations, there exists a unique deformation of its Virasoro symmetries such that they are symmetries of the deformed integrable hierarchy. Moreover, the action of the Virasoro symmetries on the tau-function Z of the deformed integrable hierarchy can be represented in the form

$$\frac{\partial Z}{\partial s_m} = L_m Z + O_m Z, \quad m \geq -1,$$

where L_m are the Virasoro operators, and O_m are some differential polynomials, and the flows $\frac{\partial}{\partial s_m}$ satisfy the Virasoro commutation relations

$$\left[\frac{\partial}{\partial s_k}, \frac{\partial}{\partial s_l}\right] = (l-k)\frac{\partial}{\partial s_{k+l}}, \quad k, l \geq -1.$$

5. The Virasoro symmetries

Linearizable Virasoro symmetries

Theorem (Liu, Wang, Z. 2021)

For a given tau-symmetric bihamiltonian deformation of the Principal Hierarchy associated with a semisimple Frobenius manifold, the Virasoro symmetries of the deformed integrable hierarchy can be linearized if and only if the central invariants of the deformed bihamiltonian structure are all equal to $\frac{1}{24}$, i.e., in this case, by choosing a suitable representative of the bihamiltonian structure, the action of the Virasoro symmetries on the tau-function Z of the deformed integrable hierarchy can be represented in the form

$$\frac{\partial Z}{\partial s_m} = L_m Z, \quad m \geq -1.$$

5. The Virasoro symmetries

Polynomiality of the bihamiltonian structure of the topological deformation of the Principal Hierarchy

Theorem (Liu, Wang, Z. 2021)

The bihamiltonian structure of the topological deformation of the Principal Hierarchy of a semisimple Frobenius manifold can be represented by differential polynomials.

6. Super variables and Hamiltonian structures

Let M be a smooth manifold of dimension n.

1. $J^\infty(M) = \varprojlim_k J^k(M)$: the associated infinite jet bundle with fiber $\mathbb{R}^\infty$
2. Local coordinate system on M: $(U; u^1, \dots, u^n)$
3. Local coordinate system on $J^\infty(M)$:

$$(U \times \mathbb{R}^\infty; u^{\alpha,p}, 1 \le \alpha \le n, \ p \ge 0).$$

Let $\hat{M} = \Pi T^* M$ be the super manifold of dimension $(n|n)$ obtained by reversing the parity of fibers of the cotangent bundle $T^* M$.

1. $J^\infty(\hat{M}) = \varprojlim_k J^k(\hat{M})$: the associated infinite jet bundle
2. Local coordinate system on $\hat{M}$: $(U; u^\alpha, \theta_\alpha, \alpha = 1, \dots, n)$
3. Local coordinate system on $J^\infty(\hat{M})$:

$$(\hat{U} \times \mathbb{R}^\infty; u^{\alpha,p}, \theta_\alpha^p, \ 1 \le \alpha \le n, \ p \ge 0).$$

6. Super variables and Hamiltonian structures

The ring of differential polynomials $\hat{\mathcal{A}}$

$$\hat{\mathcal{A}} = C^\infty(U)[[u^{\alpha,s+1}, \theta^s_\alpha \mid i = 1, \cdots, n; s \geq 0]].$$

It has the differential and the super gradations

$$\deg_x u^{\alpha,s} = \deg_x \theta^s_\alpha = s; \quad \deg_\theta u^{\alpha,s} = 0, \quad \deg_\theta \theta^s_\alpha = 1.$$

We denote the space of homogeneous elements by

$$\hat{\mathcal{A}}_d = \{f \in \hat{\mathcal{A}} \mid \deg_x f = d\}, \quad \hat{\mathcal{A}}^p = \{f \in \hat{\mathcal{A}} \mid \deg_\theta f = p\},$$

$$\hat{\mathcal{A}}^p_d = \hat{\mathcal{A}}^p \cap \hat{\mathcal{A}}_d.$$

6. Super variables and Hamiltonian structures

The space of local functionals $\hat{\mathcal{F}}$

The global vector field

$$\partial_x = \sum_{s\geq 0} u^{\alpha,s+1}\frac{\partial}{\partial u^{\alpha,s}} + \theta_\alpha^{s+1}\frac{\partial}{\partial \theta_\alpha^s}$$

on $J^\infty(\hat{M})$ induces a derivation on $\hat{\mathcal{A}}$. The space of local functionals on $\hat{M}$

$$\hat{\mathcal{F}} := \hat{\mathcal{A}}/\partial_x\hat{\mathcal{A}}$$

also admits differential and super gradation induced from that of $\hat{\mathcal{A}}$, and we denote the set of homogeneous elements of differential degree d, super degree p by

