Global surfaces of section

for Kupka-Smale Reeb vector fields of 3-dimensional closed manifolds

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Joint work with Gonzalo Contreras

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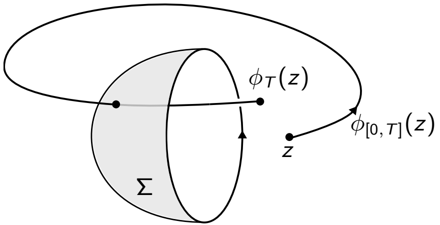
A global surface of section is a compact immersed surface $\Sigma \looparrowright N$ such that:

- $\partial\Sigma$ is tangent to X,
- $\mathrm{int}(\Sigma)$ is embedded in $N \setminus \partial\Sigma$ and transverse to X,
- for some $T > 0$, any orbit segment $\phi_{[0,T]}(z)$ intersects Σ.



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$$\tau : \mathrm{int}(\Sigma) \to (0, T], \quad \tau(z) = \min \left\{ t > 0 \;\middle|\; \phi_t(z) \in \Sigma \right\}$$

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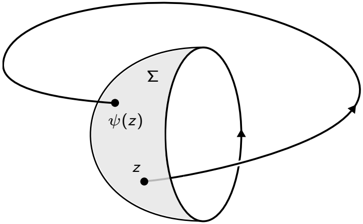
First return map:

$$\psi : \mathrm{int}(\Sigma) \to \mathrm{int}(\Sigma), \quad \psi(z) = \phi_{\tau(z)}(z)$$



Global surfaces of section of Reeb flows

(N,λ) closed contact 3-manifold, X Reeb vector field

$\lambda \wedge d\lambda$ volume form
$\lambda(X) \equiv 1, \;\; d\lambda(X,\cdot) \equiv 0$

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Example. (M, g) Riemannian surface
$N = SM$,
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- $d\lambda|_{\Sigma}$ is an area form, and therefore $\partial\Sigma \neq \varnothing$

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Remarks.

- $d\lambda|_\Sigma$ is an area form, and therefore $\partial\Sigma \neq \varnothing$
- The first return map $\psi : \mathrm{int}(\Sigma) \to \mathrm{int}(\Sigma)$ preserves $d\lambda$, and indeed $\psi^*\lambda = \lambda + d\tau$

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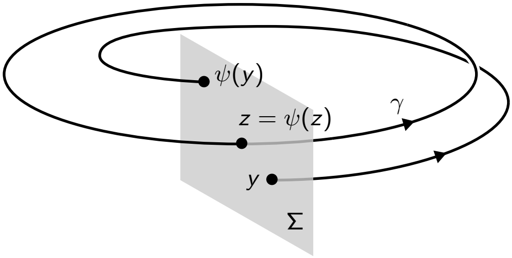
  An application: any contact convex 3-spheres has either exactly two or infinitely many closed Reeb orbits

Closed Reeb orbits

$\Sigma \subset N$ cross section at a closed Reeb orbit γ

$\psi : \Sigma \to \Sigma$, $\psi(y) = \phi_{\tau(y)}(y)$ first-return map



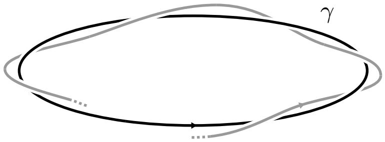
The Floquet multipliers of γ are the eigenvalues of $d\psi(z)$:

$$\sigma(d\psi(z)) = \{\lambda, \lambda^{-1}\} \subset S^1 \cup \mathbb{R}.$$

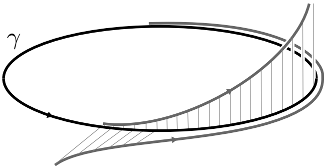
Closed Reeb orbits

The closed Reeb orbit γ is

- elliptic when its Floquet multipliers are in $S^1 \subset \mathbb{C}$



- hyperbolic when its Floquet multipliers are in $\mathbb{R} \setminus \{1, -1\}$



- non-degenerate when its Floquet multipliers are not complex roots of 1.

