Symmetries of Lundgren-Monin-Novikov equations

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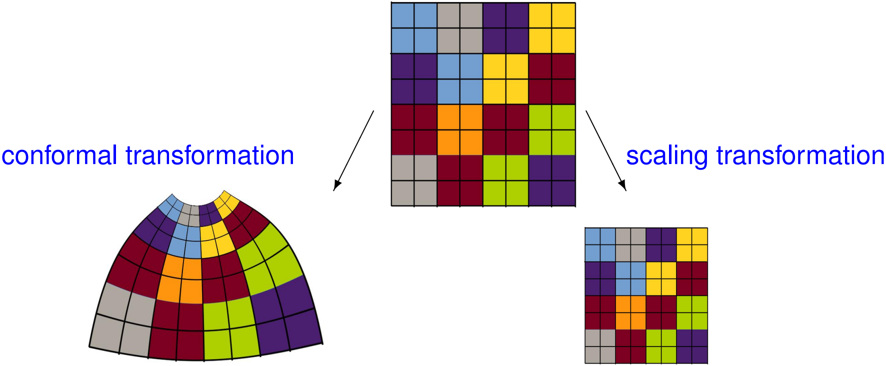
Introduction - conformal transformation

Conformal transformations: coordinate transformations $\boldsymbol{x} \rightarrow \boldsymbol{x}^*$ which preserve the angles between any two vectors.



Introduction - conformal transformation

Scaling transformation : $\boldsymbol{x}^* = \boldsymbol{x} + \epsilon \boldsymbol{x} + \mathcal{O}(\epsilon^2)$

Conformal transformation: $\boldsymbol{x}^* = \boldsymbol{x} + \epsilon \boldsymbol{\xi}(\boldsymbol{x}) + \mathcal{O}(\epsilon^2)$
(rescaling factor made an analytic function of position)

Conservation of angles - in 2D Cauchy-Riemann conditions:

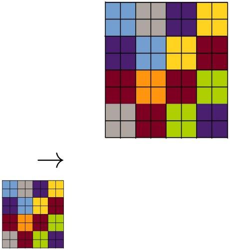
$$\frac{\partial \xi^1}{\partial x} = \frac{\partial \xi^2}{\partial y}, \qquad \frac{\partial \xi^1}{\partial y} = -\frac{\partial \xi^2}{\partial x}$$

Introduction - conformal transformation

Conformal transformation: global form

$$\text{Example:} \qquad z^* = \frac{az+b}{cz+d}, \qquad z = x + iy$$

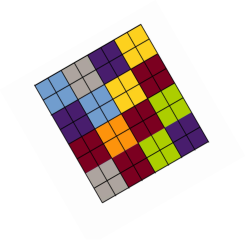
$c = 0$, $a = 1$, $d = 1$,
translation: $z^* = z + b$



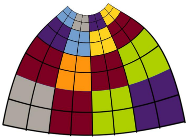
$a = const$, $c = 0$,
$b = 0$, $d = 1$,
scaling: $z^* = az$

Introduction - conformal invariance

$c = 0$, $a = \mathrm{e}^{i\theta}$, $d = 1$,
rotation: $z^* = \mathrm{e}^{i\theta} z$



$a = 0$
special conformal transformation:
$z^* = \frac{b}{cz+d}$



Conformal invariance

Polyakov [Soviet Physics JETP, 30, 1969]
Considered problems of phase transitions (Ising model)

Key observation!
At a critical point an invariance larger than scale invariance exists: the conformal invariance

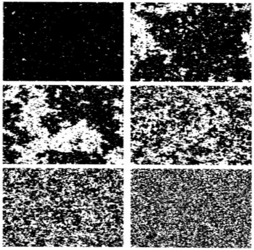


figure: http://www.faculty.umassd.edu/j.wang/vp/

Conformal invariance

Polyakov [Soviet Physics JETP, 30, 1969]

Close to a critical point clusters of same-spin particles are formed

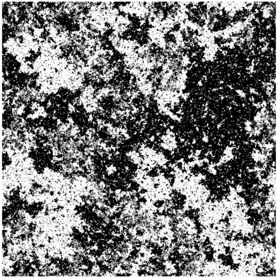


figure: http://wiki.swarma.net/index.php/ISING

Conformal invariance in turbulence

Polyakov [Nucl.Phys. B396, 1993]

Assumed that the conformal invariance may exist within certain range of scales in 2D turbulence

