On recognition of finite simple groups with connected prime graph

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Let G be a finite group, $\pi(G)$ be the set of prime divisors of its order and $\omega(G)$ be the spectrum of G, that is the set of element orders of G. The prime graph $GK(G)$ of a group G is defined as follows. The vertex set of $GK(G)$ is $\pi(G)$ and two primes $r, s \in \pi(G)$ considered as vertices of the graph are adjacent by the edge if and only if $rs \in \omega(G)$. K. W. Gruenberg and O. Kegel introduced this graph (it is also called the Gruenberg — Kegel graph) in the middle of 1970th and gave a characterization of finite groups with a disconnected prime graph (we denote the number of connected components of $GK(G)$ by $s(G)$). This deep result and a classification of finite simple groups with $s(G) > 1$ obtained by J. S. Williams and A. S. Kondrat'ev (see [1, 2]) implied a series of important corollaries.

The proof of the Gruenberg–Kegel Theorem relies substantially upon the fact that $\pi(G)$ contains an odd prime which is disconnected with 2 in $GK(G)$. It turned out that disconnectedness could be successfully replaced in most cases by a weaker condition for the prime 2 to be nonadjacent to at least one odd prime.

Denote by $t(G)$ the maximal number of primes in $\pi(G)$ pairwise non-adjacent in $GK(G)$. In other words, $t(G)$ is a maximal number of vertices in cocliques, i. e., independent sets, of $GK(G)$. In graph theory this number is usually called an independence number of the graph. By analogy we denote by $t(r, G)$ the maximal number of vertices in cocliques of $GK(G)$ containing the prime r. We call this number an r-independence number. Recently, in [3] it was given a characterization of finite groups G with $t(G) \geq 3$ and $t(2, G) \geq 2$, and in [4] it was proved that all finite nonabelian simple groups except the alternating permutation groups satisfy the condition $t(2, G) \geq 2$. Here we give a refinement of the main theorem of [3].

Theorem 1. *Let G be a finite group with $t(G) \geq 3$ and $t(2, G) \geq 2$. Then*

(1) *There exists a finite simple nonabelian group S such that $S \leq \overline{G} = G/K \leq \mathrm{Aut}(S)$ for maximal soluble normal subgroup K of G.*

(2) *For every independent subset ρ of $\pi(G)$ with $|\rho| \geq 3$ at most one prime in ρ divides the product $|K| \cdot |\overline{G}/S|$. In particular, $t(S) \geq t(G) - 1$.*

(3) *One of the following holds:*

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(a) *every prime* $r \in \pi(G)$ *non-adjacent in* $GK(G)$ *to* 2 *does not divide the product* $|K| \cdot |\overline{G}/S|$; *in particular,* $t(2, S) \geq t(2, G)$;

(b) *there exists a prime* $r \in \pi(K)$ *non-adjacent in* $GK(G)$ *to* 2; *in which case* $t(G) = 3$, $t(2, G) = 2$, *and* $S \simeq Alt_7$ *or* $A_1(q)$ *for some odd* q.

The above characterization with the description of prime graph of every finite nonabelian simple group (see [4]) can be applied to a so-called recognition problem. For a given finite group G denote by $h(G)$ the number of pairwise non-isomorphic finite groups H with $\omega(H) = \omega(G)$. The group G is called *recognizable* (by spectrum) if $h(G) = 1$, *almost recognizable* if $1 < h(G) < \infty$, and *non-recognizable* if $h(G) = \infty$. We say that for a given group G the recognition problem is solved if the value of $h(G)$ is known. Since every finite group with a nontrivial normal soluble subgroup is non-recognizable, each recognizable or almost recognizable group is an extension of the direct product M of nonabelian simple groups by some subgroup of $\mathrm{Out}(M)$. So, of prime interest is the recognition problem for simple and almost simple groups. Let L be a finite nonabelian simple group and G be a finite group with $\omega(G) = \omega(L)$. Clearly, the equality $\omega(G) = \omega(L)$ implies the coincidence of the prime graphs of G and L. Thus, if L satisfies the condition of Theorem 1, then so does G. The statement (1) of the conclusion of the theorem implies that G has the unique nonabelian composition factor S. On the other hand, the statements (2) and (3) help to prove that this factor S is isomorphic to L. If this fact is established we say that L is *quasirecognizable*. Obviously, the proof of quasirecognizability of L is a substantial step on the way to prove that L is recognizable or almost recognizable.