$$\hat{\mathcal{F}}_d, \quad \hat{\mathcal{F}}^p, \quad \hat{\mathcal{F}}_d^p.$$

6. Super variables and Hamiltonian structures

The Schouten-Nijenhuis bracket

For an element $f \in \hat{\mathcal{A}}$, we denote its image in $\hat{\mathcal{F}}$ by $\int f$ and we call it a local functional of $\hat{M}$. On the space of local functionals we can define the Schouten-Nijenhuis bracket

$$[P, Q] = \int \left(\frac{\delta P}{\delta \theta_\alpha} \frac{\delta Q}{\delta u^\alpha} + (-1)^p \frac{\delta P}{\delta u^\alpha} \frac{\delta Q}{\delta \theta_\alpha} \right), \quad \forall P \in \hat{\mathcal{F}}^p, \forall Q \in \hat{\mathcal{F}}^q.$$

Here the variational derivatives are defined by

$$\frac{\delta P}{\delta u^\alpha} = \sum_{s \geq 0} (-\partial)^s \frac{\partial \tilde{P}}{\partial u^{\alpha,s}}, \quad \frac{\delta P}{\delta \theta_\alpha} = \sum_{s \geq 0} (-\partial)^s \frac{\partial \tilde{P}}{\partial \theta_\alpha^s},$$

with $\tilde{P} \in \hat{\mathcal{A}}$ being any lift of $P \in \hat{\mathcal{F}}$.

6. Super variables and Hamiltonian structures

Hamiltonian evolutionary PDEs

For any local functional $X = \int X^\alpha \theta_\alpha \in \hat{\mathcal{F}}^1$, we can associate to it an evolutionary PDEs of the form

$$\frac{\partial u^\alpha}{\partial t} = X^\alpha, \quad \alpha = 1, \dots, n,$$

here we need to make the replacement $u^{\alpha,s} \mapsto \partial_x^s u^\alpha$.

We call $X \in \hat{\mathcal{F}}^1$ a Hamiltonian evolutionary PDE if there exist $P \in \hat{\mathcal{F}}^2$ and $H \in \hat{\mathcal{F}}^0$ such that

$$X = [H, P], \quad [P, P] = 0.$$

Here P and H are called the Hamiltonian structure and the Hamiltonian of X respectively.

6. Super variables and Hamiltonian structures

Hamiltonian and bihamiltonian structures

We can represent P and H in the form

$$P = \frac{1}{2}\int \sum_{s\geq 0} P_s^{\alpha\beta}\theta_\alpha\theta_\beta^s, \quad H = \int h(u, u_x, \dots),$$

then the Hamiltonian evolutionary PDE can be represented as

$$\frac{\partial u^\alpha}{\partial t} = \mathcal{P}^{\alpha\beta}\frac{\delta H}{\delta u^\beta}, \quad \text{with} \ \ \mathcal{P}^{\alpha\beta} = \sum_{s\geq 0} P_s^{\alpha\beta}\partial_x^s.$$

The evolutionary PDE X is called a bihamiltonian system if there exist $P_1, P_2 \in \hat{\mathcal{F}}^2$ and $H, G \in \hat{\mathcal{F}}^0$ such that

$$X = [H, P_1] = [G, P_2], \quad [P_1, P_1] = [P_2, P_2] = [P_1, P_2] = 0.$$

6. Super variables and Hamiltonian structures

Example: KdV hierarchy

For example, let M be a smooth manifold of dimension one with local coordinate u. We consider the local functionals

$$X_0 = \int u_x\theta, \quad X_n = \frac{2^n}{(2n+1)!!}\int \left(\mathcal{R}^n u_x\right)\theta, \quad n \geq 2,$$

where

$$\mathcal{R} = \frac{\varepsilon^2}{8}\partial_x^2 + u + \frac{1}{2}u_x\partial_x^{-1}.$$

These local functionals correspond to the KdV hierarchy

$$\frac{\partial u}{\partial t_0} = u_x, \quad \frac{\partial u}{\partial t_1} = uu_x + \frac{\varepsilon^2}{12}u_{xxx}, \dots .$$

6. Super variables and Hamiltonian structures

Example: bihamiltonian structure of the KdV hierarchy

The KdV hierarchy has a bihamiltonian structure given by the following local functionals

$$P_1 = \frac{1}{2}\int \theta\theta^1, \quad P_2 = \frac{1}{2}\int \left(u\theta\theta^1 + \frac{\varepsilon^2}{8}\theta\theta^3\right).$$