Similarities between the turbulent system and a system close to critical point

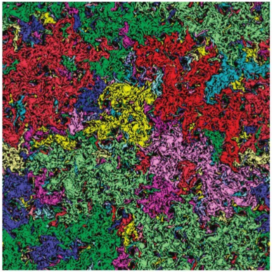


Figure: 2D turbulence, vorticity clusters

Bernard D., Boffetta G. , Celani A., Falkovich G. Conformal invariance in two-dimensional turbulence, Nature Physics, **2**, 2006

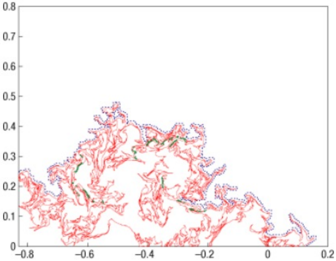
Conformal invariance in turbulence

Bernard et al. [Nature Physics, **2**, 2006]

Zero-vorticity lines belong to a class of stochastic Loewner evolution or SLE curves that can be mapped into a Brownian walk.

Their statistics are conformally invariant.

1. Bernard D., Boffetta G. , Celani A., Falkovich G. Conformal invariance in two-dimensional turbulence, Nature Physics, **2**, 2006
2. Bernard D., Boffetta G. , Celani A., Falkovich G., Inverse Turbulent Cascades and Conformally Invariant Curves, PRL, **98**, 2007
3. Falkovich G., Conformal invariance in hydrodynamic turbulence, Russian Math. Surveys, **62**, 2007
4. Falkovich G., Symmetries of the turbulent state, J. Phys. A:Math. Theor., **42**, 2009



Conformal invariance in turbulence

Falkovich G. [Russian Math. Surveys, **62**, 2007]

Can the conformal invariance of some statistical properties of turbulence be explained by analyzing the algebraic structure of the equations of hydrodynamics?

Invariance of the probability measure e.g. of vorticity:

$$f_1(\omega_1, z_1, t)d\omega_1 = f_1(\omega_1^*, z_1^*, t)d\omega_1^*,$$

f_1 - one-point probability density function
ω_1 - sample space of vorticity

Lundgren-Monin-Novikov hierarchy of equations

We consider ensemble of vorticity fields $\boldsymbol{\Omega}(\boldsymbol{x}, t)$ in 2D and for $\nu = 0$

$$\frac{\partial \Omega}{\partial t} + \boldsymbol{u} \cdot \nabla \Omega = 0, \quad \text{where,} \quad \boldsymbol{u} = \int \mathrm{d}\boldsymbol{x}' \boldsymbol{\Omega}(\boldsymbol{x}', t) \times \frac{\boldsymbol{x} - \boldsymbol{x}'}{2\pi |\boldsymbol{x} - \boldsymbol{x}'|^2},$$

Statistical description with probability density functions:

$$\frac{\partial f_1}{\partial t} + \frac{\partial J^1}{\partial x_1^1}, + \frac{\partial J^2}{\partial x_1^2} = 0,$$

$$\frac{\partial f_2}{\partial t} + \frac{\partial I^1}{\partial x_1^1} + \frac{\partial I^2}{\partial x_1^2} + \frac{\partial I^3}{\partial x_2^1} + \frac{\partial I^4}{\partial x_2^2} = 0, \quad \ldots \text{ with } f_1 = \int f_2 \mathrm{d}\omega_1$$

where

$$J^1 = -\frac{1}{2\pi} \int \mathrm{d}\mathbf{x}' \mathrm{d}\omega' \omega' \frac{x_1^2 - x'^2}{|\boldsymbol{x}_1 - \boldsymbol{x}'|^2} f_2, \quad J^2 = \frac{1}{2\pi} \int \mathrm{d}\mathbf{x}' \mathrm{d}\omega' \omega' \frac{x_1^1 - x'^1}{|\boldsymbol{x}_1 - \boldsymbol{x}'|^2} f_2, \ldots$$

Lie group analysis

Lie group analysis - provides set of symmetry transformations of a considered equation, i.e transformations of variables which do not change the form of equation

$$\mathcal{D}(t, \boldsymbol{x}, \boldsymbol{u}, \boldsymbol{u}^{(1)}, \ldots) \Longleftrightarrow \mathcal{D}(t^*, \boldsymbol{x}^*, \boldsymbol{u}^*, \boldsymbol{u}^{(1)*}, \ldots)$$

Lie group analysis of integro-differential equations
Grigoriev et al. *Symmetries of Integro-differential Equations: with Applications in Mechanics and Plasma Physics*, 2010

Lie group analysis of LMN equations

We observe:

- Symmetry transformations of local part (continuity form) contain conformal subgroup

  Fushchych and Boyko [*J.Nonl. Math. Phys.* **1-2** 1997]

- This conformal group is broken in LMN hierarchy for velocity

  Wacławczyk et al [*Phys. Rev. E* **90**, 2014]

  Wacławczyk, Grebenev, Oberlack [*J. Phys. A: Math. Theor.* **50** 2017]

What about LMN system for vorticity in 2D?