The description of prime graph [4] shows that the condition $t(2, L) \geq 2$ holds true for all finite nonabelian simple groups except the alternating groups Alt_n with n such that $n, n-1, n-2, n-3$ are not primes. On the other hand, for every finite simple group L with $t(L) < 3$ the recognition problem has been solved.

The next result shows that we can omit the exceptional case (b) of the statement (3) of Theorem 1 when we apply the theorem to the recognition of finite nonabelian simple groups.

Theorem 2. *Let* L *be a finite nonabelian simple group with* $t(L) \geq 3$ *and* $t(2, L) \geq 2$, *and* G *is a finite group with* $\omega(G) = \omega(L)$. *Then*

(1) *There exists a finite simple nonabelian group* S *such that* $S \leq \overline{G} = G/K \leq \mathrm{Aut}(S)$ *for maximal soluble normal subgroup* K *of* G.

(2) *For every independent subset* ρ *of* $\pi(G)$ *with* $|\rho| \geq 3$ *at most one prime in* ρ *divides the product* $|K| \cdot |\overline{G}/S|$. *In particular,* $t(S) \geq t(G) - 1$.

(3) *Every prime* $r \in \pi(G)$ *non-adjacent in* $GK(G)$ *to* 2 *does not divide the product* $|K| \cdot |\overline{G}/S|$. *In particular,* $t(2, S) \geq t(2, G)$.

1 Preliminaries

We begin from the main result of [3]. Note that we denote a finite simple group of Lie type accordingly to the Lie notation even so it is a classical group.

Lemma 1 [3] *Let G be a finite group with $t(G) \geq 3$ and $t(2, G) \geq 2$. Then there exists a finite nonabelian simple group S such that $S \leq \overline{G} = G/K \leq$ $\mathrm{Aut}(S)$ for the maximal normal soluble subgroup K of G. Furthermore, $t(S) \geq t(G) - 1$, and of the following statements holds:*

(1) $S \simeq Alt_7$ or $A_1(q)$ for some odd q, and $t(S) = t(2, S) = 3$.

(2) For every prime $p \in \pi(G)$ non-adjacent to 2 in $GK(G)$ a Sylow p-subgroup of G is isomorphic to a Sylow p-subgroup of S. In particular, $t(2, S) \geq t(2, G)$.

Actually the inequality $t(S) \geq t(G)-1$ in the above theorem was obtained by using the following proposition.

Lemma 2 [3, Proposition 3] *Let G be a group satisfying the conditions of Lemma 1, and the groups K, S, $\overline{G}$ are as in the conclusion of Lemma 1. Then $t(S) \geq t(G) - 1$. Moreover, for every independent subset ρ of $\pi(G)$ such that $|\rho| \geq 3$ at most one prime from ρ divides the product $|K| \cdot |\overline{G}/S|$.*

Lemma 3 [5, Lemma 1] *Let G be a finite group, K be its normal subgroup, and G/K be a Frobenius group with kernel F and cyclic complement C. If $(|F|, |K|) = 1$ and F does not lie in $KC_G(K)/K$, then $r \cdot |C| \in \omega(G)$ for some prime divisor r of $|K|$.*

Lemma 4 [6] *Let r, s be distinct primes, $H\langle x\rangle$ be a semidirect product of normal $\{2, r, s\}'$-subgroup H and group $\langle x\rangle$ of order s such that $[H, x] \neq 1$. If $H\langle x\rangle$ acts faithfully on a vector space V over the field of order r, then $C_V(x) \neq 0$.*

Now following [4] we define a notion of the primitive prime divisor which origin from well-known Zsigmondy Theorem. If q is a natural number, r is an odd prime and $(r, q) = 1$, then by $e(r, q)$ we denote the minimal natural number n with $q^n \equiv 1 (\mod r)$. If q is odd, let $e(2, q) = 1$ if $q \equiv 1 (\mod 4)$, and $e(2, q) = 2$ if $q \equiv -1 (\mod 4)$.

