RECOGNITION BY SPECTRUM FOR SIMPLE CLASSICAL GROUPS IN CHARACTERISTIC 2

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Abstract. Groups are said to be *isospectral* if their spectra (i.e., the sets of orders of elements) coincide. To solve the recognition-by-spectrum problem for a group G is to find the number $h(G)$ of pairwise nonisomorphic groups isospectral to G and, provided that $h(G)$ is finite, determine all such groups. We complete the study of the recognition-by-spectrum problem for finite simple classical groups in characteristic 2.

Keywords: simple classical group, orders of elements, recognition by spectrum.

1. Introduction

All groups considered in the paper are finite. The *spectrum* $\omega(G)$ of a group G is the set of orders of its elements. Groups are *isospectral* if their spectra coincide. To *solve the recognition-by-spectrum problem* for a group G is to find the number $h(G)$ of pairwise nonisomorphic groups isospectral to G and, provided that $h(G)$ is finite, determine all such groups. If $h(G)=1$ then we say that G is recognizable by spectrum. If G has a nontrivial normal soluble subgroup, then $h(G)=\infty$ [1]. However, if G is a nonabelian simple group then, in general, every group isospectral to G is isomorphic to some automorphic extension of G, and in particular $h(G)$ is finite (see details in [2]). In this paper, we complete the study of the recognition-by-spectrum problem for simple classical groups in characteristic 2.

Recall that there are five series of simple classical groups over fields of characteristic 2, which we denote following [3], namely, $L_n(q)$, $n\geq 2$, $U_n(q)$, $n\geq 3$, $S_{2n}(q)\simeq O_{2n+1}(q)$, $n\geq 2$, and $O^{\pm}_{2n}(q)$, $n\geq 4$. The recogition problem has been already solved for linear and unitary groups over fields of characteristic 2 (see [4,5]). The following result completes the solution for symplectic and orthogonal groups.

Theorem. *Let q be a power of 2 and let L be one of the simple groups $S_{2n}(q)$, where $n\geq 2$, or $O^{\pm}_{2n}(q)$, where $n\geq 4$. Then L is recognizable by spectrum or one of the following holds.*

(1) $L\in\{S_6(2),O_8^+(2)\}$, $\omega(S_6(2))=\omega(O_8^+(2))$ *and* $h(L)=2$.

(2) $L\in\{S_4(q),S_8(q)\}$ *and* $h(L)=\infty$.

The assertion of the theorem has already been proved for $S_4(q)$ [6], $S_6(q)$ and $O_8^+(q)$ [7], $S_8(q)$ [8]. Also it follows from [2,9] for all groups with $n\geq 20$. Moreover, to prove the theorem in the remaining case $4\leq n<20$, it is suffices to show that a nonabelian composition factor of a group isospectral to L cannot be a group of Lie type in odd characteristic. That is what the present paper is devoted to.

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2. Preliminaries

By $[m_1, m_2, \dots, m_k]$ and $(m_1, m_2, \dots, m_k)$ we denote the least common multiple and greatest common divisor of integers $m_1, m_2, \dots, m_s$ respectively. By $\pi(m)$ we denote the set of prime divisors of a positive integer m. If r is a prime, them we write $(m)_r$ for the *r-part* of m, that is, the highest power of r that divides m. And we write $(m)_{r'}$ for the r'-part of m, that is, the ratio $m/(m)_r$.

If q is an integer, r is an odd prime and $(q, r) = 1$, then $e(r, q)$ denotes the multiplicative order of q modulo r. We define $e(2, q)$ to be 1 if 4 divides $q - 1$, and 2 if 4 divides $q + 1$. A *primitive prime divisor* of $q^m - 1$, where $|q| > 1$ and $m \geq 1$, is a prime r such that $e(r, q) = m$. Note that this definition depends on q and m, not only on $q^m - 1$: 7 is a primitive prime divisor of $4^3 - 1$ and $(-2)^6 - 1$, while $2^6 - 1$ has no primitive prime divisors. We write $R_m(q)$ for the set of primitive prime divisors of $q^m - 1$, and $r_m(q)$ denotes some (generally speaking, arbitrary) element of $R_m(q)$.

Lemma 2.1 (Zsigmondy [10]). *Let q be an integer, $|q| > 1$, and $m \geq 1$. Then $R_m(q)$ is not empty except when*

$$(q, m) \in \{(2, 1), (2, 6), (-2, 2), (-2, 3), (3, 1), (-3, 2)\}.$$

The *greatest primitive divisor* of $q^m - 1$, where $|q| > 1$, is $k_m(q) = \prod_{r \in R_m(q)} |q^m - 1|_r$ for $m \neq 2$ and $k_2(q) = \prod_{r \in R_2(q)} |q + 1|_r$ for $m = 2$. For $m \geq 3$, the greatest primitive divisor can be expressed in terms of the cyclotomic polynomial $\Phi_m(x)$.

Lemma 2.2. *Let q, m be integers, $|q| > 1$, $m \geq 3$. If r is the largest prime divisor of m and $l = (m)_{r'}$, then*

$$k_m(q) = \frac{|\Phi_m(q)|}{(r, \Phi_l(q))},$$

and if l does not divide $r - 1$ then $(r, \Phi_l(q)) = 1$.

Proof. It follows from [11, Proposition 2] that

$$k_m(q) = |\Phi_m(q)|/(r, \Phi_m(q)).$$

It remains to observe that r divides $\Phi_m(q)$ if and only if r divides $\Phi_l(q)$ (see, for example, [12, Lemma 8.1]). □

A *coclique* of a graph is the set of pairwise nonadjacent vertices. By $t(\Gamma)$ we denote the independence number of a graph Γ or, in other words, the largest size of a coclique in Γ.