- Conformal symmetry broken for the first equation of the LMN
- Conformal symmetry is retained under the condition $\omega_1 = 0$
- Condition $\omega_1 = 0$ is satisfied on the characteristic lines which are iso-vorticity lines. Grebenev et al J.Phys. A: Math. Theor. 52 2019

Lie group analysis of LMN equations

Characteristic of the first LMN equation
Friedrich et al. [*C. R. Physique* **13**, 2012]

$$\begin{aligned}
\frac{d}{ds}t(s) &= 1, \\
\frac{d}{ds}\mathbf{X}(s) &= \langle \boldsymbol{u}(\boldsymbol{x},t)|\omega,\boldsymbol{x}\rangle|_{\{\omega,\boldsymbol{x}\}=\{\Omega(s),\mathbf{X}(s)\}}, \\
\frac{d}{ds}\Omega(s) &= 0,
\end{aligned}$$

Initial conditions: $\mathbf{X}(0,\{\omega^0,\boldsymbol{x}^0\}) = \boldsymbol{x}^0$, $\Omega(0,\{\omega^0,\boldsymbol{x}^0\}) = \omega^0$

Characteristics are lines of a constant $\Omega = \omega^0$!

Lie group analysis of LMN equations

Calculated symmetry transformations under condition $\omega_1 = 0$:

$$
\begin{aligned}
x_1^{1*} &= U(x^1, x^2), \quad x_1^{2*} = V(x^1, x^2) \\
\omega_1^* &= \left(U_{,1}^2 + V_{,1}^2\right)\omega_1 = F_{,z}^2(z)\omega_1 = 0 \\
f_1^* &= \left(U_{,1}^2 + V_{,1}^2\right)^{-1} f_1 = F_{,z}^{-2}(z) f_1,
\end{aligned}
$$

$U(x^1, x^2)$, $V(x^1, x^2)$ - conjugate harmonic functions,
$F(z) = U(x^1, x^2) + iV(x^1, x^2)$, $F(z)$ is a conformal map

Cauchy-Riemann conditions are satisfied!

Probability measure $\mu(t, \boldsymbol{x}, \omega_1) = f_1(t, \boldsymbol{x}, \omega_1)\mathrm{d}\omega_1$ is invariant

under conformal transformations of $\boldsymbol{x}_1$! Grebenev, Wacławczyk, Oberlack [*J. Phys. A: Math. Theor.* 50 2017] *for the inviscid and forcing-free LMN equations*

Outlook

2017 --2019

- We addressed the question about presence of CG in the underlying equations of hydrodynamics.
- Lie group analysis of the first equation from the Ludgren-Monin-Novikov hierarchy was performed.
- First equation and the probability measure of vorticity in $2D$ is invariant under CG on the characteristics $\omega = const$.
- Characteristics are isovorticity lines.

Perspectives

- What changes if we consider $\nu \neq 0 \ldots$
- $\ldots$ and forcing term?
- Can CG be observed in other systems?

The LMN equations for 2d scalar fields

LMN hierarchy extended by viscosity and large-scale friction

We consider the first equation from the LMN chain for $2d$ scalar fields ϕ under viscosity and a large-scale friction. With this, the flow is kept in a statistically steady state.

For the inviscid case we prove the CG invariance of the 1-point statistics of the zero-isolines $\boldsymbol{x}(l)$ of a scalar field, i.e. the CG invariance of the probability $f_1(\boldsymbol{x}(l), \phi_1)d\phi_1$ that a random curve $\boldsymbol{x}(l)$ passes through the point $\boldsymbol{x}_1$ with $\phi_1 = 0$ for $l = l_1$. Then we demonstrate that the CG invariance is broken if the viscous term is included into the equation.