Lemma 5 (Zsigmondy Theorem [7]) *Let q be a natural number greater than 1. Then for every $n \in \mathbb{N}$ there exists a prime r such that $e(r, q) = n$ but for the cases where $q = 2$ and $n = 1$, $q = 3$ and $n = 1$, $q = 2$ and $n = 6$.*

The prime r with $e(r, q) = i$ is said to be a *primitive prime divisor* of $q^i - 1$. By Zsigmondy theorem such a number exists except in the case mentioned above. If q is fixed, we denote by r_i any primitive prime divisor of $q^i - 1$ (obviously, $q^i - 1$ can have more than one such divisor). Note that according to our definition every prime divisor of $q - 1$ is a primitive prime divisor of $q-1$ with sole exception: 2 is not a primitive prime divisor of $q-1$ if $e(2, q) = 2$. In the last case 2 is a primitive prime divisor of $q^2 - 1$. If q is fixed, we denote by k_i the maximal divisor of $q^i - 1$ such that the set of prime divisors of k_i is the set of all primitive prime divisors of $q^i - 1$. The number k_i is called *a maximal primitive divisor* of $q^i - 1$.

2 Proof of Theorem 1

Let G be a finite group satisfying the condition of Theorem 1. By Lemma 1 the statement (1) of the conclusion of the theorem holds, and by the Lemma 2 so does the statement (2). If the item (a) of the statement (3) is not true then by Lemma 1 a nonabelian composition factor S of the group G is isomorphic to Alt_7 or $A_1(q)$ with q odd. Thus, further we assume that item (a) of the statement (3) is not true for G and prove that $t(G) = 3$ and $t(2, G) = 2$ in that case.

We start proving that $t(2, G) = 2$. In fact, we prove the following result.

Lemma 6 *If item* (a) *of the statement* (3) *of Theorem* 1 *is not true, then the soluble radical K of G contains a non-trivial normal $2'$-subgroup N of index* 2 *such that a Sylow* 2*-subgroup of G/N is a generalized quaternion group, G/N has center of order* 2*, all odd primes from $\pi(G)$, whose are non-adjacent to* 2 *in $GK(G)$, are pairwise adjacent, divide the order of K and do not divide the order of G/K; in particular, $t(2, G) = 2$.*

Proof. By our assumption there exists a prime $r \in \pi(G)$ such that r is non-adjacent to 2 in $GK(G)$ and r divides the product $|K| \cdot |\overline{G}/S|$. By [3, Lemma 1.2] the prime r cannot divide $|\overline{G}/S|$, so r belongs to $\pi(K)$. Let T be a Sylow 2-subgroup of G and H be a Hall $\{2, r\}$-subgroup of the group KT. Since a Sylow r-subgroup R of H is a Sylow r-subgroup of K, the factor-group of its normalizer $N = N_G(R)$ by $N \cap K$ is isomorphic to $\overline{G}$ and contains a subgroup isomorphic to S. If R is cyclic, $C_G(R)K/K$ has to include S and so $2r \in \omega(G)$; a contradiction. Thus, R is not cyclic, and so $O_2(H) = 1$. Therefore, H is a Frobenius group with the kernel R and the complement T. Since a Sylow 2-subgroup of nonabelian simple group S cannot be cyclic, the group T as Sylow 2-subgroup of G is not cyclic too. Hence T is a generalized quaternion group. If $M = O_{2'}(G) = O_{2'}(K)$ then by Brauer — Suzuki Theorem [8] the factor-group G/M has the center Z/M of order 2. It is easy to see that $Z = K$ and that 2 is adjacent to every odd prime divisor of $|G/K|$. Suppose that there exists a prime $s \in \pi(K)$ such that $s \neq r$ and s is non-adjacent to 2 in $GK(G)$. A Hall $\{2, r, s\}$-subgroup of K is a Frobenius group with complement of order 2. Since a Hall $\{r, s\}$-subgroup of K is the kernel of this Frobenius group, it is abelian. Therefore, r adjacent to s, and $t(2, G) = 2$. The lemma is proved.

