Cocliques of maximal size in the prime graph of a finite simple group

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Abstract

A prime graph of a finite group is defined in the following way: the set of vertices of the graph is the set of prime divisors of the group, and two distinct vertices r and s are adjacent, if there is an element of order rs in the group. In this paper we continue our investigation of the prime graph of a finite simple group started in [1], namely we describe all cocliques of maximal size for all finite simple groups.

Let G be a finite group, let $\pi(G)$ be the set of all prime divisors of its order, and let $\omega(G)$ be the spectrum of G, i. e., the set of its element orders. A graph $GK(G)$ is called the *prime graph* (or the *Gruenberg-Kegel graph*) of G, if the set of vertices of $GK(G)$ equals $\pi(G)$, and two distinct vertices r and s are adjacent in if and only if $rs \in \omega(G)$. Primes $r, s \in \pi(G)$ are called *adjacent*, if they are adjacent as vertices of $GK(G)$. Otherwise, r and s are called *non-adjacent*.

In the present paper we continue the investigation of the prime graphs of finite simple groups started in [1]. We preserve notation and agreements from [1].

Denote by $t(G)$ the maximal number of prime divisors of G that are pairwise non-adjacent in $GK(G)$. In other words $t(G)$ is the maximal number of vertices in cocliques of $GK(G)$ (a set of vertices of a graph is called a *coclique* (or *independent*), if its elements are pairwise non-adjacent). In the graph theory this number is called an *independence number* of a graph. By analogy we denote by $t(r, G)$ the maximal number of vertices in cocliques of $GK(G)$, containing a prime r. We call this number an *r-independence number*.

In [1] for every finite nonabelian simple group G we gave an arithmetical criterion of adjacency of vertices in the prime graph $GK(G)$. Using this criterion we determined the values of $t(G)$, $t(2, G)$, and in case, when G is a group of Lie type over a field of characteristic p, the value of $t(p, G)$. Denote by $\rho(G)$ and $\rho(r, G)$ a coclique of maximal size in $GK(G)$ and a coclique of maximal size, containing r, in $GK(G)$ respectively. It is not difficult to see that in general $\rho(G)$ and $\rho(r, G)$ are not uniquely determined. In [1] all cocliques $\rho(2, G)$, and also all cocliques $\rho(p, G)$ for groups G of Lie type over a field of characteristic p, were described. Moreover, in the same paper for every simple group G at least one coclique $\rho(G)$ have been determined, and this allows to calculate $t(G)$, but the problem of finding all such cocliques has not been considered.

The main goal of the present paper is to find all cocliques of maximal size in the prime graph of a finite simple group G. In order to achieve this goal we introduce two sets $\Theta(G)$ and $\Theta'(G)$ consisting of some subsets of $\pi(G)$. Then every coclique $\rho(G)$ of maximal size can be derived then as $\theta(G) \cup \theta'(G)$, where $\theta(G) \in \Theta(G)$ and $\theta'(G) \in \Theta'(G)$.

Theorem. *Suppose that G is a finite nonabelian simple group. Then every coclique of maximal size in $GK(G)$ is the union of $\theta(G) \in \Theta(G)$ and $\theta'(G) \in \Theta'(G)$. The sets $\Theta(G)$, $\Theta'(G)$ together with the maximal size $t(G)$ of cocliques in $GK(G)$ are described in Proposition* 1.1 *for alternating groups, in Table* 1 *for sporadic groups, and in Tables* 2, 3, 4 *for groups of Lie type.*

Article [1] appeared to contain several misprints and errors. Some of them were found by the authors, and others were pointed out by the readers of this paper. We are grateful to W. Shi, H. He, A. R. Moghaddamfar, A. Iranmanesh, Z. Taheri, S. Shariati, M. A. Grechkoseeva, A. A. Buturlakin, and A. Zavarnitsine for their comments. Section 4 of the present article contains the corrections to all detected inaccuracies in [1].

1 Sporadic and alternating groups

Results of the section are easily developed from known ones, and we include them here just for completeness. Let G be a finite simple sporadic or alternating group. Denote by $\theta(G)$ the intersection of all cocliques of maximal size of $GK(G)$, and by $\Theta(G)$ the set $\{\theta(G)\}$. The set $\Theta'(G)$ is defined as follows. A subset $\theta'(G)$ of $\pi(G) \setminus \theta(G)$ is an element of $\Theta'(G)$ if and only if $\rho(G) = \theta(G) \cup \theta'(G)$ is coclique of $GK(G)$ of maximal size. Obviously, the sets $\Theta(G)$ and $\Theta'(G)$ are uniquely determined, and $\Theta'(G)$ either is empty or contains at least two elements.