Conformal invariance of the 1-point statistics of the zero-isolines of $2d$ scalar fields in inverse turbulent cascades

M Wacławczyk, V N Grebenev, M Oberlack 2021 Phys. Rev. Fluids **6***(8)*

$f_1(t, \boldsymbol{x}, \phi)|_{\phi=0}$– equation of the LMN chain

$$\frac{\partial f_1(\boldsymbol{x}, \phi, t)}{\partial t} + \nabla_{\boldsymbol{x}} \cdot [\langle \mathbf{u}(\boldsymbol{x}) | \boldsymbol{x}, \phi \rangle f_1(\boldsymbol{x}, \phi, t)] = \mathcal{F}, \tag{1}$$

$\langle \mathbf{u}(\boldsymbol{x}) | \boldsymbol{x}, \phi \rangle$ is the conditional velocity field with components

$$\langle u(\boldsymbol{x}, t) | \boldsymbol{x}, \phi, t \rangle = -\int \mathrm{d}\mathbf{x}' \mathrm{d}\phi' \phi' \frac{y - y'}{|\boldsymbol{x} - \boldsymbol{x}'|^m} f_2(\phi, \boldsymbol{x}, \phi', \boldsymbol{x}', t) f_1^{-1}(\boldsymbol{x}, \phi, t) \tag{2}$$

$$\langle v(\boldsymbol{x}, t) | \boldsymbol{x}, \phi, t \rangle = \int \mathrm{d}\mathbf{x}' \mathrm{d}\phi' \phi' \frac{x - x'}{|\boldsymbol{x} - \boldsymbol{x}'|^m} f_2(\phi, \boldsymbol{x}, \phi', \boldsymbol{x}', t) f_1^{-1}(\boldsymbol{x}, \phi, t). \tag{3}$$

The random forcing, large-scale friction and viscous transport are encoded in the term $\mathcal{F}$ in Eq. (1)

$$\mathcal{F} = \alpha\frac{\partial}{\partial\phi}(\phi f_1) - \nu\frac{\partial}{\partial\phi}\left(\int \mathrm{d}\phi' \int \mathrm{d}\boldsymbol{x}'\delta(\boldsymbol{x}-\boldsymbol{x}')\phi'\triangle^2_{\boldsymbol{x}'}f_2\right) + Q(\boldsymbol{x},\boldsymbol{x})\frac{\partial^2 f_1}{\partial\phi^2}, \tag{4}$$

$Q(\boldsymbol{x},\boldsymbol{x})$ is the amplitude of the forcing which was calculated from the two-point correlation function $Q(\boldsymbol{x},\boldsymbol{x}')$,

The normalisation conditions

$$\int \mathrm{d}\phi f_1 = 1, \quad \int \mathrm{d}\phi' f_2 = f_1. \tag{5}$$

The separation condition

$$\lim_{|\boldsymbol{x}-\boldsymbol{x}'|\to\infty} f_2(\boldsymbol{x},\phi,\boldsymbol{x}',\phi',t) = f_1(\boldsymbol{x},\phi,t) f_1(\boldsymbol{x}',\phi',t), \tag{6}$$

and the coincidence condition

$$\lim_{|\boldsymbol{x}-\boldsymbol{x}'|\to 0} f_2(\boldsymbol{x},\phi,\boldsymbol{x}',\phi',t) = \delta(\phi'-\phi) f_1(\boldsymbol{x},\phi,t) \tag{7}$$

The statistics of zero-lines $\boldsymbol{x}(l)$ of the scalar fields Φ and invariance of their probability measure $f_1(\boldsymbol{x}, \phi) d\phi|_{\phi=0}$ for $\boldsymbol{x} \in \boldsymbol{x}(l)$, under the transformations of the curve $\boldsymbol{x}(l)$, as there is numerical evidence which indicate that such lines, which are boundaries of large clusters, are conformally invariant. The measure conformally invariant if it is invariant with respect to a conformal transformation $F : D \mapsto D^*$, that is, $\mu_D(\boldsymbol{x}) = \mu_{D^*}(\boldsymbol{\lambda}(\boldsymbol{x}))$.