Now we consider the value of $t(G)$. Since $t(S) = 3$, the inequality $t(S) \geq t(G) - 1$ from Lemma 1 implies that $t(G) \leq 4$. Suppose $t(G) = 4$, i. e., the maximal independent set ρ of the graph $GK(G)$ contains four primes. By Lemma 2 and equality $t(S) = 3$ exactly one of these primes divides the product $|K| \cdot |\overline{G}/S|$. Denote this prime by r. Note that r is odd, since $t(2, G) = 2$. Assume that r divides $|\overline{G}/S|$. If $S \simeq Alt_7$ then r cannot divide $|\overline{G}/S| \leq 2$. Let $S \simeq A_1(q)$ and $q = p^m$, where p is the characteristic of the base field. Since every maximal coclique in $GK(S)$ has the form $\{p, r_1, r_2\}$, where r_i is a primitive prime divisor of $q^i - 1$ for $i = 1, 2$, the prime p must be one of three primes from $\rho \cap \pi(S)$. On the other hand, since $\overline{G}/S$ is isomorphic to a subgroup of $\operatorname{Out} S$, there exists an element x of odd order r from $\overline{G} \setminus S$ which is conjugate to a field automorphism of S. Then $pr \in \omega(G)$; a contradiction. Thus, we can assume that r divides order of K.

If $S \simeq Alt_7$ then $\rho = \{3, 5, 7, r\}$. Let T be a Sylow 3-subgroup of G and H be a Hall $\{3, r\}$-subgroup of KT. Since a Sylow r-subgroup of H is a Sylow r-subgroup of K, it is not cyclic. Thus, $O_3(H) = 1$ and H is a Frobenius group with the complement T. Therefore, T must be cyclic, which is impossible, since a Sylow 3-subgroup of S is not cyclic.

Suppose that $S \simeq A_1(q)$, where $q = p^m$ and p is odd prime. Then $\rho = \{r, p, s, t\}$, where all primes are odd, s divides $q - 1$ and t divides $q + 1$. Note that by Lemma 2 (or by statement (2) of the theorem) the order of K is coprime to the product pst. Let R be a Sylow r-subgroup of K, and $N = N_G(R)$ be its normalizer in G. By Frattini argument $G/K \simeq N/N \cap K$, so we can assume without loss of generality that R is a normal subgroup of G. The group S includes a subgroup F which is a Frobenius group with a kernel of order q and a complement of order s. Since $(|K|, |F|) = 1$, by Shur — Zassenhaus Theorem the factor group G/R contains a subgroup isomorphic to F. Lemma 3 implies that G contains an element of order rs; a contradiction. Theorem 1 is proved.

