Recognition by Spectrum of $L_{16}(2^m)$

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Abstract. In this paper we prove that the simple linear groups $L_{16}(2^m)$ $(m \geq 1)$ over fields of characteristic 2 are recognizable by the sets of their element orders.

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Given a finite group $G$, denote by $\omega(G)$ the spectrum of $G$, i.e., the set of its element orders. A group $G$ is said to be recognizable by spectrum (briefly, recognizable) if every finite group $H$ with $\omega(H) = \omega(G)$ is isomorphic to $G$. Since a finite group with a non-trivial normal soluble subgroup is not recognizable [8, Corollary 4], the recognition problem for simple and almost simple groups is of prime interest.

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At present there is a vast list of finite and almost finite groups with solved recognition problem. The most recent version of this list is presented in [5, Table 1], and references to some new results can be found in [10]. We mention some results concerning the recognition of simple linear groups over fields of characteristic 2. The following groups were proved to be recognizable: $L_2(2^m)$ with $m \geq 2$ (see [7]) and $L_3(2^m)$ with $m \geq 1$ (see [6]). All these groups have disconnected prime graphs and a certain property of these groups, called quasi-recognizability, was proved with applying the Gruenberg–Kegel theorem on groups with disconnected prime graphs (see [13]). A finite non-abelian simple group $S$ is said to be quasi-recognizable if every finite group $H$ with the same spectrum as $S$ contains a unique non-abelian composition factor and this factor is isomorphic to $S$.

Recent results [10] and [12] allow considering of recognition problem for groups with connected prime graphs. In this way in a recent paper [11], the simple linear groups $L_n(2^m)$, where $n = 2^l \geq 32$, were proved to be recognizable. The cases when $n$ is a power of 2 and equals 4, 8, and 16 were left out of consideration since the corresponding groups require special methods. The point is that the less rank of a Lie type group we investigate, the more simple groups we have to consider proving quasi-recognizability. In this paper, we establish recognizability for the case $n = 16$.

**Theorem.** The simple linear groups $L = L_{16}(2^m) (m \geq 1)$ are recognizable by spectrum.

1 **Preliminaries**

Let $G$ be a finite group, and $\omega(G)$ be its spectrum. The set $\omega(G)$ is ordered by the divisibility relation and we denote by $\mu(G)$ the set of its elements that are maximal under this relation. If $p$ is a prime, then the $p$-period of $G$ is the maximal power of $p$ that belongs to $\omega(G)$.

Let $\pi(G)$ be the set of all prime divisors of the order of $G$. On the set $\pi(G)$, we define a graph with the following adjacency relation: vertices $p$ and $r$ in $\pi(G)$ are joined by an edge if and only if $pr \in \omega(G)$. This graph is called the Gruenberg–Kegel graph or prime graph of $G$ and denoted by $GK(G)$ (see [13]). Guided by the given graph conception, we say that prime divisors $p$ and $r$ of the order of $G$ are adjacent if vertices $p$ and $r$ are joined by an edge in $GK(G)$. Otherwise, primes $p$ and $r$ are said to be non-adjacent.

The set of vertices of a graph is called independent if vertices of this set are pairwise non-adjacent. The cardinality of an independent set with maximal number of vertices is usually called the independence number of the graph. Denote by $t(G)$ the independence number of the graph $GK(G)$ of $G$. By analogy, we denote by $t(2, G)$ the maximal number of vertices in independent sets of $GK(G)$ containing the vertex 2. We call this number the 2-independence number.

The following result concerning connection between the structure of a finite group and the properties of its prime graph is proved in [10].

**Lemma 1.** [10] Let $G$ be a finite group satisfying two conditions:

(a) There exist three primes in $\pi(G)$ which are pairwise non-adjacent in $GK(G)$, that is, $t(G) \geq 3$. 

(b) There exists an odd prime in \( \pi(G) \) which is non-adjacent to 2 in \( \text{GK}(G) \), that is, \( t(2, G) \geq 2 \).

Then there exists a finite non-abelian simple group \( S \) such that \( S \leq G/K \leq \text{Aut}(S) \) for a maximal normal soluble subgroup \( K \) of \( G \). Furthermore, \( t(S) \geq t(G) - 1 \) and one of the following statements holds:

1. \( S \simeq \text{Alt}_7 \) or \( L_2(q) \) for some odd \( q \) and \( t(S) = t(2, S) = 3 \).
2. For every prime \( p \) in \( \pi(G) \) non-adjacent to 2 in \( \text{GK}(G) \), the Sylow \( p \)-subgroup of \( G \) is isomorphic to the Sylow \( p \)-subgroup of \( S \). In particular, \( t(2, S) \geq t(2, G) \).