Given a finite group G, we write $\pi(G)$ for $\pi(|G|)$ and $\mu(G)$ for the set of elements of $\omega(G)$ that are maximal under divisibility in $\omega(G)$. The *prime graph* $GK(G)$ of G is an ordinary labeled graph with the vertex set $\pi(G)$ in which two distinct vertices labeled by r and s are adjacent if and only if $rs \in \omega(G)$. By $t(G)$ we denote $t(GK(G))$. For $r \in \pi(G)$, the largest size of a coclique containing r in $GK(G)$ is denoted by $t(r, G)$. By [13], a group G satisfying $t(G) \geq 3$ and $t(2, G) \geq 2$ has exactly one nonabelian composition factor. The following lemma is a consequence of this result.

Lemma 2.2 [13, 14]. *Let L be a finite nonabelian simple group with $t(L) \geq 3$ and $t(2, L) \geq 2$, and let G be a finite group such that $\omega(G) = \omega(L)$. The following then hold.*

(a) *There is a nonabelian simple group S such that*

$$S \leq \overline{G} = G/K \leq \operatorname{Aut} S,$$

where K is the largest normal soluble subgroup of G.

(b) *For every coclique ρ of $GK(G)$ whose size is larger than 2, at most one element of ρ divides $|K| \cdot |\overline{G}/S|$. In particular, $t(S) \geq t(G) - 1$.*

(c) *Every prime $r \in \pi(G)$ that is not adjacent to 2 in $GK(G)$ 2, is coprime to $|K| \cdot |\overline{G}/S|$. In particular, $t(2, S) \geq t(2, G)$.*

Lemma 2.4 [15, Lemma 3]. *Let G be a finite group, K a normal soluble subgroup of G and $S \leq \overline{G} = G/K \leq \operatorname{Aut} S$ for some nonabelian simple group S. Suppose that $\pi(S) \setminus \pi(K)$ contains numbers t and s whose neighbourhoods in $GK(G)$ are disjoint. If $r \in \pi(K)$ is not adjacent to both t and s in $GK(G)$, and S has a Frobenius subgroup with cyclic complement C and kernel F such that $(|F|, r) = 1$, then $r|C| \in \omega(G)$.*

Lemma 2.5. *Let q be even.*

(1) *If $L = S_4(q)$ then $h(L) = \infty$* [6].

(2) *If $L \in \{S_6(2), O_8^+(2)\}$ then $h(L) = 2$, and $\omega(S_6(2)) = \omega(O_8^+(2))$* [16, 17].

(3) *If $L \in \{S_6(q), O_8^+(q)\}$, where $q > 2$, then $h(L) = 1$* [7].

(4) *If $L = S_8(q)$ then $h(L) = \infty$* [8].

(5) *If $L = O_{2n}^-(2)$, where $n = 2^m + 1 \geq 5$, then $h(L) = 1$* [17, 18].

A nonabelian simple group L is said to be *quasirecognizable* by spectrum if every finite group isospectral to L has exactly one nonabelian composition factor and this factor is isomorphic to L.

Lemma 2.6. *The following groups are quasirecognizable by spectrum.*

(1) *$S_{2n}(q)$, $O_{2n}^{\pm}(q)$, where $n \geq 20$* [2, 19].

(2) *$O_{2n}^-(q)$, where $n \geq 16$ is even* [13, 20].

(3) *$S_{2n}(q)$, where $n = 2^m \geq 8$, $O_{2n}^-(q)$, where $n = 2^m \geq 4$* [18].

(4) *$S_{2p}(2)$, where $p \geq 5$ is a prime* [21].

(5) *$O_{2p}^+(2)$, where $p \geq 5$ is a prime* [22, 23].

(6) *$O_{2p+2}^+(2)$, where $p \geq 5$ is a prime* [22, 24].

3. Some properties of simple groups of Lie type

We use the standard notation $L_n^{\pm}(q)$ and $E_6^{\pm}(q)$, where $L_n^+(q) = L_n(q)$, $L_n^-(q) = U_n(q)$, $E_6^+(q) = E_6(q)$, and $E_6^-(q) = {}^2E_6(q)$. If $\varepsilon \in \{+, -\}$, we write ε instead of $\varepsilon 1$ in arithmetic expressions.

Lemma 3.1 [25, Corollary 3]. *Let $L = S_{2n}(q)$, where q is even and $n \geq 2$. The set $\omega(L)$ consists of all divisors of the following numbers:*

(1) *$[q^{n_1} \pm 1, \dots, q^{n_s} \pm 1]$, where $s \geq 1$, $n_i > 0$ for $1 \leq i \leq s$, and $n_1 + \dots + n_s = n$;*

(2) *$2[q^{n_1} \pm 1, \dots, q^{n_s} \pm 1]$, where $s \geq 1$, $n_i > 0$ for $1 \leq i \leq s$, and $1 + n_1 + \dots + n_s = n$;*

(3) *$2^k[q^{n_1} \pm 1, \dots, q^{n_s} \pm 1]$, where $k \geq 2$, $s \geq 1$, $n_i > 0$ for $1 \leq i \leq s$, and $2^{k-2} + 1 + n_1 + \dots + n_s = n$;*

(4) *2^k if $n = 2^{k-2} + 1$ for some $k \geq 2$.*

Lemma 3.2 [25, Corollary 4]. *Let $L = O_{2n}^{\varepsilon}(q)$, where q is even, $n \geq 4$ and $\varepsilon \in \{+, -\}$. The set $\omega(L)$ consists of all divisors of the following numbers:*

(1) *$[q^{n_1} - \tau_1, \dots, q^{n_s} - \tau_s]$, where $s \geq 1$, $n_i > 0$ and $\tau_i \in \{+, -\}$ for $1 \leq i \leq s$, $n_1 + \dots + n_s = n$ and $\tau_1 \dots \tau_s = \varepsilon$;*

(2) *$2[q^{n_1} \pm 1, \dots, q^{n_s} \pm 1]$, where $s \geq 1$, $n_i > 0$ for $1 \leq i \leq s$, and $2 + n_1 + \dots + n_s = n$;*

(3) $2^k[q^{n_1} \pm 1, \dots, q^{n_s} \pm 1]$, *where* $k \geq 2$, $s \geq 1$, $n_i > 0$ *for* $1 \leq i \leq s$ *and* $2^{k-2} + 2 + n_1 + \dots + n_s = n$;