Symmetry transformations

$$x^* = \mathcal{X}(x, y, a), \quad y^* = \mathcal{Y}(x, y, a), \tag{8}$$

$$x'^* = \mathcal{X}(x, y, a) + \gamma^{1/3}(\boldsymbol{x}) \left[(x' - x)\frac{\partial \mathcal{X}}{\partial x} + (y' - y)\frac{\partial \mathcal{X}}{\partial y} \right], \tag{9}$$

$$y'^* = \mathcal{Y}(x, y, a) + \gamma^{1/3}(\boldsymbol{x}) \left[(x' - x)\frac{\partial \mathcal{Y}}{\partial x} + (y' - y)\frac{\partial \mathcal{Y}}{\partial y} \right], \tag{10}$$

$$\phi^* = \gamma^{-1}(\boldsymbol{x})\phi, \tag{11}$$

$$\phi'^* = \gamma^{-m/6}(\boldsymbol{x})\phi', \tag{12}$$

$$f_1^* = \gamma(\boldsymbol{x}) f_1, \tag{13}$$

$$f_2^* = \gamma^{(6+m)/6}(\boldsymbol{x}) f_2. \tag{14}$$

For the inviscid case we prove the CG invariance of the 1-point statistics of the zero-isolines $\boldsymbol{x}(l)$ of a scalar field, i.e. the CG invariance of the probability $f_1(\boldsymbol{x}(l), \phi)d\phi$ that a random curve $\boldsymbol{x}(l)$ passes through the point $\boldsymbol{x}$ with $\phi = 0$ for $l = l_1$. The group G acts conformally with respect to the spatial variable $\boldsymbol{x}$ and transforms invariantly only "a fragment"of the first LMN equation i.e. the $f_1(\boldsymbol{x}, \omega, t)|_{\omega=0}$ – equation. CG invariance is broken if the viscous term is included into the equation.

$f_n(\boldsymbol{x}_{(1)}, \omega_{(1)}, \dots, \boldsymbol{x}_{(n)}, \omega_{(n)}, t)$– equation of the inviscid forcing-free LMN chain

$$
\begin{aligned}
\frac{\partial f_n}{\partial t} + \frac{\partial}{\partial x^{1}_{(1)}} \int \mathrm{d}\mathbf{x}_{(n+1)} \mathrm{d}\omega_{(n+1)} \omega_{(n+1)} \alpha^{1}(\mathbf{r}_{(1,n+1)}) f_{n+1} \\
+ \frac{\partial}{\partial x^{2}_{(1)}} \int \mathrm{d}\mathbf{x}_{(n+1)} \mathrm{d}\omega_{(n+1)} \omega_{(n+1)} \alpha^{2}(\mathbf{r}_{(1,n+1)}) f_{n+1} \\
+ \dots \quad \dots + \\
\frac{\partial}{\partial x^{1}_{(n)}} \int \mathrm{d}\mathbf{x}_{(n+1)} \mathrm{d}\omega_{(n+1)} \omega_{(n+1)} \alpha^{1}(\mathbf{r}_{(n,n+1)}) f_{n+1} \\
+ \frac{\partial}{\partial x^{2}_{(n)}} \int \mathrm{d}\mathbf{x}_{(n+1)} \mathrm{d}\omega_{(n+1)} \omega_{(n+1)} \alpha^{2}(\mathbf{r}_{(n,n+1)}) f_{n+1} = 0,
\end{aligned}
\tag{1}
$$

$n = 1, \dots n$

$$\boldsymbol{r}_{(sd)} = \boldsymbol{x}_{(s)} - \boldsymbol{x}_{(d)},\ \alpha^{1}(\boldsymbol{r}_{(s,d)}) = -\frac{1}{2\pi}\frac{r^{2}_{(s,d)}}{|\boldsymbol{r}_{(s,d)}|^{2}},\ \alpha^{2}(\boldsymbol{r}_{(s,d)}) = \frac{1}{2\pi}\frac{r^{1}_{(s,d)}}{|\boldsymbol{r}_{(s,d)}|^{2}}. \tag{2}$$

Here

$$\alpha^{1}(\mathbf{r}_{(1,n+1)}) = \alpha^{1}_{(1,n+1)}, \quad \alpha^{2}(\mathbf{r}_{(1,n+1)}) = \alpha^{2}_{(1,n+1)}. \tag{3}$$

The normalisation conditions

$$\int d\omega_{(1)} \dots d\omega_{(n)} f_n = 1, \quad \int d\omega_{(n+1)} f_{n+1} = f_n; \tag{4}$$