It can be represented as

$$\frac{\partial u}{\partial t_p} = \mathcal{P}_1 \frac{\delta H_p}{\delta u} = \left(p+\frac{1}{2}\right)^{-1} \mathcal{P}_2 \frac{\delta H_{p-1}}{\delta u}, \quad p \geq 0,$$

where the Hamiltonian operators and the Hamiltonians are given by

$$\mathcal{P}_1 = \partial_x, \quad \mathcal{P}_2 = u\partial_x + \frac{1}{2}u_x + \frac{\varepsilon^2}{8}\partial_x^3,$$

$$H_{-1} = \int u, \quad H_0 = \int \left(\frac{1}{2}u^2 + \frac{\varepsilon^2}{12}u_{xx}\right), \dots.$$

7. Hamiltonian cohomologies

The space of super derivations on $\mathrm{Der}(\hat{\mathcal{A}})$

Denote by $\mathrm{Der}(\hat{\mathcal{A}})$ the space of super derivations (vector fields on $J^\infty(\hat{M})$).

$$\Delta\colon \hat{\mathcal{A}} \to \hat{\mathcal{A}}.$$

A derivation of $\deg = p$ satisfies the condition

$$\Delta(f\cdot g) = \Delta(f)g + (-1)^{pk} f\Delta(g), \quad \forall f \in \hat{\mathcal{A}}^k(M), g \in \hat{\mathcal{A}}(M).$$

It has a graded Lie algebra structure

$$[\Delta_1, \Delta_2] = \Delta_1\Delta_2 - (-1)^{p_1 p_2}\Delta_2\Delta_1, \quad \deg\Delta_1 = p_1,\ \deg\Delta_2 = p_2.$$

Denote

$$\mathrm{Der}(\hat{\mathcal{A}})^\partial = \{X \in \mathrm{Der}(\hat{\mathcal{A}}) \mid [X, \partial] = 0\}.$$

7. Hamiltonian cohomologies

The subspace $\mathrm{Der}(\hat{\mathcal{A}})^D$

The canonical symplectic structure $\varpi = \sum_\alpha \delta u^\alpha \wedge \delta\theta_\alpha$ on $\hat{M}$ induces a map

$$D\colon \hat{\mathcal{F}} \to \mathrm{Der}(\hat{\mathcal{A}}), \quad P \mapsto D_P,$$

with

$$D_P = \sum_{s\geq 0} \partial^s \left(\frac{\delta P}{\delta\theta_\alpha}\right) \frac{\partial}{\partial u^{\alpha,s}} + (-1)^p \partial^s \left(\frac{\delta P}{\delta u^\alpha}\right) \frac{\partial}{\partial \theta_\alpha^s}, \quad P \in \hat{\mathcal{F}}^p.$$

We denote

$$\mathrm{Der}(\hat{\mathcal{A}})^D = \mathrm{Im}\, D \subset \mathrm{Der}(\hat{\mathcal{A}})^\partial.$$

The map D satisfies

$$[P, Q] = \int D_P(\tilde{Q})$$

7. Hamiltonian cohomologies

Let $P \in \hat{\mathcal{F}}^2$ be a Hamiltonian structure of hydrodynamic type, d_P be its adjoint action, i.e.

$$d_P Q = [P, Q], \quad Q \in \hat{\mathcal{F}}.$$

Then we have

$$d_P^2 = 0, \quad D_P^2 = 0,$$

Thus we have two complexs $(\hat{\mathcal{F}}, d_P)$ and $(\hat{\mathcal{A}}, D_P)$. They are in fact DGLAs, and the DGLA $(\hat{\mathcal{F}}, [\ ,\ ], d_P)$ controls the deformation theory of the Hamiltonian structure P.

The exact sequence of complexes

$$0 \longrightarrow (\hat{\mathcal{A}}/\mathbb{R}, D_P) \xrightarrow{\partial} (\hat{\mathcal{A}}, D_P) \xrightarrow{\int} (\hat{\mathcal{F}}, d_P) \longrightarrow 0,$$

helps us to compute the cohomology groups of $(\hat{\mathcal{F}}, d_P)$ from that of $(\hat{\mathcal{A}}, D_P)$, and we have

$$H^p_{d>0}(\hat{\mathcal{A}}, D_P) = 0, \quad H^p_{d>0}(\hat{\mathcal{F}}, d_P) = 0, \quad p \geq 0.$$

7. Hamiltonian cohomologies

Bihamiltonian cohomologies

Let (P_1, P_2) be a bihamiltonian structure, i.e., they are Hamiltonian structures, and $[P_1, P_2] = 0$. Denote

$$D_1 = D_{P_1},\ D_2 = D_{P_2},\ d_1 = d_{P_1},\ d_2 = d_{P_2}.$$

Then we have

$$D_1 D_2 + D_2 D_1 = 0,\ d_1 d_2 + d_2 d_1 = 0,$$

so we obtain two double complexes $(\hat{\mathcal{A}}, D_1, D_2)$ and $(\hat{\mathcal{F}}, d_1, d_2)$.