(4) $2[q \pm 1, q^{n_1} - \tau_1, \dots, q^{n_s} - \tau_s]$, *where* $s \geq 1$, $n_i > 0$ *and* $\tau_i \in \{+, -\}$ *for* $1 \leq i \leq s$, $2 + n_1 + \dots + n_s = n$ *and* $\tau_1 \dots \tau_s = \varepsilon$;

(5) $4[q - \tau, q^{n_1} - \tau_1, \dots, q^{n_s} - \tau_s]$, *where* $s \geq 1$, $\tau \in \{+, -\}$, $n_i > 0$ *and* $\tau_i \in \{+, -\}$ *for* $1 \leq i \leq s$, $3 + n_1 + \dots + n_s = n$ *and* $\tau\tau_1 \dots \tau_s = \varepsilon$;

(6) 2^k *if* $n = 2^{k-2} + 2$ *for some* $k \geq 3$.

Let $m(n, q)$ be the largest number of the form $[q^{n_1} \pm 1, \dots, q^{n_s} \pm 1]$, where $s \geq 1$, $n_i > 0$ for $1 \leq i \leq s$, and $n_1 + \dots + n_s = n$.

Lemma 3.3. *Let q be a power of* 2.

(a) *For every positive integer n, we have* $m(n, q) \leq (q^{n+1} - 1)/(q - 1)$. *If n is odd and $q > 2$ or if $n \in \{9, 2^l - 1, 3 \cdot 2^l - 1\}$ for some l, then* $m(n, q) = (q^{n_1} + 1) \dots (q^{n_s} + 1)$, *where $n_1 + \dots + n_s$ is the binary expansion of n.*

(b) *If L is one of the groups $S_{2n}(q)$, where $n \geq 2$, or $O^{\pm}_{2n}(q)$, where $n \geq 4$, then the orders of elements of L do not exceed $2q^n$.*

Proof. (a) The first assertion is [26, Lemma 1.2]. The second one follows from [27, Propositions 1 and 3].

(b) This follows from Lemmas 3.1, 3.2 and ( ). See also [26, Lemma 1.3]. □

Now we consider cocliques of $GK(G)$ that consist of odd numbers r such that $4r \notin \omega(G)$, and denote by $t^*(4, G)$ the largest size of such a coclique.

Table 1

L	$\rho^*(2, L)$	$\rho^*(4, L)$
$S_{2n}(q)$, $n \geqslant 4$ even	$\{r_{2n}(q)\}$	$\{r_{2n}(q),\ r_{2(n-1)}(q),\ r_{n-1}(q)\}$
$S_{2n}(q)$, n odd	$\{r_{2n}(q), r_n(q)\}$	$\{r_{2n}(q),\ r_n(q),\ r_{2(n-1)}(q)\}$
$O^+_{2n}(q)$, n even	$\{r_{2(n-1)}(q), r_{n-1}(q)\}$	$\{r_{2(n-1)}(q),\ r_{n-1}(q),\ r_{2(n-2)}(q)\}$
$O^+_{2n}(q)$, n odd	$\{r_n(q), r_{2(n-1)}(q)\}$	$\{r_n(q), r_{2(n-1)}(q),\ r_{n-2}(q)\}$
$O^-_{2n}(q)$, n even	$\{r_{2n}(q), r_{2(n-1)}(q), r_{n-1}(q)\}$	$\{r_{2n}(q), r_{2(n-1)}(q), r_{n-1}(q)\}$
$O^-_{2n}(q)$, n odd	$\{r_{2n}(q), r_{2(n-1)}(q)\}$	$\{r_{2n}(q), r_{2(n-1)}(q), r_{n-2}(q)\}$

Lemma 3.4. *Let q be even and L a simple symplectic or orthogonal group over a field of order q other than $S_4(q)$, $S_6(2)$, $S_8(2)$, $O^{\pm}_8(2)$. Then the sets $\rho^*(2, L)$ and $\rho^*(4, L)$ defined in Table* 1 *have the following properties:*

(a) $\rho^*(2, L) \subseteq \rho^*(4, L)$ *and* $\rho^*(4, L)$ *is a coclique in* $GK(L)$;

(b) *if* $r \in \rho^*(4, L)$ *then* $r > 3$ *and* $4r \notin \omega(L)$, *and if in addition* $r \in \rho^*(2, L)$ *then* $2r \notin \omega(L)$; *moreover,* $|\rho^*(2, L)| + 1 = t(2, L)$;

(c) *if* $|\rho^*(2, L)| \geq 2$ *then* $\rho^*(2, L)$ *contains two numbers that have no common neighbours in* $GK(L)$, *and if* $\rho^*(2, L) = \{t\}$ *then t has no common neighbours with any element of* $\rho^*(4, L) \setminus \{t\}$ *in* $GK(L)$.

In particular, $t^*(4, L) \geq 3$.

Proof. (a) and (b) follows from Lemmas 3.1 and 3.2 together with the observation that 3 lies in $R_1(q) \cup R_2(q)$ by Fermat's Little Theorem. Cocliques of maximal size among those not containing 2 are also described in [20, Table 4].

(c) Let $L = S_{2n}(q)$, where $n \geq 4$ is even. We claim that $r_{2n}(q)$ has no common neighbours with $r_{n-1}(q)$ or $r_{2(n-1)}(q) = r_{n-1}(-q)$. Suppose that $r \in \pi(L)$. By Lemma 3.1, if $rr_{2n}(q) \in \omega(L)$ then r divides q^n+1. And if $rr_{n-1}(\tau q) \in \omega(L)$, where $\tau \in \{+,-\}$, then r divides $2(q^{n-1}-\tau)(q \pm 1)$. Since $(q^n+1, q^{n-1}-\tau) = (q^n+1, q-\tau) = 1$, the result follows.

Let $L = S_{2n}(q)$, where n is odd. Let $r \in \pi(L)$. If $rr_{2n}(q) \in \omega(L)$, then r divides q^n+1, and if $rr_n(q) \in \omega(L)$ then r divides q^n-1. Since $(q^n+1, q^n-1) = 1$, the numbers $r_{2n}(q)$ and $r_n(q)$ have no common neighbours.