The separation condition

$$\lim_{|\boldsymbol{x}_{(n)} - \boldsymbol{x}_{(n+1)}| \to \infty} f_{n+1}(\boldsymbol{x}_{(1)}, \omega_{(1)}, \dots, \boldsymbol{x}_{(n+1)}, \omega_{(n+1)}, t) = \tag{5}$$

$$f_1(\boldsymbol{x}_{(n+1)}, \omega_{(n+1)}, t) \cdot f_n(\boldsymbol{x}_{(1)}, \omega_{(1)}, \dots, \boldsymbol{x}_{(n)}, \omega_{(n)}, t) \tag{6}$$

The coincidence condition

$$\lim_{|\boldsymbol{x}_{(n)} - \boldsymbol{x}_{(n+1)}| \to 0} f_{n+1} = \delta(\omega_{(n+1)} - \omega_{(n)}) f_n. \tag{7}$$

The characteristics of 1) satisfy to the dynamical system which describes the evolution of Lagrangian particles according to the conditional velocity field

$$\frac{d}{ds}t(s) = 1, \tag{8}$$

$$\frac{d}{ds}\boldsymbol{X}_{n(j)}(s) = \langle \boldsymbol{u}(\boldsymbol{x}_{(j)}, t) | \omega_{(l)}, \boldsymbol{x}_{(l)} \rangle \Big|_{\{\omega_{(l)}, \boldsymbol{x}_{(l)}\} = \left\{ \Omega_{(l)}(s), \boldsymbol{X}_{n(l)}(s) \right\}}, \tag{9}$$

$$\frac{d}{ds}\Omega_{n(j)}(s) = 0, \tag{10}$$

$j = 1, \ldots, n,\ l = 1, \ldots, n.$

Equation (1) along the characteristics (1) reads

$$\frac{\mathrm{d}f_n(s)}{\mathrm{d}s} =$$

$$-f_n(s) \sum_{j=1}^{n} \left[\frac{\partial}{\partial x_{(j)}} \langle u(\boldsymbol{x}_{(j)}, t) | \left\{\omega_l, \boldsymbol{x}_{(l)}\right\}\rangle + \frac{\partial}{\partial y_{(j)}} \langle v(\boldsymbol{x}_{(j)}, t) | \left\{\omega_{(l)}, \boldsymbol{x}_{(l)}\right\}\rangle\right]\Bigg|_{\{\omega_{(l)}, \boldsymbol{x}_{(l)}\}}$$

where

$$\langle u(\boldsymbol{x}_{(j)}, t) | \left\{\omega_{(l)}, \boldsymbol{x}_{(l)}\right\}\rangle = \tag{12}$$

$$\int \mathrm{d}\boldsymbol{x}_{(n+1)} \mathrm{d}\omega_{(n+1)} \omega_{(n+1)} \alpha^1 (\boldsymbol{x}_{(l)} - \boldsymbol{x}_{(n+1)}) \frac{f_{n+1}(\boldsymbol{x}_{(n+1)}, \omega_{(n+1)}, \{\boldsymbol{x}_{(l)}, \omega_{(l)}\}, t)}{f_n(\{\boldsymbol{x}_{(l)}, \omega_{(l)}\}, t)}$$

$$\langle v(\boldsymbol{x}_{(j)}, t) | \left\{\omega_{(l)}, \boldsymbol{x}_{(l)}\right\}\rangle = \tag{13}$$

$$\int \mathrm{d}\boldsymbol{x}_{(n+1)} \mathrm{d}\omega_{(n+1)} \omega_{(n+1)} \alpha^2 (\boldsymbol{x}_{(j)} - \boldsymbol{x}_{(n+1)}) \frac{f_{n+1}(\boldsymbol{x}_{(n+1)}, \omega_{(n+1)}, \{\boldsymbol{x}_{(l)}, \omega_{(l)}\}, t)}{f_n(\{\boldsymbol{x}_{(l)}, \omega_{(l)}\}, t)}$$