Let $(C, \partial_1, \partial_2)$ be the double complex $(\hat{\mathcal{A}}, D_1, D_2)$ or $(\hat{\mathcal{F}}, d_1, d_2)$, its bihamiltonian cohomology is defined as

$$BH_d^p(C, \partial_1, \partial_2) = \frac{C_d^p \cap \operatorname{Ker} \partial_1 \cap \operatorname{Ker} \partial_2}{C_d^p \cap \operatorname{Im}(\partial_1 \partial_2)}.$$

$$BH_d^2(\hat{\mathcal{F}}) = 0,\ \text{if } d \geq 2,\ d \neq 3, \quad BH_3^2(\hat{\mathcal{F}}) \cong \bigoplus_{i=1}^{n} C^\infty(\mathbb{R}), \quad BH_{\geq 5}^3(\hat{\mathcal{F}}) = 0.$$

8. Variational forms on the infinite jet bundle

Let $\mathcal{E}$ be the space of differential forms on $J^\infty(\hat{M})$, locally it is the space

$$\mathcal{E} = \wedge^* \left(\mathrm{Span}_{\hat{\mathcal{A}}} \{\delta\theta_\alpha^s, \delta u^{\alpha,s} \mid \alpha = 1, \cdots, n,\ s \geq 0\}\right).$$

Here we adopt Deligne's sign rule for these variational forms, for example

$$\theta_i^s \delta\theta_j^t = -\delta\theta_j^t \theta_i^s, \quad \delta\theta_i^s \delta\theta_j^t = \delta\theta_j^t \delta\theta_i^s, \quad \delta u^{i,s} \delta\theta_j^t = -\delta\theta_j^t \delta u^{i,s}, \quad \delta u^{i,s} \delta u^{j,t} = -\delta u^{j,t} \delta u^{i,s}.$$

The space of 1-forms is given by

$$\Omega = \left\{ \sum_{\alpha;s\geq 0} g_{\alpha,s} \delta u^{\alpha,s} + h_s^\alpha \delta\theta_\alpha^s \mid g_{\alpha,s}, h_s^\alpha \in \hat{\mathcal{A}} \right\}.$$

It has the gradations induced from that of $\hat{\mathcal{A}}$, and denote its homogeneous subspaces by

$$\Omega_d, \quad \Omega^p, \quad \Omega_d^p.$$

8. Variational forms on the infinite jet bundle

A derivation $X \in \mathrm{Der}(\hat{\mathcal{A}})$ induces an action on Ω by the Lie derivative

$$\mathcal{L}_X = \delta \circ \iota_X + \iota_X \circ \delta,$$

where the de Rham differential is given by

$$\delta = \sum_{\alpha; s \geq 0} \delta u^{\alpha,s} \frac{\partial}{\partial u^{\alpha,s}} + \delta\theta_\alpha^s \frac{\partial}{\partial \theta_\alpha^s}.$$

We still denote by ∂_x the action induced by $\partial_x \in \mathrm{Der}(\hat{\mathcal{A}})$. We define

$$\bar{\Omega} = \Omega / \partial_x \Omega,$$

and represent its elements in the form $\int f$ with $f \in \Omega$.

8. Variational forms on the infinite jet bundle

Relation of $\mathrm{Der}(\hat{\mathcal{A}})^\partial$ with $\bar{\Omega}$

An element $X \in \mathrm{Der}(\hat{\mathcal{A}})^\partial$ has the expression

$$X = \sum_{\alpha; s\geq 0} \partial_x^s \left(X(u^\alpha)\right) \frac{\partial}{\partial u^{\alpha,s}} + \partial_x^s \left(X\left(\theta_\alpha\right)\right) \frac{\partial}{\partial \theta_\alpha^s}.$$

The canonical symplectic structure $\varpi = \sum_\alpha \delta u^\alpha \wedge \delta\theta_\alpha$ on $\hat{M}$ induces a map

$$\Phi : \mathrm{Der}(\hat{\mathcal{A}})^\partial \to \bar{\Omega}; \quad \Phi : X \mapsto \int \iota_\varpi \left(X|_{\hat{M}}\right) = \int \sum_i X(u^\alpha)\delta\theta_\alpha - X(\theta_\alpha)\delta u^\alpha.$$

There is a natural cochain map

$$\delta : \quad \hat{\mathcal{F}} \to \bar{\Omega}, \quad \delta X = (-1)^{p-1}\Phi(D_X), \quad X \in \hat{\mathcal{F}}^p$$

It corresponds to the embedding

$$\mathrm{Der}(\hat{\mathcal{A}})^D \hookrightarrow \mathrm{Der}(\hat{\mathcal{A}})^\partial.$$

9. Variational Hamiltonian cohomologies

If $X \in \mathrm{Der}(\hat{\mathcal{A}})^\partial$, then $\mathcal{L}_X$ also induces an action on $\bar{\Omega}$ which we still denote by $\mathcal{L}_X$.