Let $L = O^{\varepsilon}_{2n}(q)$, where n is odd. By Lemma 3.2, if $rr_n(\varepsilon q) \in \omega(L)$ then r divides $q^n - \varepsilon$. If $rr_{2(n-1)} \in \omega(L)$ then r divides $(q^{n-1}+1)(q+\varepsilon)$. The equality $(q^n-\varepsilon, q+\varepsilon) = 1$ implies the desired result.

Let $L = O^{+}_{2n}(q)$, where n is even. If $rr_{n-1}(\tau q) \in \omega(L)$ for $\tau \in \{+,-\}$ then r divides $[q^{n-1}-\tau, q-\tau] = q^{n-1}-\tau$. Since $q^{n-1}+1$ and $q^{n-1}-1$ are coprime, $r_{2(n-1)}(q)$ and $r_{n-1}(q)$ have no common neighbours.

Let finally $L = O^{-}_{2n}(q)$, where n is even. Since $\omega(L) \subseteq \omega(S_{2n}(q))$, by the above result $r_{2n}(q)$ has common neighbours with neither $r_{n-1}(q)$ nor $r_{2(n-1)}(q)$. □

Lemma 3.5. *Let u be a power of an odd prime v and S one of the following simple groups of Lie type:*

(1) *$G_2(u)$, ${}^3D_4(u)$, $F_4(u)$, $E^{\pm}_6(u)$;*

(2) *$L^{\pm}_n(u)$, where $n \geq 4$; $L^{\varepsilon}_3(u)$, where $u \equiv \varepsilon \pmod 4$;*

(3) *$S_{2n}(u)$, $O_{2n+1}(u)$, $O^{\pm}_{2n}(u)$.*

Then $4v \in \omega(S)$ and $t^*(4, S) \leq 2$.

Proof. Denote the set of odd primes r such that $r \in \pi(S)$ and $4r \notin \omega(S)$ by $\tilde{\pi}(S)$.

(1) Since $v(u\pm1) \in \omega(G_2(u))$ by [28, Lemma 1.4] and $G_2(u) < {}^3D_4(u) < F_4(u) < E^{\pm}_6(u)$ by [29], it follows that $4v \in \omega(S)$. If r is an odd prime such that $4r \notin \omega(S)$, then 4 does not divide any exponent of maximal torus of S that is divisible by r. Furthermore, if r and s divide the same exponent then $rs \in \omega(S)$. Thus $t^*(4, S)$ is at most the number of maximal under divisibility exponents of maximal tori that are not divisible by 4. The structure of maximal tori of the groups under consideration is well known (see [30–32] for $G_2(u)$, ${}^3D_4(u)$, $E^{\pm}_6(u)$ respectively and [33, pp. 94–96] for $F_4(u)$).

If $S = G_2(u)$ then the maximal under divisibility exponents of maximal tori not divisible by 4 are u^2+u+1 and u^2-u+1.

If $S = {}^3D_4(u)$ or $S = F_4(u)$ then the corresponding exponents lie in $\{u^4-u^2+1, u^4+1\}$.

If $S = E^{\varepsilon}_6(q)$ then the corresponding exponents (in the universal covering of S) are $u^6+\varepsilon u^3+1$ and $(u^2+\varepsilon u+1)(u^4-u^2+1)$.

In any case, the number of the required exponents is at most 2, whence $t^*(4, S) \leq 2$.

(2) Let $S = L^{\varepsilon}_n(u)$, where $n \geq 3$. The spectrum of L can be found in [34]. Denote $(n, u-\varepsilon)$ by d.

If $n \geq 4$ then the spectrum of S contains the following numbers: $v(u-\varepsilon)$, $v(u^2-1)/d$, and $[u^k-\varepsilon^k, u-\varepsilon]$, $[u^k-\varepsilon^k, u^2-1]/d$ for all $k \leq n-2$. Since 4 divides at least one of the numbers $u-\varepsilon$ and $(u^2-1)/d$, it follows that $4v, 4r_k(\varepsilon u) \in \omega(L)$ for all $k \leq n-2$. Thus $\tilde{\pi}(S) \subseteq R_n(\varepsilon u) \cup R_{n-1}(\varepsilon u)$.

If $S = L_3^\varepsilon(u)$ then $\omega(S)$ contains $(u^2-1)/d$ and $v(u-\varepsilon)/d$. Therefore, $4r_1(u)$ and $4r_2(u)$ lie in $\omega(S)$, and if $u \equiv \varepsilon \pmod 4$ then $4v \in \omega(S)$. Thus $\tilde{\pi}(S) \subseteq R_3(\varepsilon u)$.

In any case, $t^*(4, S) \leq 2$.

(3) The spectra of symplectic and orthogonal groups can be found in [25]. For any such group S, we have that $v(u \pm 1) \in \omega(S)$, whence $4v \in \omega(S)$.

If $S = S_{2n}(u)$, where $n \geq 2$, or $S = O_{2n+1}(u)$, where $n \geq 3$, then $[u^k \pm 1, u \pm 1] \in \omega(S)$ for all $k \leq n-1$, and so $\tilde{\pi}(S) \subseteq R_{2n}(u) \cup R_n(u)$.

If $S = O_{2n}^\varepsilon(u)$ then $[u^k \pm 1, u \pm 1] \in \omega(S)$ for all $k \leq n-2$. Therefore $\tilde{\pi}(S) \subseteq R_n(\varepsilon u) \cup R_{2(n-1)}(u)$ if n is odd, and $\tilde{\pi}(S) \subseteq R_{n-1}(u) \cup R_{2(n-1)}(u)$ if n is even and $\varepsilon = +$. If n is even and $\varepsilon = -$ then $[u^{n-1} \pm 1, u \mp 1] \in \omega(S)$, whence $\tilde{\pi}(S) \subseteq R_{2n}(u) \cup R_{n-1}(\tau u)$, where τ is defined by $u \equiv \tau 1 \pmod 4$.