3 Proof of Theorem 2

Let L be a finite nonabelian simple group, G be a finite group with $\omega(G) = \omega(L)$. Theorem 1 implies that $S \leq \overline{G} = G/K \leq \mathrm{Aut}(S)$, where K is the soluble radical of G, and S is a finite nonabelian simple group. Moreover, if we assume that for G the statement (a) of item (3) of Theorem 1 does not hold, then S is isomorphic to Alt_7 or $A_1(q)$ for odd q; $t(L) = t(G) = 3$, $t(2, L) = t(2, G) = 2$. By [9] group S can not be isomorphic to Alt_7 (in such case $L \simeq Alt_7$ and $K = 1$), so we can assume that $S \simeq A_1(q)$. Lemma 6 implies that every prime r non-adjacent to 2 in $GK(G)$ divides only the order of K. Since in [4] the values of independent and 2-independent numbers were determined for all finite nonabelian simple groups, we can list all such groups L with $t(L) = 3$ and $t(2, L) = 2$. Using [4] one can verify that the every maximal coclique $\rho(L)$ of $GK(L)$ contains the prime r non-adjacent to 2 in $GK(L)$. Since r divides the order of K, any other prime from $\rho(L)$ divides only the order of S.

Let $S \simeq A_1(q)$ with $q = p^m$ for an odd prime p. As it was mentioned above, every maximal coclique in $GK(S)$ has the form $\{p, r_1, r_2\}$, where r_i is a primitive prime divisor of $q^i - 1$ for $i = 1, 2$. Let $\rho(G) = \rho(L) = \{r, s, t\}$ be a maximal coclique and $\rho(2, G) = \rho(2, L) = \{2, r\}$ be a maximal coclique, containing 2, of $GK(L)$ and so of $GK(G)$. Then $s, t \in \{p, r_1, r_2\}$.

Suppose that $s = r_1$ is a primitive prime divisor of $q - 1$. Taken a factor group of G by $O_{r'}(K)$ and then a factor group of $G/O_{r'}(K)$ by Frattini subgroup of its maximal normal r-subgroup, we may assume that $O_{r'}(K) = 1$, $V = O_r(K)$ is nontrivial normal elementary abelian r-subgroup of G and $C_G(V) = V$. Denote by $\widetilde{G}$ and $\widetilde{K}$ factor groups of G and K by V. Let $\widetilde{S}$ be the preimage of S in $\widetilde{G}$, P be a Sylow p-subgroup of $\widetilde{S}$. Put $\widetilde{P} = P \cap \widetilde{K}$ and $N = N_{\widetilde{S}}(\widetilde{P})$. Since by Frattini argument $N/N \cap \widetilde{K} \simeq \widetilde{S}/\widetilde{K}$, we can assume that $\widetilde{P}$ is normal in $\widetilde{S}$ and so $N_{\widetilde{S}}(P)/\widetilde{K} = N_S(U)$, where $U = P/\widetilde{P}$ is a Sylow p-subgroup of S. The normalizer $N_S(U)$ contains an element y of order s, and $U\langle y \rangle$ is a Frobenius group with kernel U and complement $\langle y \rangle$; in particular $[U, y] \neq 1$. Therefore, $N_{\widetilde{S}}(P)$ contains an element x of order s and $[P, x] \neq 1$. Since $C_G(V) = V$, the group $P\langle x \rangle$ acts faithfully on the group V, which can be considered as a vector space over the field of order r. Lemma 4 implies $C_V(x) \neq 1$. Hence, $sr \in \omega(G)$; a contradiction.

Thus, $s, t \in \{p, r_2\}$. Let $s = p$, and $t = r_2$ be an odd divisor of $q + 1$. If $q > p$ then abelian Sylow p-subgroup U of S is not cyclic. Considering the action of U on normal r-subgroup of K, we obtain that G contains an element of order pr, which is impossible, since $pr \notin \omega(L)$. Therefore, $q = p$ and $S \simeq A_1(p)$ for some odd prime p.

If the prime graph of L is disconnected then so is a prime graph of G and its soluble radical K is nilpotent (by Thompson Theorem on the nilpotency of a group admitting the fixed-point-free automorphism of prime order). On the other hand, by Lemma 6 the element of order 2 lies in K. Therefore, in that case a prime r non-adjacent to 2 in $GK(G)$ can not divide the order of K; contrary to our assumption. Thus, the prime graph of L must be connected.