Equation (9) in the complex variables takes the form

$$X^{1}_{n(j),s} + iX^{2}_{n(j),s} = \frac{1}{2}\int(\bar{z}_{(n+1)}\mathsf{d}\mathsf{z}_{(\mathsf{n}+1)} + z_{(n+1)}\mathsf{d}\bar{\mathsf{z}}_{(\mathsf{n}+1)})\mathsf{d}\varphi_{(\mathsf{n}+1)}\omega_{(\mathsf{n}+1)}\mathsf{d}\omega_{(\mathsf{n}+1)}\alpha^{1}_{(\mathsf{j},\mathsf{n}+1)})\mathsf{f}_{\mathsf{n}+1}\mathsf{f}_{\mathsf{n}}^{-1} + \frac{i}{2}\int(\bar{z}_{(n+1)}\mathsf{d}\mathsf{z}_{(\mathsf{n}+1)} + z_{(n+1)}\mathsf{d}\bar{\mathsf{z}}_{(\mathsf{n}+1)})\mathsf{d}\varphi_{(\mathsf{n}+1)}\omega_{(\mathsf{n}+1)}\mathsf{d}\omega_{(\mathsf{n}+1)}\alpha^{2}_{(\mathsf{j},\mathsf{n}+1)})\mathsf{f}_{\mathsf{n}+1}\mathsf{f}_{\mathsf{n}}^{-1}. \tag{14}$$

Equation (14) describes the dynamics of jth Lagrangian particle of the jth component of nth dimensional complex space $C^n = C_{(1)} \times \cdots \times C_{(n)}$, $C_{(j)} \simeq C$. Here $Z_{(n)j}(s) = X^{1}_{n(j),s} + iX^{2}_{n(j),s}$ is a curve given on $C_{(j)}$.

Symmetry transformations

Symmetry operator $S_{(n)}$ takes the form

$$\begin{aligned} S_{(n)} &= \xi^{1}\frac{\partial}{\partial x^{1}_{(1)}} + \xi^{2}\frac{\partial}{\partial x^{2}_{(1)}} + \xi^{3}\frac{\partial}{\partial \omega_{(1)}} + \dots \\ &+ \xi^{3n-2}\frac{\partial}{\partial x^{1}_{(n)}} + \xi^{3n-1}\frac{\partial}{\partial x^{2}_{(n)}} + \xi^{3n}\frac{\partial}{\partial \omega_{(n)}} + \eta^{1}_{(n)}\frac{\partial}{\partial f_n} \\ &+ \xi^{3n+1}\frac{\partial}{\partial x^{1}_{(n+1)}} + \xi^{3n+2}\frac{\partial}{\partial x^{2}_{(n+1)}} + \xi^{3n+3}\frac{\partial}{\partial \omega_{(n+1)}} + \eta^{2}_{(n)}\frac{\partial}{\partial f_{n+1}}. \end{aligned} \quad (15)$$

Here the coordinates of the infinitesimal operator read

$$
\begin{aligned}
\xi^{1} &= c^{11}(\boldsymbol{x}_{(1)})x^{1}_{(1)} + c^{12}(\boldsymbol{x}_{(1)})x^{2}_{(1)} + d^{1}(\boldsymbol{x}_{(1)}) && (16)\\
\xi^{2} &= c^{21}(\boldsymbol{x}_{(1)})x^{1}_{(1)} + c^{22}(\boldsymbol{x}_{(1)})x^{2}_{(1)} + d^{2}(\boldsymbol{x}_{(1)}), && (17)\\
\xi^{3} &= \left[6c^{11}(\boldsymbol{x}_{(j)})\right]\omega_{(1)}, && (18)\\
& \quad \ldots \qquad \ldots \qquad \ldots \\
\xi^{3k-2} &= c^{11}(\boldsymbol{x}_{(k)})x^{1}_{(k)} + c^{12}(\boldsymbol{x}_{(k)})x^{2}_{(k)} + d^{1}(\boldsymbol{x}_{(k)}), && (19)\\
\xi^{3k-1} &= c^{21}(\boldsymbol{x}_{(k)})x^{1}_{(k)} + c^{22}(\boldsymbol{x}_{(k)})x^{2}_{(k)} + d^{2}(\boldsymbol{x}_{(k)}), && (20)\\
\xi^{3k} &= \left[6c^{11}(\boldsymbol{x}_{(k\neq j)})\right]\omega_{(k)}, && (21)\\
& \quad \ldots \qquad \ldots \qquad \ldots \\
\xi^{3n-2} &= c^{11}(\boldsymbol{x}_{(n)})x^{1}_{(n)} + c^{12}(\boldsymbol{x}_{(n)})x^{2}_{(n)} + d^{1}(\boldsymbol{x}_{(n)}), && (22)\\
\xi^{3n-1} &= c^{21}(\boldsymbol{x}_{(n)})x^{1}_{(n)} + c^{22}(\boldsymbol{x}_{(n)})x^{2}_{(n)} + d^{2}(\boldsymbol{x}_{(n)}), && (23)
\end{aligned}
$$