We start with alternating groups. Let $G = Alt_n$ be the alternating group of degree n, and $n \geqslant 5$. In order to describe cocliques of maximal size in $GK(G)$ we present the following notation. For every prime r define $e(r) = r$ if r is odd, and $e(r) = 4$ if $r = 2$. Denote by $\tau(n)$ the set of all primes r with $n/2 \leqslant e(r) \leqslant n$, and by s_n and s'_n the smallest elements of $\tau(n)$ and $\tau(n) \setminus \{s_n\}$ respectively. Define the sets $\tau'(n)$ and $\tau''(n)$ as follows. A prime r lies in $\tau'(n)$ if and only if $e(r) < n/2$ and $e(r) + e(s_n) > n$, and r lies in $\tau''(n)$ if and only if $e(r) < n/2$ and $e(r) + e(s'_n) > n$.

Proposition 1.1. *Let G be an alternating group of degree n, and $n \geqslant 5$.*

1. *If $\tau'(n) = \tau''(n) = \varnothing$, then $\theta(G) = \tau(n)$, and $\Theta'(G) = \varnothing$.*

2. *If $\tau'(n) = \varnothing$ and $\tau''(n) \neq \varnothing$, then $\theta(G) = \tau(n) \setminus \{s_n\}$, and $\Theta'(G) = \{\{r\} \mid r \in \tau''(n) \cup \{s_n\}\}$.*

3. *If $|\tau'(n)| = 1$, then $\theta(G) = \tau(n) \cup \tau'(n)$, and $\Theta'(G) = \varnothing$.*

4. *If $|\tau'(n)| \geqslant 2$, then $\theta(G) = \tau(n)$, and $\Theta'(G) = \{\{r\} \mid r \in \tau'(n)\}$.*

In all cases every coclique of maximal size in $GK(G)$ is of the form $\theta(G) \cup \theta'(G)$, where $\theta(G) \in \Theta(G)$, and $\theta'(G) \in \Theta'(G)$. The set $\Theta(G) = \{\theta(G)\}$ is one-element, and all elements $\theta'(G)$ of $\Theta'(G)$ are one-element subsets of $\pi(G)$.

Proof. An adjacency criterion for vertices of $GK(G)$ [1, Proposition 1.1] can be formulated as follows. Distinct primes $r, s \in \pi(G)$ are non-adjacent in $GK(G)$ if and only if $e(r) + e(s) > n$. Therefore, $\tau(n)$ is a coclique of $GK(G)$, and $\pi(G) \setminus \tau(n)$ is a clique. Moreover, if

r, s, t are distinct primes from $\pi(G)$, and $e(r) < e(s)$, then the adjacency of s and t implies the adjacency of r and t, as well as the non-adjacency of r and t implies the non-adjacency of s and t. These simple observations allow to verify the assertion easily. □

Proposition 1.2. *Let G be a simple sporadic group. If $\Theta'(G) = \varnothing$, then $\theta(G)$ is the unique coclique of maximal size in $GK(G)$. If $\Theta'(G) \neq \varnothing$, then every coclique of maximal size is of the form $\theta(G) \cup \theta'(G)$, where $\theta'(G) \in \Theta'(G)$. If $G \neq M_{23}$ then every $\theta'(G)$ of $\Theta'(G)$ contains precisely one element. The sets $\Theta(G)$ and $\Theta'(G)$, as well as the value of $t(G)$, are listed in Table* 1.

Proof. The proposition is easy to verify using [2] or [3]. □

Remark. Note that in Columns 3 and 4 of Table 1 we list the elements of $\Theta(G)$ and $\Theta'(G)$, that is sets $\theta(G) \in \Theta(G)$ and $\theta'(G) \in \Theta'(G)$, and omit the braces for one-element sets. In particular, for group $G = M_{11}$ we have $\Theta(G) = \{\theta(G)\} = \{\{5, 11\}\}$ and $\Theta'(G) = \{\{2\}, \{3\}\}$, while for $G = M_{23}$ we have $\Theta(G) = \{\theta(G)\} = \{\{11, 23\}\}$ and $\Theta'(G) = \{\{2, 5\}, \{3, 7\}\}$.