Since all sporadic simple groups have the disconnected prime graphs no one of them can be a counterexample. Among the alternating groups with $t(L) = 3$ and $t(2, L) = 2$ only the group Alt_{16} has a connected prime graph. However, this group is recognizable by its spectrum [10]. All exceptional groups of Lie type except the groups of type E_7 also have a disconnected prime graph. Since for $L \simeq E_7(q)$ the equality $t(L) = 8$ holds, we can assume that L is a classical group of Lie type. Using a condition of connectedness of the prime graph together with equalities $t(L) = 3$ and $t(2, L) = 2$ we obtain that the groups that we have to consider are contained among the following groups: $A_3(u)$, $A_5(u)$, ${}^2A_3(u)$, ${}^2A_5(u)$, $B_3(u)$, $C_3(u)$, $D_4(u)$; and $B_4(2)$, $C_4(2)$.

Let L be isomorphic to $B_4(2)$ or $C_4(2)$. The prime graph $GK(L) = GK(G)$ has the maximal coclique $\rho(L) = \{5, 7, 17\}$ and the maximal coclique $\rho(2, L) = \{2, 17\}$ containing 2. Since 17 divides only the order of soluble radical K, primes $5, 7$ divides only the order of $S \simeq A_1(p)$ and so $5, 7 \in \{p, r_2\}$. If $p = 5$ then 7 must divide $p + 1 = 6$. If $p = 7$ then 5 must divide $p + 1 = 8$. Both cases are impossible.

Let L be isomorphic to one of the group $A_3(u)$, $A_5(u)$, ${}^2A_3(u)$, ${}^2A_5(u)$, $B_3(u)$, $C_3(u)$, $D_4(u)$, where $u = v^m$ and v is a prime. We may suppose that v is odd, since otherwise $t(2, L) = 3$. Let w_i be a primitive prime divisor of $u^i - 1$ and k_i be the maximal primitive divisor of $u^i - 1$.

Lemma 7 *Let v be an odd prime and $u = v^m$. Then the following statements holds.*

(1) If $L \simeq A_3(u)$, then for arbitrary w_3 and w_4 the set $\{v, w_3, w_4\}$ is the maximal coclique of $GK(L)$.

(2) If $L \simeq {}^2A_3(u)$, then for arbitrary w_4 and w_6 the set $\{v, w_4, w_6\}$ is the maximal coclique of $GK(L)$.

(3) If $L \simeq A_5(u)$, then for arbitrary w_4, w_5 and w_6 the sets $\{v, w_5, w_6\}$ and $\{w_4, w_5, w_6\}$ are the maximal cocliques of $GK(L)$.

(4) If $L \simeq {}^2A_5(u)$, then for arbitrary w_3, w_4 and w_{10} the sets $\{v, w_3, w_{10}\}$ and $\{w_4, w_3, w_{10}\}$ are the maximal cocliques of $GK(L)$.

(5) If $L \simeq B_3(u)$, $C_3(u)$ or $D_4(u)$, then for arbitrary w_3 and w_6 the set $\{v, w_3, w_6\}$ is the maximal coclique of $GK(L)$.

(6) $k_3=(u^2+u+1)/(3,u-1)$, $k_4=(u^2+1)/2$, $k_5=(u^4+u^3+u^2+u+1)/(5,u-1)$, $k_6=(u^2-u+1)/(3,u+1)$, $k_{10}=(u^4-u^3+u^2-u+1)/(5,u+1)$.

Proof. The values of k_i can be calculated directly. The rest holds by [4]. The lemma is proved.

The prime graph of every our group L has the maximal coclique ρ of the form $\{v, w_i, w_j\}$ from Lemma 7. As it was mentioned, the coclique ρ contains exactly one prime r which is non-adjacent to 2 and this prime is not the characteristic v, which is obviously adjacent to 2. For the definiteness let w_j is adjacent and w_i is non-adjacent to 2. By our assumption w_j divides the order of K and $(w_j, |G/K|)=1$. On the other hand, w_i and v divide the order of S and $(w_iv, |K|\cdot|G/S|)=1$. As it was proved $S\simeq A_1(p)$ for some odd prime p, and the primes $w_i, v\in\{p, r_2\}$, where r_2 is an odd prime divisor of $p+1$.