To apply this result, we use the values of independence and 2-independence numbers of finite simple groups calculated in [12].

We use the following number-theoretic notation. If \( n \) is a natural number, then \( \pi(n) \) is the set of prime divisors of \( n \). If \( p \in \pi(n) \), then \( n_p \) is the maximal \( p \)-power that divides \( n \). By \( \lfloor x \rfloor \) we denote the integer part of \( x \). If \( q \) is a natural number, \( r \) is an odd prime and \( (q, r) = 1 \), then by \( e(r, q) \) we denote the smallest natural number \( m \) such that \( q^m \equiv 1 \pmod{r} \). Given an odd \( q \), put \( e(2, q) = 1 \) if \( q \equiv 1 \pmod{4} \) and put \( e(2, q) = 2 \) if \( q \equiv -1 \pmod{4} \).

The following number-theoretic result is of fundamental importance for investigations on the structure of prime graphs of finite simple groups of Lie type.

**Lemma 2.** [14] Let \( q \) be a natural number greater than 1. Then for every natural number \( l \), there exists a prime \( r \) such that \( e(r, q) = l \), except for the following cases:

1. \( l = 6 \) and \( q = 2 \);
2. \( l = 2 \) and \( q = 2^m - 1 \) for some natural number \( m \).

The prime \( r \) with \( e(r, q) = l \) is called a primitive prime divisor of \( q^l - 1 \). If \( q \) is fixed, we denote by \( r_l \) any primitive prime divisor of \( q^l - 1 \) (obviously, \( q^l - 1 \) can have more than one primitive prime divisor).

**Lemma 3.** [4, Lemma 1] Let \( G \) be a finite group, \( K \lhd G \), and \( G/K \) be a Frobenius group with kernel \( F \) and a cyclic complement \( C \). If \( (|F|, |K|) = 1 \) and \( F \) does not lie in \( KC_G(K)/K \), then \( r \cdot |C| \in \omega(G) \) for some prime divisor \( r \) of \( |K| \).

**Lemma 4.** Let \( q \) be a power of a prime \( p \) and let \( r_{2n-2} \) be a primitive prime divisor of \( q^{2n-2} - 1 \). The group \( 2D_n(q) \), where \( (n, q) \neq (4, 2) \), contains a Frobenius subgroup whose kernel is an elementary abelian \( p \)-group and complement is cyclic of order \( r_{2n-2} \).

**Proof.** By [3, Part 8, A], there is a parabolic subgroup in \( 2D_n(q) \) whose Levi radical \( U \) is an elementary abelian \( p \)-group and whose Levi subgroup contains \( 2D_{n-1}(q) \). The group \( 2D_{n-1}(q) \) contains an element \( x \) of order \( r_{2n-2} \). Since \( pr_{2n-2} \notin \omega(2D_{n-1}(q)) \) (see [12, Proposition 3.1]), the element \( x \) acts on \( U \) regularly. Thus, \( U \cdot \langle x \rangle \) is a desired Frobenius group.

**Lemma 5.** [11, Proposition 1] Let \( L = L_n(q) \), where \( n = 2^m \geq 4 \) and \( q = 2^k \geq 2 \). Let \( G \) be a finite group and \( K \) be its non-trivial normal soluble subgroup satisfying \( L \leq G/K \leq \text{Aut}(L) \). Then \( \omega(G) \not\subseteq \omega(L) \).
Lemma 6. [11, Proposition 2] Let $L = L_n(q)$, where $n \geq 10$, $q = 2^k \geq 2$, and $(q - 1, n) = 1$. Suppose that $L < G \leq \text{Aut}(L)$. Then $\omega(G) \not\subseteq \omega(L)$.

2 Proof of the Theorem

Let $L = L_4(2^m) = A_{15}(2^m)$, where $m \geq 1$. By [12, §8], we have $t(L) = 8$ and $t(2, L) = 3$. Furthermore, by [9, Proposition 0.5], the 2-period of $L$ is equal to 16.