It follows that $t^*(4, S) \leq 2$. □

Lemma 3.6. *Let S be a simple group from the statement of Lemma* 3.5.

(a) *If $S \leq G \leq \operatorname{Aut} S$ and $|G/S|$ is divisible by an odd prime r larger than* 3 *then $4r \in \omega(G)$.*

(b) *S has a Frobenius subgroup whose kernel is a v-group and whose complement is cyclic of order* 4.

Proof. (a) Suppose that r does not divide $(n, u-\varepsilon)$ if $S = L_n^\varepsilon(u)$. Then G contains a field automorphism of order r. The centralizer of this field automorphism is a group of the same Lie type as S but over some subfield of $GF(u)$, so it has an element of order 4.

Let $S = L_n^\varepsilon(u)$ and r divides $(n, u-\varepsilon)$. Then $n \geq r \geq 5$, and so u^2-1, which is divisible by $4r$, lies in $\omega(S)$.

(b) By [35, Lemma 2.1], $G_2(u)$ has a Frobenius subgroup with kernel of order u^2 and cyclic complement of order u^2-1. By the embeddings $G_2(u) < {}^3D_4(u) < F_4(u)$, ${}^3D_4(u)$ and $F_4(u)$ also contain such a Frobenius subgroup. By [36, Lemma 3.5], $S_4(u)$ has a Frobenius subgroup with kernel of order u^2 and cyclic complement of order $(u^2-1)/2$. In the remaining cases, S has a subgroup isomorphic to $L_3(q)$ or $SL_3(q)$, and hence it has a Frobenius subgroup with kernel of order u^2 and cyclic complement of order $(u^2-1)_{3'}$ [37, Lemma 5]. □

4. Proof of the theorem

Let $q = 2^k$ and L be one of the simple groups $S_{2n}(q)$, where $n \geq 2$, or $O_{2n}^\pm(q)$, where $n \geq 4$. If L is one of the groups $S_4(q)$, $S_6(q)$, $S_8(q)$, and $O_8^+(q)$, then the assertion follows from Lemma 2.5. Thus throughout the proof we assume that L differs from these groups.

Suppose that G is a finite group such that $\omega(G) = \omega(L)$. Since $t(L) \geq 3$ and $t(2, L) \geq 2$ (here and in what follows we use the information about cocliques of prime graphs of simple groups obtained in [20, 38]), Lemma 2.3 implies that there is a nonabelian simple group S such that

$$S \leq \overline{G} = G/K \leq \operatorname{Aut} S,$$

where K is the soluble radical of G.

Our aim is to show that $S \simeq L$. By Lemma 2.6, we may assume that $n \geq 5$. By [39, Theorems 1, 2], it follows that S is neither sporadic nor alternating. By [2, Theorem 2], if S is a group of Lie type over a field of characteristic 2 then $S \simeq L$. Thus it remains to establish that S cannot be a group of Lie type over a field of odd characteristic.

Proposition 1. *If $n \geq 5$ then S is not a group from the statement of Lemma* 3.5.

Proof. Suppose the assertion is false. Let $\rho = \rho^*(4, L)$ be a coclique defined in Lemma 3.4. Recall that for every $r \in \rho$, we have $r > 3$ and $4r \notin \omega(L)$. If $\rho \subseteq \pi(S)$ then $t^*(4, S) \geq 3$, which contradicts Lemma 3.5. Thus there is $r \in \rho$ such that $r \in \pi(K)$ or $r \in \pi(\overline{G}/S)$. By Lemma 3.6(a), the latter condition yields $4r \in \omega(G) \setminus \omega(L)$, which is a contradiction.

Suppose that $r \in \pi(K)$. Then two elements of $\rho \setminus \{r\}$ lie in $\pi(S) \setminus \pi(K)$ by Lemma 2.3 and have disjoint neighbourhoods in $GK(G)$ by Lemma 3.4(c). If $r = v$ then by Lemma 3.5, it follows that $4r \in \omega(S)$, and hence we may assume that $r \neq v$. By Lemma 3.6(b), S has a Frobenius subgroup whose kernel is a v-group and whose complement is a cyclic group of order 4. Applying Lemma 2.4, we conclude that $4r \in \pi(G)$, a contradiction. □

Observe that the conclusion of Proposition 1 for $n \geq 20$ is proved in [19, Proposition 6]. On the other hand, our proof is suitable for all $n \geq 5$, so we do not use the result of [19, Proposition 6], however, following the lines of its proof.

Lemma 4.1. *Let m be the largest $\pi(m)$-number in $\omega(G)$, $(m, |K| \cdot |\overline{G}/S|) = 1$ and let S have a cyclic $\pi(m)$-Hall subgroup. If $r \in \pi(\overline{G}/S)$ then either r divides $m-1$ or $rs \in \omega(\overline{G})$ for some $s \in \pi(m)$.*

Proof. Suppose that H is a cyclic $\pi(m)$-Hall subgroup of S. Then $|H|$ is the largest $\pi(m)$-number in $\omega(G)$, and hence $|H| = m$. Let $s \in \pi(m)$ and P be a Sylow s-subgroup of S. By the Frattini argument, $\overline{G} = SN_{\overline{G}}(P)$, therefore, $N_{\overline{G}}(P)$ has an element of order r. If $rs \notin \omega(\overline{G})$ then this element acts fixed-point-freely on P, whence $|P| \equiv 1 \pmod r$. Thus if $rs \notin \omega(\overline{G})$ for every $s \in \pi(m)$ then $|H| \equiv 1 \pmod r$. □

Proposition 2. *If $n \geq 5$ then S is not one of the groups $L_2(u)$, $L_3^{\pm}(u)$, ${}^2G_2(u)$, $E_7(u)$, or $E_8(u)$, where u is odd.*

Proof. Suppose that the assertion is false. By Lemma 2.3, we have $t(L) \leq t(S) + 1$ and $t(2, L) \leq t(2, S)$.