Suppose that $v=p$. Then w_i must divide $p+1$. On the other hand w_i is a primitive prime divisor of $u^i-1=p^{mi}-1$, which is impossible, since $i>2$ by Lemma 7. Thus, v divides $p+1$ and $w_i=p$. Consider the maximal primitive prime divisor k_i of u^i-1. Since L contains an element of order k_i, so does G. On the other hand, $(k_i, |K|\cdot|G/S|)=1$. Hence S contains an element of order k_i. Since by Lemma 7 the equality $w_i=p$ holds for arbitrary primitive prime divisor w_i of u^i-1, the maximal primitive prime divisor k_i is equal to p.

Let $L\simeq B_3(u)$, $C_3(u)$ or $D_4(u)$. Then $p=k_3$ or k_6 by Lemma 7.

Suppose that $p=k_3=(u^2+u+1)/(3,u-1)$. Since v divides $p+1$, prime v divides u^2+u+2 if $(3,u-1)=1$, and v divides u^2+u+4 if $(3,u-1)=3$. In both cases $v=2$; a contradiction.

Suppose that $p=k_6=(u^2-u+1)/(3,u+1)$. Then v divides u^2-u+2 if $(3,u+1)=1$, and v divides u^2-u+4 otherwise. Again $v=2$; a contradiction.

Let $L\simeq A_3(u)$ or ${}^2A_3(u)$. Since the equalities $p=k_3$ and $p=k_6$ lead to contradiction, we may assume that $p=k_4=(u^2+1)/2$. Therefore, v divides $p+1=(u^2+3)/2$, and so $v=3$. The group L contains the element of order 9. On the other hand, $v=3$ divides only the order of S. It follows that 9 divides $p+1=(u^2+3)/2=(3^{2m}+3)/2$; a contradiction.

Let $L\simeq A_5(u)$. If $p=k_5=u^4+u^3+u^2+u+1$, then $v=2$, which is impossible. If $p=k_5=(u^4+u^3+u^2+u+1)/5$, then v divides $u^4+u^3+u^2+u+6$, and so $v=3$. By Lemma 7 the graph $GK(L)$ contains the coclique $\{w_4, w_5, w_6\}$ which is also maximal. Since $w_5=p$ and $w_6\in\pi(K)$, all the primitive prime divisors w_4 of u^4-1 must divide $p+1$. The group L contains an element of order k_4 and so does S, hence, $k_4=(u^2+1)/2$ divides $p+1=u^4+u^3+u^2+u+6$. Therefore, u^2+1 divides $2(u^4+u^3+u^2+u+6)$ and so u^2+1 divides 12; a contradiction.

Let $L \simeq {}^2A_5(u)$. If $p = k_{10} = u^4 - u^3 + u^2 - u + 1$, then $v = 2$, which is impossible. If $p = k_5 = (u^4 - u^3 + u^2 - u + 1)/5$, then v divides $u^4 - u^3 + u^2 - u + 6$, and so $v = 3$. By Lemma 7 the graph $GK(L)$ contains the coclique $\{w_4, w_{10}, w_3\}$ which is also maximal. Since $w_{10} = p$ and $w_3 \in \pi(K)$, all the primitive prime divisors w_4 of $u^4 - 1$ must divide $p + 1$, and so, as in the previous case, $k_4 = (u^2 + 1)/2$ divides $p + 1 = u^4 - u^3 + u^2 - u + 6$. Therefore, $u^2 + 1$ divides $2(u^4 - u^3 + u^2 - u + 6)$ and so $u^2 + 1$ divides 12; a contradiction.

Theorem 2 is proved.

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