Let $G$ be a finite group with $\omega(G) = \omega(L)$ and $K$ be the maximal normal soluble subgroup of $G$. By Lemma 1, there is a finite non-abelian simple group $S$ such that $S \leq G/K \leq \text{Aut}(S)$, moreover, $t(S) \geq t(G) + 1$ and either $t(S) = t(2, S) = 3$ or $t(2, S) \geq 2, G$. Since $t(G) = t(L) = 8$ and $t(2, G) = t(2, L) = 3$, the group $S$ must satisfy $t(S) \geq 7$ and $t(2, S) \geq 3$. By using [12, §8], we make a table of all the finite non-abelian simple groups satisfying these conditions. For every group $S$, the table shows the 2-independence number and some independent set of $S$ with maximal number of vertices among those containing the vertex 2. Furthermore, for every classical group of Lie type, the table gives the independence number as a function of Lie rank; and for sporadic groups and exceptional groups, this number is given explicitly.

The proof of quasi-recognizability relies on an case by case analysis of all possibilities for $S$ from this table. The cases of alternating groups and classical groups over fields of characteristic 2 have been considered in [11]; all of them except for the case $S \simeq L$ lead to a contradiction. We examine only the rest cases. Through this paragraph $r_1$ denotes a prime such that $e(r_1, 2^m) = l$.

Let $S = A_n^\omega(q)$ with odd $q$. Then $n/2 = (q - 1)/2 > 2$ and $t(S) = n/2$. Since $t(S) \geq t(G) + 1$ and $t(G) = 8$, we have $n'/2 \geq 7$, whence $n \geq 14$. Therefore, $S$ contains a cyclic subgroup of order $q^8 - 1$. In view of

$$(q^8 - 1)| = (q - 1)|_2(q + 1)|_2(q^2 + 1)|_2(q^4 + 1)|_2 \geq 4 \cdot 2^3 = 32,$$

we have $32 \in \omega(S)$; a contradiction.

Let $S = D_n^\omega(q)$ with odd $q$. Then $q - 1 \equiv 4 \pmod{8}$, $n' \equiv 1 \pmod{2}$ and $t(S) \leq [(3n + 4)/4]$. Since $t(S) \geq t(G) + 1$ and $t(G) = 8$, we have $(3n + 3)/4 \geq 7$, which implies $n \geq 8$. Actually, $n \geq 9$ since $n'$ is odd. Suppose that $S \not= 2D_9(q)$. Then $S$ contains the universal covering of $A_8(q)$ and thus $S$ contains an element of order $q^8 - 1$. By repeating the above argumentation, we have $32 \in \omega(S)$; a contradiction.

Let $S = 2D_9(q)$, where $q = p^k$ and $p$ is odd. Since $\rho(2, L) = \{2, r_{15}, r_{16}\}$, it follows from Lemma 1 that $r_{15}$ and $r_{16}$ divide $|S|$, and they are not adjacent to 2 in $\text{GK}(S)$. Therefore, by [12, Proposition 6.7] and Table 1 below, we have $\{e(r_{15}, q), e(r_{16}, q)\} = \{16, 18\}$. Let $r'_{14} \in \{r_{15}, r_{16}\}$ and $e(r'_{14}, q) = 16$.

Denote by $r$ a primitive prime divisor $r_{14}$ of $q^{14} - 1$. Suppose that $r$ divides $|S|$. Since the primes $r, r_{15}, r_{16}$ are pairwise non-adjacent in $\text{GK}(L)$, they are pairwise non-adjacent in $\text{GK}(S)$ as well. Hence, $e(r, q) \not\in \{16, 18\}$. As one can verify using [1, Proposition 10], the last condition implies that $4r \in \omega(S)$. On the other hand, it follows from [1, Proposition 7] that $4r \not\in \omega(L)$. Since $\omega(S) \not\subseteq \omega(L)$, we have a contradiction. Thus, $r$ does not divide $|S|$. 