Let P be a Hamiltonian structure of hydrodynamic type, we will use $\tilde{D}_P$ to denote the action $\mathcal{L}_{D_P}$ on the space Ω and $\bar{\Omega}$. By using the identity

$$\Phi\left([D_P,\ X]\right) = \mathcal{L}_{D_P}\Phi(X).$$

we know that $\tilde{D}_P \circ \tilde{D}_P = 0$, so $\tilde{D}_P$ is a differential on the spaces Ω and $\bar{\Omega}$.

Definition

The variational Hamiltonian cohomology of Ω (and of $\bar{\Omega}$, respectively) is defined to be the cohomology of the complex $(\Omega^\bullet, \tilde{D}_P)$ (and of $(\bar{\Omega}^\bullet, \tilde{D}_P)$, respectively) given by

$$H^p_d(\Omega, \tilde{D}_P) = \frac{\Omega^p_d \cap \ker \tilde{D}_P}{\Omega^p_d \cap \mathrm{Im}\, \tilde{D}_P}; \quad H^p_d(\bar{\Omega}, \tilde{D}_P) = \frac{\bar{\Omega}^p_d \cap \ker \tilde{D}_P}{\bar{\Omega}^p_d \cap \mathrm{Im}\, \tilde{D}_P}.$$

9. Variational Hamiltonian cohomologies

The variational bihamiltonian cohomologies

Let (P_1, P_2) be a semisimple bihamiltonian structure of hydrodynamic type. We will use $\tilde{D}_1$ and $\tilde{D}_2$ to denote $\tilde{D}_{P_1}$ and $\tilde{D}_{P_2}$ respectively. Then we have $\tilde{D}_i\tilde{D}_j + \tilde{D}_j\tilde{D}_i = 0$ for $i, j = 1, 2$.

Definition

The variational bihamiltonian cohomology for (P_1, P_2) is defined to be the following groups:

$$VBH^p_d(\Omega, \tilde{D}_1, \tilde{D}_2) = \frac{\Omega^p_d \cap \ker \tilde{D}_1 \cap \ker \tilde{D}_2}{\Omega^p_d \cap \mathrm{Im}\, \tilde{D}_1\tilde{D}_2};$$

$$VBH^p_d(\bar{\Omega}, \tilde{D}_1, \tilde{D}_2) = \frac{\bar{\Omega}^p_d \cap \ker \tilde{D}_1 \cap \ker \tilde{D}_2}{\bar{\Omega}^p_d \cap \mathrm{Im}\, \tilde{D}_1\tilde{D}_2}.$$

9. Variational Hamiltonian cohomologies

Theorem

We have the following triviality of the variational bihamiltonian cohomology groups:

$$VBH^1_{d\geq 2}(\bar{\Omega}, \tilde{D}_1, \tilde{D}_2) = 0, \quad VBH^2_{d\geq 4}(\bar{\Omega}, \tilde{D}_1 \tilde{D}_2) = 0.$$

Proposition

Let (P_1, P_2) be a semisimple bihamiltonian structure of hydrodynamic type, and let $X \in \mathrm{Der}(\hat{\mathcal{A}})^0_1$ commute with ∂_x, D_{P_1} and D_{P_2}. Then for any deformation $(\tilde{P}_1, \tilde{P}_2)$ of (P_1, P_2), there exists a unique $\tilde{X} \in \mathrm{Der}(\hat{\mathcal{A}})^0_{d\geq 1}$ with leading term given by X such that $\tilde{X}$ commutes with ∂_x, $D_{\tilde{P}_1}$ and $D_{\tilde{P}_2}$.

9. Variational Hamiltonian cohomologies

Remark

The above-mentioned result is a generalization of the fact that a bihamiltonian vector field is uniquely determined by its leading term. This generalized version can be applied to the case when the flows are not in the space $\mathrm{Der}(\hat{\mathcal{A}})^D$. *Typical examples of such kind of flows arise when we consider Virasoro symmetries of deformations of the Principal Hierarchies.*

Thanks