The Kupka-Smale condition

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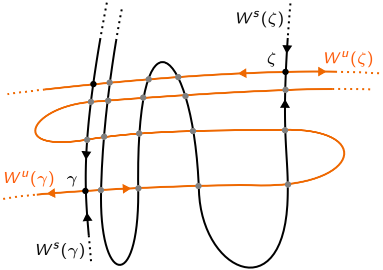

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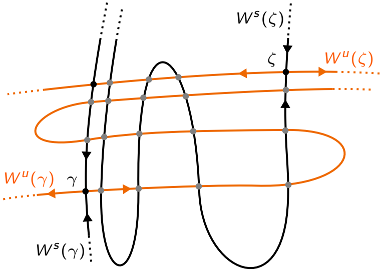

Remarks. Kupka-Smale holds for:

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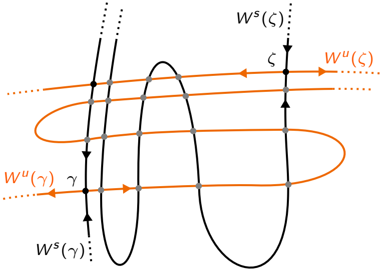

Remarks. Kupka-Smale holds for:

- (Robinson) the Reeb vector field of a C^∞ generic contact form on a closed 3-manifold;
- (Contreras-Paternain) the geodesic vector field of a C^∞ generic Riemannian metric on a closed surface.

Main theorem

Theorem (Contreras-Mazzucchelli). *Any Kupka-Smale Reeb vector field on a closed 3-manifold admits a global surface of section.*

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Remarks.

- Colin-Dehornoy-Hryniewicz-Rechtman recently provided an alternative proof for Corollary (i), based on Irie's equidistribution theorem.

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Remarks.

- The existence of a global surface of section with non-degenerate boundary is a C^1-open condition in the vector field (thus a C^2-open condition in the contact form)
- The Kupka-Smale condition in the above theorem is only required on a suitable finite collection of hyperbolic periodic orbits.

Towards global surfaces of section

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$$\tau_- : \mathrm{int}(\Sigma) \to [-\infty, 0), \quad \tau_+ : \mathrm{int}(\Sigma) \to (0, \infty],$$
such that:

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  (i) Every Reeb orbit intersects Σ at least once.
  (ii) There is a finite collection $K := \gamma_1 \cup ... \cup \gamma_k \subset \partial\Sigma$ of hyperbolic closed Reeb orbits such that
$$\tau_-^{-1}(-\infty) \subset W^u(K), \quad \tau_+^{-1}(\infty) \subset W^s(K).$$

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Remark. If $K = \varnothing$ then Σ is a global surface of section.

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By pushing forward Σ with the flow, one obtains a broken book decomposition of (N^3, λ):

- A family $\mathcal{F}$ of surfaces of section (pages) with the same boundary of Σ, whose interiors foliate $N \setminus \partial\Sigma$.

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- Near any connected component $\gamma \subset K$ (broken binding component) the pages arrive as follows:

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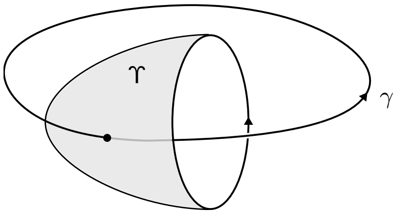
Lemma (Colin-Dehornoy-Rechtman) *If there exists another surface of section* Υ *such that* $\partial\Upsilon \cap \partial\Sigma = \varnothing$ *and whose interior* $\mathrm{int}(\Upsilon)$ *intersects some* $\gamma \subset K$*, then there exists a new broken book decomposition with broken binding* $K \setminus \{\gamma\}$*.*



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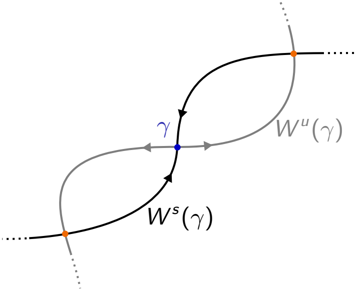

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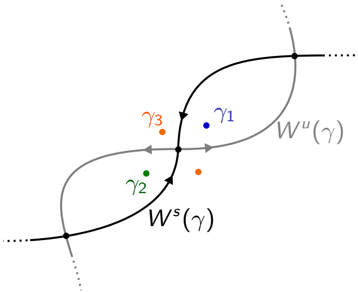

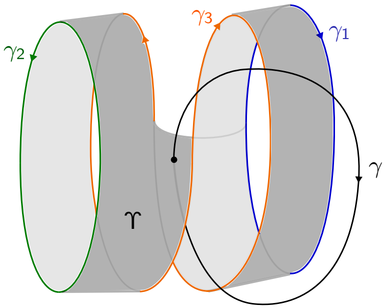

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In order to find a surface of section, we have to show that there is always some broken binding component $\gamma \subset K$ with transverse homoclinics in all separatrices.