Table 1. Simple groups $S$ with $t(S) \geq 7$ and $t(2, S) \geq 3$

<table>
<thead>
<tr>
<th>$S$</th>
<th>Conditions</th>
<th>$t(2, S)$</th>
<th>$\rho(2, S) \setminus {2}$</th>
<th>$t(S)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J_4$</td>
<td>none</td>
<td>3</td>
<td>${23, 29, 31, 37, 43}$</td>
<td>7</td>
</tr>
<tr>
<td>$F_4$</td>
<td></td>
<td>5</td>
<td>${29, 41, 59, 71}$</td>
<td>11</td>
</tr>
<tr>
<td>$F_2$</td>
<td></td>
<td>3</td>
<td>${31, 47}$</td>
<td>8</td>
</tr>
<tr>
<td>$\text{Alt}_n$</td>
<td>$n, n - 2$ are prime</td>
<td>3</td>
<td>${n, n - 2}$</td>
<td>—</td>
</tr>
<tr>
<td>$n \geq 47$</td>
<td>$n - 1, n - 3$ are prime</td>
<td>3</td>
<td>${n - 1, n - 3}$</td>
<td>—</td>
</tr>
<tr>
<td>$A_{n-1}(q)$</td>
<td>$2 &lt; (q - 1)</td>
<td>2 = n</td>
<td>2$</td>
<td>3</td>
</tr>
<tr>
<td>$n \geq 13$</td>
<td>$q$ even</td>
<td>3</td>
<td>${r_{n-1}, r_n}$</td>
<td>—</td>
</tr>
<tr>
<td>$A_{n-1}(q)$</td>
<td>$2 &lt; (q + 1)</td>
<td>2 = n</td>
<td>2$</td>
<td>3</td>
</tr>
<tr>
<td>$n \geq 13$</td>
<td>$q$ even, $n \equiv 0 \pmod{4}$</td>
<td>3</td>
<td>${r_{2n-2}, r_n}$</td>
<td>—</td>
</tr>
<tr>
<td>$A_{n-1}(q)$</td>
<td>$n \equiv 1 \pmod{4}$</td>
<td>3</td>
<td>${r_{2n-2}, r_n}$</td>
<td>—</td>
</tr>
<tr>
<td>$A_{n-1}(q)$</td>
<td>$n \equiv 2 \pmod{4}$</td>
<td>3</td>
<td>${r_{2n-2}, r_n/2}$</td>
<td>—</td>
</tr>
<tr>
<td>$A_{n-1}(q)$</td>
<td>$n \equiv 3 \pmod{4}$</td>
<td>3</td>
<td>${r_{(n-1)/2}, r_{2n}}$</td>
<td>—</td>
</tr>
<tr>
<td>$B_n(q)$, $n \geq 8$</td>
<td>$q$ even</td>
<td>3</td>
<td>${r_n, r_{2n}}$</td>
<td>$\frac{3n+5}{4}$</td>
</tr>
<tr>
<td>$D_n(q)$</td>
<td>$q \equiv 5 \pmod{8}$, $n \equiv 1 \pmod{2}$</td>
<td>3</td>
<td>${r_n, r_{2n-2}}$</td>
<td>$\frac{3n+1}{4}$</td>
</tr>
<tr>
<td>$n \geq 9$</td>
<td>$q$ even, $n \equiv 0 \pmod{2}$</td>
<td>3</td>
<td>${r_{n-1}, r_{2n-2}}$</td>
<td>—</td>
</tr>
<tr>
<td>$D_n(q)$</td>
<td>$n \equiv 1 \pmod{2}$</td>
<td>3</td>
<td>${r_n, r_{2n-2}}$</td>
<td>—</td>
</tr>
<tr>
<td>$D_n(q)$</td>
<td>$q \equiv 3 \pmod{8}$, $n \equiv 1 \pmod{2}$</td>
<td>3</td>
<td>${r_{2n-2}, r_{2n}}$</td>
<td>$\frac{3n+4}{4}$</td>
</tr>
<tr>
<td>$n \geq 8$</td>
<td>$q$ even, $n \equiv 0 \pmod{2}$</td>
<td>3</td>
<td>${r_{2n-2}, r_{2n}}$</td>
<td>—</td>
</tr>
<tr>
<td>$E_7(q)$</td>
<td>$q \equiv 1 \pmod{4}$</td>
<td>3</td>
<td>${r_{14}, r_{18}}$</td>
<td>8</td>
</tr>
<tr>
<td>$q \equiv 3 \pmod{4}$</td>
<td>3</td>
<td>${r_7, r_9}$</td>
<td>—</td>
<td></td>
</tr>
<tr>
<td>$q$ even</td>
<td>5</td>
<td>${r_7, r_9, r_{14}, r_{18}}$</td>
<td>—</td>
<td></td>
</tr>
<tr>
<td>$E_8(q)$</td>
<td>none</td>
<td>5</td>
<td>${r_{15}, r_{20}, r_{24}, r_{30}}$</td>
<td>11</td>
</tr>
</tbody>
</table>

