Locally finite groups with bounded centralizer chains

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Abstract. The c-dimension of a group G is the maximal length of a chain of nested centralizers in G. We prove that a locally finite group of finite c-dimension k has less than $5k$ nonabelian composition factors.

Keywords: locally finite group, nonabelian simple group, lattice of centralizers, c-dimension.

Introduction

Let G be a group and $C_G(X)$ be the centralizer of a subset X of G. Since $C_G(X) < C_G(Y)$ if and only if $C_G(C_G(X)) > C_G(C_G(Y))$, it follows that the minimal and the maximal conditions for centralizers are equivalent. Thus the length of every chain of nested centralizers in a group with the minimal condition for centralizers is finite. If a uniform bound for the lengths of chains of centralizers of a group G exists, then we refer to maximal such length as c-dimension of G following [1]. The same notion is also known as the height of the lattice of centralizers. It is worth to observe that the class of groups of finite c-dimension includes abelian groups, torsion-free hyperbolic groups, linear groups over fields and so on. In addition, it is closed under taking subgroups and finite direct products, but the c-dimension of a homomorphic image of a group from this class is not necessary finite.

In 1979 R. Bryant and B. Hartley [2] proved that a periodic locally soluble group with the minimal condition for centralizers is soluble. In 2009 E. I. Khukhro published the paper [3], where, in particular, he proved that a periodic locally soluble group of finite c-dimension k has the derived length bounded in terms of k. The same paper contains the conjecture attributed to A. V. Borovik, which asserts that the number of nonabelian composition factors of a locally finite group of finite c-dimension k is bounded in terms of k. The purpose of our work is to prove this conjecture.

Theorem. *Let G be a locally finite group of c-dimension k. Then the number of nonabelian composition factors of G is less than $5k$.*

§ 1. Preliminaries

Given a locally finite group G, denote by $\eta(G)$ the number of nonabelian composition factors of G.

The following well-known fact (see, for example, [4, Corollary 3.5]) helps us to derive the theorem from the corresponding statement for finite groups.

Lemma 1.1. *If G is a locally finite locally soluble simple group, then G is cyclic.*

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Recall that the factor group of a finite group G by its soluble radical R is an automorphism group of a direct product of nonabelian simple groups. Thus, if the socle $Soc(G/R)$ is a direct product of nonabelian simple groups $S_1, S_2, \dots, S_n$, then G/R is a subgroup of the semidirect product $(\mathrm{Aut}(S_1) \times \mathrm{Aut}(S_2) \times \cdots \times \mathrm{Aut}(S_n)) \leftthreetimes Sym_n$, where Sym_n permutes $S_1, S_2, \dots, S_n$. By the classification of the finite simple groups, the group of outer automorphisms of a finite simple group is soluble. Therefore, every nonabelian composition factor of G is either a composition factor of $Soc(G/R)$, or a composition factor of the corresponding subgroup of Sym_n.

Next three lemmas give an upper bound for the number of nonabelian composition factors of a subgroup of Sym_n. We denote by $\mu(G)$ the degree of the minimal faithful permutation representation of a finite group G.

Lemma 1.2 ([5], Theorem 2)**.** *Let G be a finite group. Let $\mathfrak{L}$ be a class of finite groups closed under taking subgroups, homomorphic images and extensions. If N is the maximal normal $\mathfrak{L}$-subgroup of G, then $\mu(G) \geqslant \mu(G/N)$.*

Lemma 1.3 ([6], Theorem 3.1)**.** *Let $S_1, S_2, \dots, S_r$ be simple groups. Then $\mu(S_1 \times S_2 \times \cdots \times S_r) = \mu(S_1) + \mu(S_2) + \cdots + \mu(S_r)$.*

Lemma 1.4. *If G is a subgroup of a symmetric group Sym_n, then $\eta(G) \leqslant (n-1)/4$.*

Proof. We proceed by induction on n. If R is the soluble radical of G, then Lemma 1.2 implies that $\mu(G/R)$ does not exceed $\mu(G)$. Hence, we may assume that the soluble radical of G is trivial. Let the socle $Soc(G)$ of G be the direct product of nonabelian simple groups $S_1, S_2, \dots, S_l$. It follows from Lemma 1.3 that $l \leqslant n/5$. Again G is a subgroup of the semidirect product $(\mathrm{Aut}(S_1) \times \mathrm{Aut}(S_2) \times \cdots \times \mathrm{Aut}(S_l)) \leftthreetimes Sym_l$. By inductive hypothesis, $\eta(G) \leqslant n/5 + (n/5 - 1)/4 = (n-1)/4$.

Remark. The group Sym_n, where $n = 5^k$ with $k \geqslant 1$, contains a subgroup G isomorphic to the permutation wreath product $(\dots((Alt_5 \wr Alt_5) \wr Alt_5)\dots)$, where the wreath product is applied $k-1$ times. We have $\eta(G) = \frac{5^k-1}{5-1} = \frac{n-1}{4}$.

The following lemma is a key for bounding the number of composition factors of $Soc(G/R)$ for a finite group G.