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Theorem (Contreras-Mazzucchelli) *Let* (N, λ) *be a Kupka-Smale closed contact 3-manifold, with a broken book decomposition. Any broken binding component has homoclinics in all separatrices.*

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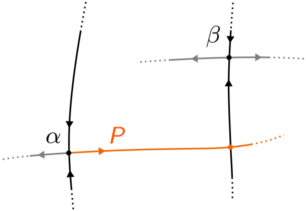

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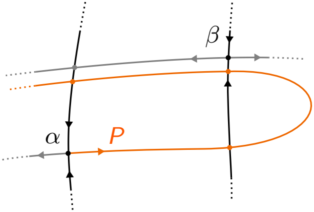

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- For every $\alpha \subset K$, every connected component $P \subset W^u(\alpha) \setminus \alpha$ satisfies $P \cap W^s(K) \neq \varnothing$.
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$$\gamma_{-1} \rightsquigarrow \gamma \rightsquigarrow \gamma_1$$

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$$\gamma_{-2} \rightsquigarrow \gamma_{-1} \rightsquigarrow \gamma \rightsquigarrow \gamma_1 \rightsquigarrow \gamma_2$$

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Theorem (Contreras-Mazzucchelli) *Let* (N, λ) *be a Kupka-Smale closed contact 3-manifold, with a broken book decomposition. Any broken binding component has homoclinics in all separatrices.*

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- This theorem provides a confirmation of the C^2-stability conjecture for Riemannian geodesic flows:

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A characterization of Anosov Reeb flows

Theorem (Contreras-Mazzucchelli). *Let (N, λ) be a closed contact 3-manifold such that:*

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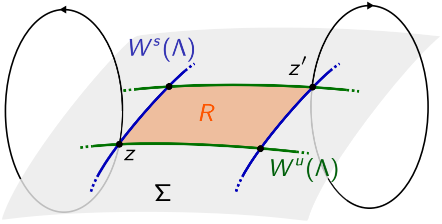
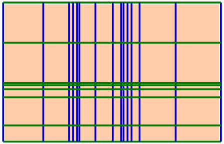
- We fix a small heteroclinic rectangle $R \subset \mathrm{int}(\Sigma)$:



$z, z' \in \Lambda \cap \mathrm{Per}(X)$

A characterization of Anosov Reeb flows

- $R \cap \left(W^s(\Lambda) \cup W^u(\Lambda)\right)$ is compact and connected



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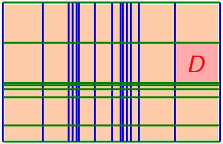

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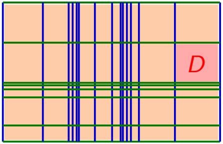

- $D \subset R \setminus \big(W^s(\Lambda) \cup W^u(\Lambda)\big)$ connected component
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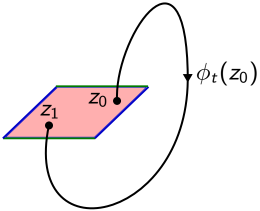

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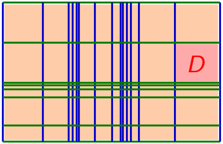

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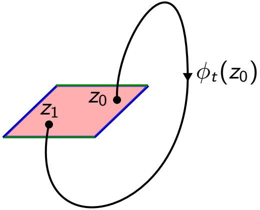

- We extend the map $z_0 \mapsto z_1$ to a smooth return map $\psi : \mathrm{int}(\Sigma) \to \mathrm{int}(\Sigma)$.

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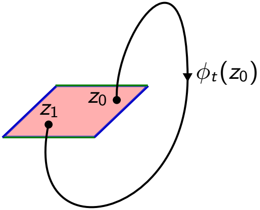

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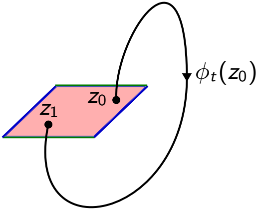

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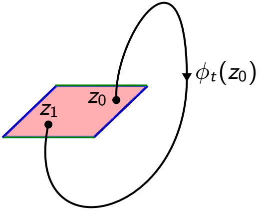

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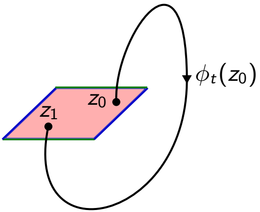

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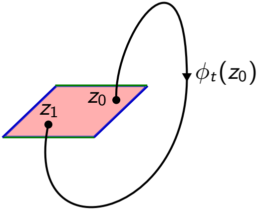

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- $\psi|_D : D \to D$ preserves the area form $d\lambda|_D$.
- (Brower translation theorem) ψ has a fixed point z.
- Thus $z \in D \cap \mathrm{Per}(X)$. But $D \cap \mathrm{Per}(X) \subset D \cap \Lambda = \varnothing$. $\square$

Thank you for your attention!