RECOGNIZABILITY OF THE FINITE SIMPLE GROUPS BY SPECTRUM AND ORDER

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The spectrum $\omega(G)$ of a finite group G is the set of its element orders. In 1987 Chinese mathematician Wujie Shi conjectured that every finite simple group is uniquely determined by its spectrum and order in the class of all groups. The conjecture was written into Kourovka Notebook [1, Question 12.39] in the following form.

Shi's Question. *Is it true that a finite group and a finite simple group are isomorphic if they have the same orders and sets of element orders?*

In this paper we give a positive answer to Shi's question.

Theorem. *Let L be a finite simple group, G be a finite group such that $\omega(G) = \omega(L)$ and $|G| = |L|$. Then G is isomorphic to L.*

Since every finite abelian simple group is isomorphic to a cyclic group of prime order and uniquely determined by the order, the theorem has to be proved for simple non-abelian groups only. At the same time as formulating the question, Shi checked out his conjecture for all sporadic groups [2]. J. Bi and Shi supplied a proof for the simple linear groups [3], for the Ree and Suzuki groups [4], and for the alternating groups [5]. Next Shi handled the exceptional groups of Lie type [6], the simple unitary groups (jointly with H. Cao) [7], and finally, the simple orthogonal groups $D_{2n+1}(q)$ and ${}^2D_n(q)$ (jointly with M. Xu) [8]. Results of the works [3,4,6-8], which concern groups of Lie type, were obtained by the technique Bi and Shi developed in [3]. An attempt to transfer this technique to the case of remaining simple groups, these are $B_n(q)$, $C_n(q)$ and $D_{2n}(q)$, faced substantial obstacles relating to structural peculiarities in spectra of these groups. In the present paper we develop another approach that enables us to establish the recognizability of groups $B_n(q)$, $C_n(q)$, $D_{2n}(q)$ by spectrum and order, and thus to give an ultimate answer to Shi's question.

We study Shi's conjecture in a wider context relating to the problem of recognition just by spectrum. Two groups are said to be isospectral if they have the same spectra. We say that two distinct primes from $\omega(G)$ are adjacent if their product lies in $\omega(G)$. We write $t(G)$ to denote the maximal number of pairwise non-adjacent primes from $\omega(G)$. It follows from the results of K. Gruenberg, O. Kegel and J. Williams [9], A.S. Kondrat'ev [10], A.V. Vasil'ev and E.P. Vdovin [11,12] that a finite group which is isospectral to a simple finite group of Lie type other than $A_2(3)$, ${}^2A_3(3)$, and $B_2(3)$, has exactly one non-abelian composition factor and the spectrum of this factor satisfies a number of conditions.

Lemma 1. *Let L be a finite simple group of Lie type other than $A_2(3)$, ${}^2A_3(3)$, and $B_2(3)$, and let G be a finite group with $\omega(G) = \omega(L)$. Then there exists a simple non-abelian group S such that $S \leq \overline{G} = G/K \leq \operatorname{Aut} S$, where K is the largest normal soluble subgroup of G. Furthermore, if $r_1, \dots, r_s$, where $s \geq 3$, are distinct pairwise non-adjacent primes from $\omega(L)$, then at most one of these numbers divides $|K| \cdot |\overline{G}/S|$; in particular, $t(S) \geq t(L) - 1$.*

Proof of the theorem. In view of the results from [2–8], we can assume that L is a simple orthogonal or symplectic group. It is not hard to check that the assertion of the theorem holds when $L = B_2(3)$. For other groups by applying Lemma 1, we have $S \leq G/K \leq \mathrm{Aut}\, S$, where S is a simple non-abelian group and K is the largest normal soluble subgroup of G. Now it is sufficient to show that S is isomorphic to L. Indeed, this isomorphism and the condition $|G| = |L|$ will yield the isomorphism of G and S, and hence the required isomorphism of G and L.

The biggest difficulties arise in proving that S cannot be isomorphic to a group of Lie type over a field of characteristic other than the defining characteristic of L. This proof essentially relies on the equality of orders of G and L, and is based on the following known properties of groups of Lie type.

Lemma 2. *If L is a simple symplectic or orthogonal group of rank n over a field of characteristic p and order q, v is a prime other than p, then the order of Sylow v-subgroup of L is at most $(2(q+1))^{3n/2}$.*

Lemma 3. *If S is a simple group of Lie type of rank m over a field of characteristic v and order u, then the order of Sylow v-subgroup of S is at least $u^{m^2/2}$.*

Lemma 4. *If L is a simple symplectic or orthogonal group of rank n, then $t(L) \geq (3n-4)/4$.*

Lemma 5. *If S is a simple group of Lie type of rank m, then $t(S) \leq 3m$. If, in addition, $t(S) > 4$, then $t(S) \leq 3m/2$.*

Lemma 6. *If S is a simple group of Lie type of rank m over a field of order u, then orders of elements of S are at most $2u^m$.*

Proposition 1. *Let q be a power of a prime p, L be one of the simple* *groups $B_n(q)$ with $n \geq 2$ and $(n,q) \neq (2,3)$, $C_n(q)$ with $n \geq 3$, and $D_n(q)$ with $n \geq 4$ even, and let G be a finite group with $\omega(G) = \omega(L)$ and $|G| = |L|$. Then a group of Lie type defined over a field of characteristic $v \neq p$ cannot be isomorphic to a composition factor of G.*