1. Suppose that $S = E_8(u)$. Then $t(S) = 12$, and so $t(L) \leq 13$. Furthermore, $u^8 - 1 \in \omega(S)$ by [32]. Since 32 divides $u^8 - 1$ and $32 \neq u^8 - 1$, it follows that $32 \in \omega(S) \setminus \mu(S)$. Thus $32 \in \omega(L) \setminus \mu(L)$. Applying formulas for $t(L)$ and Lemmas 3.1, 3.2, we conclude that L is one of the groups $S_{2n}(q)$, where $10 \leq n \leq 16$, $O_{2n}^+(q)$, where $11 \leq n \leq 18$, or $O_{2n}^-(q)$, where $11 \leq n \leq 17$. Moreover, by Lemma 2.6 we may assume that $L \neq S_{32}(q), O_{32}^-(q)$.

By Lemmas 3.1 and 3.2, elements of $\omega(L)$ that are divisible by 32 do not exceed $32 \cdot m(n-9, q)$. By Lemma 3.3, the last number is at most $(q^{n-8} - 1)/(q - 1)$. Thus

$$u^8 - 1 \leq 32(q^{n-8} - 1)/(q - 1). \tag{4.1}$$

Assume, first, that $q \geq 4$. Then $(q^k - 1)/(q - 1) < 4q^{k-1}/3$ for every positive integer k. It follows that $u^8 - 1 \leq 32(q^{n-8} - 1)/(q - 1) < 32 \cdot 4q^{n-9}/3$, whence

$$u^8 \leq 43q^{n-9}.$$

Choose i to be the largest prime that does not exceed n if L is a symplectic group, and the largest prime less than n if L is an orthogonal group. Then $7 \leq i \leq 17$ and $i \geq n - 4$. In particular, $i > n/2$, and hence $r_i(q)$ and $r_i(-q)$ lie in some coclique of size 3 in $GK(L)$. By Lemma 2.3, we have that $k_i(\epsilon q)$ lies in $\omega(S)$ for some $\epsilon \in \{+, -\}$. Orders of elements of S do not exceed $2u^8$ (see, for example, [40]). Thus

$$\frac{q^{i-1}}{2(i, q-\epsilon)} \leq \frac{q^i - \epsilon}{(q-\epsilon)(i, q-\epsilon)} = k_i(\epsilon q) \leq 2u^8 \leq 86q^{n-9}.$$

Combining this with the inequality $i \geq n-4$, we calculate that

$$q^4 \leq q^{i+8-n} \leq 172(i, q-\epsilon) \leq 172 \cdot 17.$$

Now $q^4 \leq 172 \cdot 17$ yields $q = 4$. But then $(i, q \pm 1) = 1$, and so $256 = q^4 \leq 172$, a contradiction.

Now let $q = 2$. By (4.1), it follows that $u = 3$ and $n = 17, 18$. But then $r_7(3) = 1093 \in \pi(S) \setminus \pi(L)$, a contradiction.

2. Suppose that $S = E_7(u)$. Then $t(S) = 8$ and $t(2, S) = 3$, and hence $t(L) \leq 9$ and $t(2, L) \leq 3$. Thus $n \leq 11$ if L is symplectic, and $n \geq 13$ is L is orthogonal; moreover, $L \neq O^-_{2n}(q)$ with n even. Observe that every coclique of $GK(S)$ that has size 3 and contains 2 is of the form $\{2, r_7(\epsilon u), r_9(\epsilon u)\}$, where $\epsilon \in \{+, -\}$ is defined by $u + \epsilon 1 \equiv 0 \pmod 4$.

We begin with the case $t(2, L) = 2$, that is, when L is one of the groups $S_{12}(q)$, $S_{16}(q)$, and $S_{20}(q)$. By Lemma 2.6, we may assume that $L \neq S_{16}(q)$.

Let $L = S_{12}(q)$. It follows from [32] that $\omega(S)$ contains $(u^4+1)(u^2-1)$. This number is divisible by 16. By Lemma 3.1, elements of $\omega(L)$ that are divisible by 16 divide $16(q \pm 1)$. Thus

$$(u^4+1)(u^2-1) \leq 16(q+1). \tag{4.2}$$

Elements of $R_{12}(q)$ are not adjacent to 2 in $GK(L)$, therefore, $k_{12}(q)$ divides one of the numbers $k_7(\epsilon u)$ and $k_9(\epsilon u)$. If follows that

$$k_{12}(q) \leq \frac{u^7-1}{u-1} < \frac{3}{2}u^6.$$

So we have

$$q^4 - q^2 \leq q^4 - q^2 + 1 = k_{12}(q) < \frac{3}{2}u^6 \leq 3(u^4+1)(u^2-1) \leq 48(q+1),$$

which yields $q^2(q-1) < 48$, whence $q = 2$. But then $(u^4+1)(u^2-1) \geq (3^4+1)(3^2-1) > 48 = 16(q+1)$, contrary to (4.2).

If $L = S_{20}(q)$ then by Lemma 3.1 and Lemma 3.3( ), we conlude in similar manner that

$$(u^4+1)(u^2-1) \leq 16(q^4+1)(q+1) \tag{4.3}$$

and

$$k_{20}(q) < \frac{3}{2}u^6.$$

Thus

$$\frac{q^{10}+1}{(5, q^2+1)(q^2+1)} = k_{20}(q) < \frac{3}{2}u^6 \leq 3(u^4+1)(u^2-1) \leq 48(q^4+1)(q+1),$$

whence $q^{10} < 48 \cdot (5, q^2+1)(q^8-1)/(q-1)$, and so $q \leq 4$. If $q = 4$ then $(5, q^2+1) = 1$, and, therefore, we have $4^{10} < 16(4^8-1)$, which is false. If $q = 2$ then (4.3) implies that $u = 3$. But then $r_7(3) = 1093 \in \pi(S) \setminus \pi(L)$, a contradiction.