Suppose first that $r \in \pi(\overline{G}/S)$ and $\alpha$ is the element in $\overline{G} \setminus S$ of the corresponding order. We may assume that $\alpha$ is a product of a diagonal automorphism $\delta$ and a field automorphism $\varphi$. The group of diagonal automorphisms of $S$ is cyclic of order 4. If $\varphi = 1$, then $|\alpha|$ divides 4, but $r$ is odd. Thus, $\varphi \neq 1$. The element $\varphi$ normalizes the subgroup of diagonal automorphisms. Since this subgroup is cyclic, we have $\delta^\varphi = \delta^l$, where $\delta$ is the image of $\delta$ in $\overline{G}/S$ and $l$ is a natural number. As $|\delta|$ divides 4, the number $l$ must equal 1, i.e., $\varphi$ centralizes $\delta$ and $|\alpha| = |\delta| \cdot |\varphi|$. Thus, $\delta = 1$. The centralizer $C$ of $\alpha$ in $S$ contains the group $^2D_8(q_0)$, where $q_0 = p^{k/r}$. Since $^2D_8(q_0)$ contains an element of order 4, the group $\overline{G}$ contains an element of order 4$r$; a contradiction.

Now suppose that $r \in \pi(K)$. Let $\tilde{G} = G/O_{r'}(K)$ and $\tilde{K} = K/O_{r'}(K)$. Then $R = O_{r'}(K) \neq 1$. Suppose that $\tilde{K} = R$. The group $S$ acts faithfully on $\tilde{K}$. Otherwise, in view of its simplicity, $S$ centralizes $\tilde{K}$, therefore $G$ contains an element of order $r_{16} \cdot r$. By Lemma 4(1), the group $S$ contains a Frobenius group $F$ whose kernel is an elementary abelian $p'$-group and complement is a cyclic group of order
$r'_{16}$. By applying Lemma 3 to the preimage of $F$ in $\tilde{G}$, we obtain $r'_{16} \cdot r \in \omega(G)$; a contradiction. Let $\tilde{K} \neq R$. There is a prime $t$ such that $T = O_t(\tilde{K}/R)$ is nontrivial. Since $O_t(\tilde{K}) = 1$, the group $T$ acts faithfully on $R$. Then $T$ acts faithfully on $\hat{R} = R/\Phi(R)$ as well, where $\Phi(R)$ is the Frattini subgroup of $R$. Denote by $\hat{G}$ the factor group $\hat{G}/\Phi(R)$. By [11, Lemma 4(3)], at least one of the primes $r_{16}$ and $r_{15}$ is non-adjacent to $t$ in $\omega(\hat{G})$. Denote this prime by $s$. Let $x$ be an element of order $s$ in $\hat{G}/\hat{R}$. Then $H = T(x)$ is a Frobenius subgroup in $\hat{G}/\hat{R}$. The preimage of $H$ in $\hat{G}$ satisfies conditions of Lemma 3, hence $G$ contains an element of order $r \cdot s$; a contradiction.

Let $S = E_7(q)$, where $q = p^k$ is odd. Recall that $r_{15}$ and $r_{16}$ divide $|S|$, and they are not adjacent to 2 in $\text{GK}(S)$. Therefore, by [12, Proposition 6.7] and Table 1, we have that the set $\{e(r_{15}, q), e(r_{16}, q)\}$ coincides with $\{14, 18\}$ if $q \equiv 1 \pmod{4}$, or $\{7, 9\}$ if $q \equiv 3 \pmod{4}$.

Suppose that $q \equiv 1 \pmod{4}$. Let $t \in \pi(S)$ and $x$ be an element of order $t$ in $S$. If $e(t, q) = 14$, then $x$ lies in the unique (up to conjugation) maximal torus of maximal period $n_{14} = (q^7 + 1)/2$; if $e(t, q) = 18$, then $x$ lies in the unique maximal torus of maximal period $n_{18} = (q^9 - q^3 + 1)(q + 1)/2(3, q + 1)$ (see [2]). The numbers $n_{14}$ and $n_{18}$ have a common prime divisor. Denote this divisor by $s$. Then $s$ is adjacent in $\text{GK}(S)$ to every prime divisor of $n_{14}$ and to every prime divisor of $n_{18}$. Hence, both numbers $r_{15}$ and $r_{16}$ are adjacent to $s$ in $\text{GK}(S)$. However, by [11, Lemma 4], there is no number in $\pi(L)$ adjacent to both primes $r_{15}$ and $r_{16}$ in $\text{GK}(L)$; a contradiction.