$$
\begin{aligned}
\xi^{3n} &= \left[6c^{11}(\boldsymbol{x}_{(n)})\right]\omega_{(n)}, && (24)\\
\xi^{3n+1} &= c^{11}(\boldsymbol{x}_{(j)})x^{1}_{(n+1)} + c^{12}(\boldsymbol{x}_{(j)})x^{2}_{(n+1)} + d^{1}(\boldsymbol{x}_{(j)}), && (25)\\
\xi^{3n+2} &= c^{21}(\boldsymbol{x}_{(j)})x^{1}_{(n+1)} + c^{22}(\boldsymbol{x}_{(j)})x^{2}_{(n+1)} + d^{2}(\boldsymbol{x}_{(j)}),, && (26)\\
\xi^{3n+3} &= \left[2c^{11}(\boldsymbol{x}_{(j)})\right]\omega_{(n+1)}, && (27)
\end{aligned}
$$

$k = 1, \ldots, n$, $c^{11} = c^{22}$, $c^{12} = -c^{21}$, c^{11}, c^{12} — arbitrary harmonic functions,

$$
\begin{aligned}
d^{1}_{1}(\boldsymbol{y}) &= 2c^{11}(\boldsymbol{y}) - c^{11}_{1}(\boldsymbol{y})y^{1} - c^{12}_{1}(\boldsymbol{y})y^{2}, && (28)\\
d^{1}_{2}(\boldsymbol{y}) &= -c^{11}_{2}(\boldsymbol{y})y^{1} - c^{12}_{2}(\boldsymbol{y})y^{2}, && (29)\\
d^{2}_{1}(\boldsymbol{y}) &= c^{12}_{1}(\boldsymbol{y})y^{1} - c^{11}_{1}(\boldsymbol{y})y^{2}, && (30)\\
d^{2}_{2}(\boldsymbol{y}) &= 2c^{11}(\boldsymbol{y}) + c^{12}_{2}(\boldsymbol{y})y^{1} - c^{22}_{2}(\boldsymbol{y})y^{2}. && (31)
\end{aligned}
$$

The operator $S_{(n)}$ generates a Lie group G_j:

$$z^*_{(1)} = F_{(1)}(z_{(1)}), \tag{32}$$

$$\omega^*_{(1)} = |F_{(1)z_{(2)}}|^2 \omega_{(1)}, \tag{33}$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$z^*_{(k)} = F_{(1)}(z_{(k)}), \tag{34}$$

$$\omega^*_{(k)} = |F_{(1)z_{(k\neq j)}}|^2 \omega_{(k)}, \tag{35}$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$z^*_{(n+1)} = F'(z_{(j)}, z_{(n+1)}, a), \tag{36}$$

$$\omega^*_{(n+1)} = |F_{(1)z_{(j)}}|^{2/3} \omega_{(n+1)}, \tag{37}$$

$$f^*_n = |F_{(1)z_{(1)}}|^{-2} \ldots |F_{(1)z_{(n)}}|^{-2} f_n, \tag{38}$$

$$f^*_{n+1} = |F_{(1)z_{(j)}}|^{-8/3} \ldots |F_{(1)z_{(k\neq j)}}|^{-2} f_{n+1}, \tag{39}$$

$F_{(1)} : C \mapsto C$ is the conformal mapping defined on C.

The characteristic equations describe the averaged dynamics of Lagrangian particles. Their motion is defined on $\mathbf{D} = \mathbf{D}_{(1)} \times \cdots \times \mathbf{D}_{(n)}$:

$$\mathbf{D}_{(j)} = \{C_{(j)}, C_{(n+1)}, X_{(j)s}, \omega_{(j+1)}, \mathrm{d}z_{(n+1)}, \mathrm{d}\bar{z}_{(n+1)}, \mathrm{d}\omega_{(n+1)}, f_n, f_{n+1}\}. \tag{40}$$

The group of transformations G defined on $\mathbf{D}$ is the direct product of Lie groups $G_{(j)}$ i.e. $G = G_{(1)} \times \cdots \times G_{(n)}$ and G is again a Lie group.

With this, the group G invariant transforms the characteristics (8–10). Moreover, $f_n(\boldsymbol{x}_{(1)}, \omega_{(1)}, \ldots, \boldsymbol{x}_{(n)}, \omega_{(n)}, t)$–equation is invariant transformed along the zero-vorticity characteristics. The group G breaks this equation along other characteristics.