Now suppose that $t(2, L) = 3$ and let $\{2, r, s\}$ be a coclique in $GK(L)$. By Lemma 3.4, the numbers r and s have no common neighbours in $GK(L)$ and by Lemma 2.3, both these numbers lie in $\pi(S)$. It follows that one of them divides $k_7(\epsilon u)$, while another divides $k_9(\epsilon u)$. However, if $u > 3$ then $r_7(\epsilon u)$ and $r_9(\epsilon u)$ have a common neighbour $r_1(\epsilon u)$ in $GK(S)$ (see [38, Fig. 4]). Thus $u = 3$. The only numbers not adjacent to 2 in $GK(S)$ are $1093 = k_7(3)$ and $757 = k_9(3)$. Observe that $e(1093, 2) = 364$ and $e(757, 2) = 756$. If $L = O^+_{2n}(q)$, where n is odd and $q = 2^k$, then $r_{nk}(2)$ and $r_{2(n-1)k}(2)$ are not adjacent to 2 in

$GK(L)$, and hence $\{r_{nk}(2), r_{2(n-1)k}(2)\} = \{1093, 757\}$, which yields $2(n-1)/n = 756/364$. Since $756/364 = 27/13$, the last equation has no positive integer solutions. For other groups, we derive similar equations with $2n/n$, $n/(n-1)$, or $2(n-1)/(n-1)$ in place of $2(n-1)/n$ in the left-hand side, and these equations have no solutions either.

3. Suppose that $S = L_3^{\pm}(u)$. Then $t(GK(S) \setminus \{3\}) \leq 3$ and $t(2, S) = 2$. If $t(L) > 4$ then $GK(L)$ has a coclique of size 5 that consists of numbers larger than 3, and hence $t(L) \leq 4$. It follows that L is one of the groups $S_{10}(2)$, $O_{10}^+(q)$, $O_{12}^+(q)$, and $O_{10}^-(q)$. But then $t(2, L) = 3$, a contradiction.

4. Suppose that $S = L_2(u)$, where $u = v^m \geq 5$ and v is a prime. Then $\mu(S) = \{v, (u-1)/2, (u+1)/2\}$. Since $t(S) = 3$, we see that L is one of the groups $S_{10}(2)$, $O_{10}^+(q)$, $O_{12}^+(q)$, and $O_{10}^-(q)$. By Lemmas 2.5 and 2.6, we may assume that $L \neq S_{10}(2)$, $O_{10}^{\pm}(2)$.

If $L = O_{10}^{\varepsilon}(q)$, where $q > 2$, then define $\rho = \{r_5(\varepsilon q), r_8(q), r_6(q), r_3(q)\}$, and if $L = O_{12}^+(q)$ then define $\rho = \{r_{10}(q), r_5(q), r_8(q), r_3(q)\}$. Then ρ is a coclique in $GK(L)$ and it contains numbers t and s that are not adjacent to 2 in $GK(L)$. Suppose that some element of ρ, say r, divides $|K|$. One of the three remaining numbers, say w, divides $(u-1)/2$. Two eventually remaining numbers divides u and $(u+1)/2$, in particular $r \neq v$. The group S has a Frobenius subgroup with kernel of order u and cyclic complement of order $(u-1)/2$. The numbers t and s lie in $\pi(S) \setminus \pi(K)$ by Lemma 2.3 and have disjoint neighbourhoods in $GK(L)$ by Lemma 3.4. Applying Lemma 2.4, we conclude that $rw \in \omega(G)$, a contradiction. Thus none of the elements of ρ divides $|K|$. It follows that some element of $\rho \setminus \{t, s\}$ divides $|\overline{G}/S|$, which yields $m \geq 7$.

Let $L = O_{10}^{\varepsilon}(q)$, where $q > 2$. Then $r_5(\varepsilon q)$ and $r_8(q)$ are not adjacent to 2, and hence they are coprime to $|K||\overline{G}/S|$. Thus $k_5(\varepsilon q)$ and $k_8(q)$ lie in $\omega(S)$, and one of them is equal to v. Since

$$k_5(\varepsilon q) = \frac{q^5 - \varepsilon}{(5, q-\varepsilon)(q-\varepsilon)} \text{ and } k_8(q) = q^4 + 1,$$

we see that $v > q^4/10$. It follows that $u \geq v^7 > q^{28}/10^7$. On the other hand, Lemma 3.3 implies that $(u+1)/2 \leq 2q^5$. Thus $q^{28}/10^7 < u < 4q^5$, but this contradicts the condition $q > 2$.

If $L = O_{12}^+(q)$ then similar reasoning shows that $k_5(q)$ and $k_{10}(q)$ lie in $\omega(S)$ and one of them is equal to v, and so $v > q^4/10$. Then $q^{28}/10^7 < u < 4q^6$, whence $q = 2$. Observe that $k_5(2) = 31$ and $k_{10}(2) = 11$. It follows that $31 \in \pi(S) \subseteq \pi(O_{10}^+(2)) \subseteq \{2, \ldots, 31\}$. By [41, Table 1], we see that $S = L_2(31)$ or $S = L_2(5^3)$, but then $11 \notin \pi(S)$, a contradiction.

5. Suppose that $S = {}^2G_2(u)$, where $u = 3^m$ and m is odd. Then $t(GK(S) \setminus \{3\}) = 4$ and $t(2, S) = 3$. It follows that L is one of the groups $S_{10}(q)$, $S_{12}(q)$, $O_{10}^+(q)$, $O_{12}^+(q)$, $O_{10}^-(q)$, and $O_{14}^-(2)$. We may assume that $L \neq S_{10}(2)$, $O_{10}^{\pm}(2)$.

The order of S is equal to $u^3(u^3+1)(u-1)$. Exponents of maixmal tori of S are $u-1$, $(u+1)/2$, and $u \pm \sqrt{3u}+1$ [42]. In particular, if π is a subset of $\pi(u \pm 1)$ or $\pi(u \pm \sqrt{3u}+1)$ and $2, 3 \notin \pi$, then S has a cyclic π-Hall subgroup. Thus if $(k_8(q), |K||\overline{G}/S|) = 1$ then by Lemma 4.1, we have $rr_8(q) \in \omega(G)$ for every $r \in \pi(\overline{G}/S)$.