Lemma 1.5 ([3], Lemma 3)**.** *If an elementary abelian p-group E of order p^n acts faithfully on a finite nilpotent p'-group Q, then there exists a series of subgroups $E = E_0 > E_1 > E_2 > \cdots > E_n = 1$ such that all inclusions $C_Q(E_0) < C_Q(E_1) < \cdots < C_Q(E_n)$ are strict.*

As usual, $O_p(G)$ stands for the largest normal p-subgroup of a finite group G, while $O_{p'}(G)$ denotes the largest normal p'-subgroup of G. If a series of commutator subgroups of a group G stabilizes, then we denote by $G^{(\infty)}$ the last subgroup of this series. A quasisimple group is a perfect central extension of a nonabelian simple group. The layer $E(G)$ is the subgroup of G generated by all subnormal quasisimple subgroups of G, the latter are called components of G. Recall that the layer is a central product of components of G.

§ 2. Proofs

Proposition 2.1. *Let G be a finite group of c-dimension k. Then $\eta(G) < 5k$.*

Proof. Let R be the soluble radical of G. If P is a Sylow subgroup of R, then $G/R \simeq N_G(P)/(R \cap N_G(P))$, so nonabelian composition factors of $N_G(P)$ and G coincide. On the other hand, c-dimension of $N_G(P)$ as a subgroup of G is at most k. Therefore, we may assume that $N_G(P) = G$ for every Sylow subgroup P of R, i. e. that R is nilpotent.

Obviously, we suppose that $R \neq G$. Put $\overline{G} = G/R$. The socle $\overline{L}$ of $\overline{G}$ is the direct product of nonabelian simple groups $S_1, S_2, \ldots, S_n$. As observed in preliminaries, the group $\overline{G}/\overline{L}$ is an extension of a normal soluble subgroup by a subgroup of the symmetric group Sym_n. By Lemma 1.4, an arbitrary subgroup of Sym_n has less than $n/4$ nonabelian composition factors. Thus, it is sufficient to show that $\eta(\overline{L}) = n \leqslant 4k$. In particular, we may assume that G coincides with L, the preimage of $\overline{L}$ in G, and nonabelian composition factor of G are the groups $S_1, S_2, \ldots, S_n$.

Let $K = C_G(R)$. The normal subgroup $\overline{K} = KR/R$ of $\overline{G}$ is a direct product of nonabelian simple group. Without loss of generality, we may suppose that $\overline{K} = S_1 \times S_2 \times \ldots \times S_l$ for some $1 \leq l \leq n$. For $i = 1, \ldots, l$ denote by K_i the preimage of S_i in K. Then subgroup $H_i = K_i^{(\infty)}$ is normal in K and is a perfect central extension of S_i, so it is a component of K. Therefore, if $E(K)$ is the layer of K, then $KR = E(K)R$ and $E(K)$ is a central product of $H_1, H_2, \ldots, H_l$. Hence $\eta(K) = \eta(E(K)) = l$. Since $[H_i, H_j] = 1$ for $i \neq j$, all inclusions $C_{E(K)}(H_1) < C_{E(K)}(H_1H_2) < \cdots < C_{E(K)}(H_1H_2 \ldots H_l)$ are strict. Thus, $l \leqslant k$.

Let P be a Sylow p-subgroup of G and $\overline{P}$ be the image of P in $\overline{G}$. Since $O_p(R) \leq C_G(O_{p'}(R))$, the action of P on $O_{p'}(R)$ by conjugation induces the action of $\overline{P}$ on $O_{p'}(R)$. Given a prime p, define the set $\mathcal{F}_p$ as follows: a subgroup S_i of $\overline{G}$ lies in $\mathcal{F}_p$ whenever there is an element g of order p in S_i acting faithfully on $O_{p'}(R)$. Lemma 1.5 yields that $|\mathcal{F}_p| \leqslant k$ for every prime p. On the other hand, if S_i does not lie in $\mathcal{F}_p$, then S_i is a subgroup of $C_G(O_{p'}(R))R/R$. It follows from the classification of finite simple groups that the order of every nonabelian finite simple group is an even number which is a multiple of 3 or 5. Since $R = O_2(R) \times O_{2'}(R)$, every S_i either belongs to $\mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_5$, or is a subgroup of $\overline{K} = C_G(R)R/R$. Thus, $\eta(G) \leq |\mathcal{F}_2| + |\mathcal{F}_3| + |\mathcal{F}_5| + \eta(K) \leq 4k$, as required.

Proof of the theorem. Now G is locally finite group. Assume $\eta(G) \geq 5k$. Let $\{G_i\}_{i \in I}$ be a composition series of G, where G_i is a proper subgroups of G_j for $i < j$. Let $S_1, S_2, \ldots, S_{5k}$ be pairwise distinct nonabelian composition factors of G. By Lemma 1.1 every locally finite nonabelian simple group contains a finite insoluble subgroup. Thus, we may choose finite subsets $X_1, X_2, \ldots, X_{5k}$ of G such that the image of X_i in S_i generates an insoluble group. Suppose that H is the finite subgroup of G generated by the union of the sets $X_1, X_2, \ldots, X_{5k}$. Then $\{G_i \cap H\}_{i \in I}$ is a subnormal series of H having at least $5k$ insoluble factors. This contradicts Proposition 2.1. The theorem is proved.

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