Suppose that $q \equiv 3 \pmod{4}$. Let $t \in \pi(S)$ and $x$ be an element of order $t$ in $S$. If $e(t, q) = 7$, then $x$ lies in the unique (up to conjugation) maximal torus of maximal period $n_7 = (q^7 - 1)/2$; if $e(t, q) = 9$, then $x$ lies in the unique maximal torus of maximal period $n_9 = (q^9 + q^3 + 1)(q - 1)/2(3, q - 1)$ (see [2]). The numbers $n_7$ and $n_9$ have a common prime divisor except when $q = 3$. Hence, if $q > 3$, then we proceed as in the previous paragraph.

Let $S = E_7(3)$. The unique primitive prime divisors of $3^7 - 1$ and $3^9 - 1$ are 1093 and 757, respectively. Therefore, for any primitive prime divisors $r_{15}$ and $r_{16}$, the set $\{r_{15}, r_{16}\}$ must coincide with $\{1093, 757\}$. Since $e(757, 2) = 756$, either $15m$ or $16m$ is divisible by 756. Hence, $m \geq 189$. The set $\pi(L)$ contains a prime $r$ with $e(r, 2) = 16m$. Since $e(r, 2) \leq r - 1$, we have $r > 16m \geq 3024$. On the other hand, $r \in \{1093, 757\}$; a contradiction.

Let $S = E_8(q)$, where $q$ is odd. Since $S$ contains a torus of order $q^8 - 1$, we have $32 \in \omega(S)$; a contradiction.

Let $S$ be $E_7(2^k)$ or $E_8(2^k)$. Choose primitive prime divisors $r_{16}$ and $r_{15}$ of $q^{16} - 1$ and $q^{15} - 1$ such that $e(r_{16}, 2) = 16k$ and $e(r_{15}, 2) = 15k$, respectively. By Lemma 1, the primes $r_{16}$ and $r_{15}$ divide the order of $S$. Put $e_{16} = e(r_{16}, 2^k)$ and $e_{15} = e(r_{15}, 2^k)$. Suppose that $e_{16} > 16m$. Then a prime $r$ with $e(r, 2) = e_{16}k$ divides the order of $S$ and does not divide the order of $L$. So $r \in \omega(S) \setminus \omega(G)$, which is impossible. Thus, $e_{16}k = 16m$. Suppose that $e_{16} > 15m$. Then $e_{15}k > 30m > 16m$ and the similar argumentation leads us to a contradiction. Thus, $e_{15}k = 15m$. Therefore, $e_{16}/e_{15} = 16/15$. On the other hand, since $r_{16}$ and $r_{15}$ are nonadjacent to 2 in $\text{GK}(S)$, by [12, Proposition 3.2], we have that $e_{16}$ and $e_{15}$ belong
to \{7, 9, 14, 18\} in the case of \(E_7(2^k)\), and to \{15, 20, 24, 30\} in the case of \(E_8(2^k)\); an easy contradiction.

Let \(S\) be a sporadic group. Choose \(r_{16}\) and \(r_{15}\) as above. By Lemma 1, the primes \(r_{16}\) and \(r_{15}\) divide the order of \(S\) and are non-adjacent to 2 in \(\text{GK}(S)\). All primes non-adjacent to 2 in \(\text{GK}(S)\) belong to the set \(\rho(2, S)\) from Table 1. Therefore, \(16k, 15k \in \epsilon(S) = \{\epsilon(l, 2) \mid l \in \rho(2, S)\}\). Since \(\epsilon(S)\) is equal to \{5, 11, 14, 28, 36\} if \(S = J_4\), to \{20, 28, 35, 58\} if \(S = F_1\), and to \{5, 23\} if \(S = F_2\), we have a contradiction.

Thus, \(S \simeq L\) and quasi-recognizability is proved. Applying Lemma 5 and Lemma 6, we complete the proof of the theorem. \(\square\)

References