Let $t(L) = 5$. Then $L = S_{10}(q)$, where $q > 2$, or $L = S_{12}(q)$, and we can take $\{r_{10}(q), r_5(q), r_8(q), r_6(q), r_3(q)\}$ or $\{r_{12}(q), r_{10}(q), r_5(q), r_8(q), r_3(q)\}$, respectively, as a coclique ρ of largest size in $GK(L)$. Since $t(GK(S) \setminus \{3\}) = 4$, some element of ρ, say r, divides either $|K|$ or $|\overline{G}/S|$, while the other elements are coprime to $|K| \cdot |\overline{G}/S|$. Suppose that r divides $|K|$. One of the four remaining numbers, say w, divides $(u-1)/2$. Furthermore, by Lemma 3.4, two of the four remaining numbers have disjoint neighbourhoods in $GK(L)$: these are $r_{10}(q)$ and $r_5(q)$ for $n=5$; $r_{12}(q)$ and $r_{10}(q)$ for $n=6$, $r\neq r_{10}(q)$; and $r_{12}(q)$ and $r_5(q)$ for $n=6$, $r=r_{10}(q)$. Since S has a subgroup isomorphic to $L_2(u)$, and hence has a Frobenius subgroup with kernel of order u and cyclic complement of order $(u-1)/2$, it follows by Lemma 2.4 that $rw\in\omega(G)$. This contradiction shows that $r\in\pi(\overline{G}/S)$. Assume that $r\notin R_8(q)$. Then, as we remarked earlier, $rr_8(q)\in\omega(G)$, a contradiction. Thus $k_8(q)\in\omega(\overline{G}/S)$, whence $m\geq q^4+1$. Furthermore, $u+\sqrt{3u}+1\leq 2q^6$ by Lemma 3.3. It follows that $2q^6\geq u+\sqrt{3u}+1>u\geq 3^{q^4+1}$. However, $2q^6<3^{q^4+1}$ for all $q\geq 2$.

Let $L=O_{12}^+(q)$ and consider cocliques of the form $\rho=\{r_5(q),r_{10}(q),r_8(q),r_3(q)\}$ in $GK(L)$. Assume that any such coclique is disjoint from $\pi(K)\cup\pi(\overline{G}/S)$. Then $k_8(q)$ lies in $\omega(S)$ and divides $u-1$ or $(u+1)/2$. It follows that $u-1$ or $(u+1)/2$ is divisible by $2(q^4+1)$. On the other hand, Lemma 3.2 implies that $2(q^4+1)\in\mu(L)$. Thus $u-1=2(q^4+1)$ or $(u+1)/2=2(q^4+1)$, whence $u=2q^4+3$ or $u=4q^4+3$, which is impossible since $u=3^m$. This contradiction shows that one of the numbers $r_8(q)$ and $r_3(q)$, say r, divides $|K|$ or $|\overline{G}/S|$. Suppose that $r\in\pi(K)$. Observe that $8r\notin\omega(L)$. On the other hand, $16\in\omega(L)$, and hence $8\in\omega(K)$. Let H be a $\{2,r\}$-Hall subgroup of K. Then $G=KN_G(H)$, therefore, $N_G(H)$ has an element of order $r_5(q)$. This element acts fixed-point-freely on H, and thus H is nilpotent and $8r\in\omega(K)$, a contradiction. It follows that $r\in\pi(\overline{G}/S)$ and so, reasoning as in the previous paragraph, we derive a contradiction.

Let $L=O_{10}^{\varepsilon}(q)$, where $q>2$. A coclique of largest size in $GK(L)$ is the union of $\{r_5(\varepsilon q),r_8(q),r_3(\varepsilon q)\}$ and $\{r_4(q)\}$ or $\{r_6(\varepsilon q)\}$. And we have that $k_8(q)$ and $k_5(\varepsilon q)$ are coprime to $|K||\overline{G}/S|$. Applying Lemma 4.1 to $k_8(q)$, we conlude that every $r\in\pi(\overline{G}/S)$ is adjacent to $r_8(q)$ in $GK(L)$, and hence $\pi(\overline{G}/S)\subseteq R_1(-\varepsilon q)$. Assume that $k_4(q)k_6(\varepsilon q)$ is coprime to $|K|$. Then $r_4(q)r_6(\varepsilon q)$ divides either $u-1$ or $(u+1)/2$ and, therefore, $2r_4(q)r_6(\varepsilon q)\in\omega(S)\setminus\omega(L)$, a contradiction. Thus one of the numbers $r_4(q)$ and $r_6(\varepsilon q)$, say r, divides $|K|$. Then $k_3(\varepsilon q)$ is coprime to $|K|$, and so $r_3(\varepsilon q)$ divides either $u-1$ or $(u+1)/2$, while $r_5(\varepsilon q)$ divides $u+\sqrt{3u}+1$ or $u-\sqrt{3u}+1$. In any case, $r_3(\varepsilon q)$ and $r_5(\varepsilon q)$ have no common neighbours in $GK(S)$. On the other hand, $r_1(\varepsilon q)r_3(\varepsilon q)$ and $r_1(\varepsilon q)r_5(\varepsilon q)$ lie in $\omega(G)$. It follows that $r_1(\varepsilon q)\in\pi(K)$. Furthermore, $4\in\omega(K)$ since $8\in\omega(L)$. Taking a $\{2,r,r_1(\varepsilon q)\}$-Hall subgroup of K and acting on it by an element of order $r_8(q)$, we derive that $4rr_1(\varepsilon q)\in\omega(G)\setminus\omega(L)$.

Let, finally, $L=O_{14}^-(2)$. Then $43\in\pi(S)\subseteq\{2,\dots,43\}$. By [41, Table 1], none of the groups ${}^2G_2(u)$ meets this condition. □

By Propositions 1 and 2, the group S cannot be a group of Lie type over a field of odd characteristic, and thus $S\simeq L$. Then $K=1$ by [43, Theorem 1.1], and hence up to isomorphism $L\leq G\leq \operatorname{Aut} L$. It follows from [9] that $G\simeq L$. Thus L is recognizable by spectrum, and the proof is complete.

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