Proof of Proposition 1. Assume to the contrary that $S \leq G/K \leq \operatorname{Aut} S$, where S is a simple group of Lie type of rank m over a field of characteristic v and order u. Then the order of S divides that of L, and so the order of Sylow v-subgroup of S is at most the order of that of L. Using Lemmas 2 and 3, we derive that

$$u^{m^2/2} \leq (2(q+1))^{3n/2}.$$

Suppose $n > 16$. Then it follows from Lemmas 1, 4, and 5 that $3m/2 \geq t(S) \geq t(L) - 1 \geq (3n-8)/4$. Hence $3n/2 \leq 3m + 4$, and in particular, $m \geq 7$. Substituting the estimate of $3n/2$ into the right-hand side of the previous inequality, extracting the $(m/2)$th root of both sides and exploiting $m \geq 7$, we derive

$$u^m \leq (2(q+1))^{6+8/m} \leq (2(q+1))^8.$$

In view of Lemma 6 this yields $\max \omega(S) \leq 2(2(q+1))^8$. Thus orders of elements of S are bounded by a function that does not depend on the rank of L.

Now we show that S must contain elements of large order. Since $n > 16$, the interval $(n/2, n]$ contains three different primes i, j, and k. As follows from [12], if r_i, r_j, r_k are arbitrary prime divisors of the numbers

$$a_i = \frac{q^i - 1}{(q-1)(i, q-1)}, \quad a_j = \frac{q^j - 1}{(q-1)(j, q-1)}, \quad a_k = \frac{q^k - 1}{(q-1)(k, q-1)}$$

respectively, then r_i, r_j, and r_k are pairwise non-adjacent elements of $\omega(L)$. Applying Lemma 1, we infer that at least two of a_i, a_j, and a_k lie in $\omega(S)$. If $i < j < k$, then $a_j > q^{j-2} \geq q^i > q^{n/2}$, and $a_k > q^{n/2}$ as well. Thus $\max \omega(S) > q^{n/2}$. Therefore,

$$q^{n/2} < 2(2(q+1))^8.$$

Since $n > 16$, the final inequality is impossible if q is sufficiently large. We handle the cases of small n and q following the same scheme but using more flexible estimates.

Now to prove the theorem, it remains to check that S can be neither an alternating group nor a sporadic group, and if S is a group of Lie type in the same characteristic as L, then S is isomorphic to L. As stated above, we consider Shi's conjecture within the framework of a more general problem of recognition by spectrum. For that reason, the check is carried out for all symplectic and orthogonal groups and does not use the condition $|G| = |L|$ unless it is necessary. Applying methods developed through our study of the recognition-by-spectrum problem in [13–15], we manage to obtain the following results.

Proposition 2. *Let L be one of the groups $B_n(q)$ with $n \geq 2$ and $(n, q) \neq (2, 3)$, $C_n(q)$ with $n \geq 3$, and $D_n(q)$, ${}^2D_n(q)$ with $n \geq 4$. Then there are no alternating groups among non-abelian composition factors of finite groups isospectral to L.*

Proposition 3. *Let L be one of the groups $B_n(q)$ with $n \geq 2$ and $(n, q) \neq (2, 3)$, $C_n(q)$ with $n \geq 3$, and $D_n(q)$, ${}^2D_n(q)$ with $n \geq 4$. Then there are no sporadic groups nor the Tits group ${}^2F_4(2)'$ among non-abelian* *composition factors of finite groups isospectral to L.*

Proposition 4. *Let q be a power of a prime p, L be one of the simple groups $B_n(q)$ with $n \geq 2$ and $(n,q) \neq (2,3)$, $C_n(q)$ with $n \geq 3$, and $D_n(q)$, ${}^2D_n(q)$ with $n \geq 4$, and let G be a finite group with $\omega(G) = \omega(L)$. Suppose that there is a factor S isomorphic to a group of Lie type over a field of characteristic p among compositions factors of G.*

(1) *If $L = B_2(q)$, where $q > 3$, then S is isomorphic to one of the groups $A_1(q^2)$ and $B_2(q)$.*

(2) *If $L \in \{B_3(q), C_3(q), D_4(q)\}$, then S is isomorphic to one of the groups $A_1(q^3)$, $B_3(q)$, $C_3(q)$, $D_4(q)$, and $G_2(q)$.*

(3) *If $n \geq 4$ and $L \in \{B_n(q), C_n(q), {}^2D_n(q)\}$, then S is isomorphic to one of the groups $B_n(q)$, $C_n(q)$, and ${}^2D_n(q)$.*

(4) *If $n \geq 6$ is even and $L = D_n(q)$, then S is isomorphic to one of the groups $B_{n-1}(q)$, $C_{n-1}(q)$, and $D_n(q)$.*

(5) *If $n \geq 5$ is odd and $L = D_n(q)$, then $S \simeq L$.*

If, in addition, $|G| = |L|$, then $S \simeq G \simeq L$.

The main theorem follows by Propositions 1–4.

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Abstract

The spectrum of a finite group is the set of its element orders. We prove that a finite group and a finite simple group are isomorphic if they have the same spectra and orders. In other words, we show that every finite simple group is uniquely determined by its spectrum and order in the class of all groups. This provides a positive answer to Question 12.39 from Kourovka Notebook.

Key words: finite group, simple group, orthogonal group, symplectic group, element orders