Table 1. Cocliques of sporadic groups

G	$t(G)$	$\Theta(G)$	$\Theta'(G)$
M_{11}	3	$\{5, 11\}$	$2, 3$
M_{12}	3	$\{3, 5, 11\}$	$\varnothing$
M_{22}	4	$\{5, 7, 11\}$	$2, 3$
M_{23}	4	$\{11, 23\}$	$\{2, 5\}$, $\{3, 7\}$
M_{24}	4	$\{5, 7, 11, 23\}$	$\varnothing$
J_1	4	$\{7, 11, 19\}$	$2, 3, 5$
J_2	2	7	$2, 3, 5$
J_3	3	$\{17, 19\}$	$2, 3, 5$
J_4	7	$\{11, 23, 29, 31, 37, 43\}$	$5, 7$
Ru	4	$\{7, 13, 29\}$	$3, 5$
He	3	$\{5, 7, 17\}$	$\varnothing$
McL	3	$\{7, 11\}$	$3, 5$
HN	3	$\{11, 19\}$	$3, 5, 7$
HiS	3	$\{7, 11\}$	$2, 3, 5$
Suz	4	$\{5, 7, 11, 13\}$	$\varnothing$
Co_1	4	$\{11, 13, 23\}$	$5, 7$
Co_2	4	$\{7, 11, 23\}$	$3, 5$
Co_3	4	$\{5, 7, 11, 23\}$	$\varnothing$
Fi_{22}	4	$\{5, 7, 11, 13\}$	$\varnothing$
Fi_{23}	5	$\{11, 13, 17, 23\}$	$5, 7$
Fi'_{24}	6	$\{11, 13, 17, 23, 29\}$	$5, 7$
O'N	5	$\{7, 11, 19, 31\}$	$3, 5$
LyS	6	$\{5, 7, 11, 31, 37, 67\}$	$\varnothing$
F_1	11	$\{11, 13, 19, 23, 29, 31, 41, 47, 59, 71\}$	$7, 17$
F_2	8	$\{7, 11, 13, 17, 19, 23, 31, 47\}$	$\varnothing$
F_3	5	$\{5, 7, 13, 19, 31\}$	$\varnothing$

In addition, we notice another substantial property of prime graphs of groups under consideration.

Proposition 1.3. *Suppose that G is either an alternating group of degree n, $n \geqslant 5$, or a sporadic group distinct from M_{23}. Then the set $\pi(G) \setminus \theta(G)$ is a clique of $GK(G)$.*

Proof. This follows from [2] and [1, Proposition 1.1]. □

2 Preliminary results for groups of Lie type

We write $[x]$ for the integer part of a rational number x. The set of prime divisors of a natural number m is denoted by $\pi(m)$. By $(m_1, m_2, \ldots, m_s)$ we denote the greatest common divisor of numbers $m_1, m_2, \ldots, m_s$. For a natural number r, the r-part of a natural number m is the greatest divisor t of m with $\pi(t) \subseteq \pi(r)$. We write m_r for the r-part of m and $m_{r'}$ for the quotient m/m_r.

If q is a natural number, r is an odd prime and $(q, r) = 1$, then $e(r, q)$ denotes a multiplicative order of q modulo r, that is a minimal natural number m with $q^m \equiv 1 \pmod r$. For an odd q, we put $e(2, q) = 1$ if $q \equiv 1 \pmod 4$, and $e(2, q) = 2$ otherwise.

Lemma 2.1. (Corollary of Zsigmondy's theorem [4]) *Let q be a natural number greater than 1. For every natural number m there exists a prime r with $e(r, q) = m$ but for the cases $q = 2$ and $m = 1$, $q = 3$ and $m = 1$, and $q = 2$ and $m = 6$.*

Remark. In conclusion of the same corollary [1, Lemma 1.4] in our previous article we miss two exceptions: $m = 1$ and $q = 2$, and $m = 1$ and $q = 3$. However, these exceptions don't arise in all proofs and arguments from [1], that use the corollary to Zsigmondy's theorem.

A prime r with $e(r, q) = m$ is called a *primitive prime divisor* of $q^m - 1$. By Lemma 2.1 such a number exists except for the cases mentioned in the lemma. Given q we denote by $R_m(q)$ the set of all primitive prime divisors of $q^m - 1$ and by $r_m(q)$ any element of $R_m(q)$. If $m \neq 2$ then a divisor $k_m(q)$ of $q^m - 1$ is said to be the *greatest primitive divisor* if $\pi(k_m(q)) = \pi(R_m(q))$ and $k_m(q)$ is the greatest divisor with this property, i.e., $k_m(q) = (q^m - 1)_t$, where $t = \prod_{s \in R_m(q)} s$. The *greatest primitive divisor* $k_2(q)$ of $q^2 - 1$ is the greatest divisor of $q + 1$ with $\pi(k_2(q)) = R_2(q)$. The singularity in the definition of the greatest primitive divisor in case $m = 2$ appears because of the singularity of the definition for $e(2, q)$. Following our definition of $e(2, q)$, we derive that $k_1(q) = (q - 1)/2$ if $q \equiv -1 \pmod 4$, and $k_1(q) = q - 1$ otherwise; $k_2(q) = (q + 1)/2$ if $q \equiv 1 \pmod 4$, and $k_2(q) = q + 1$ otherwise. The following lemma provides a formula for expressing greatest primitive divisors k_m, $m \geqslant 3$, in terms of cyclotomic polynomials $\phi_m(x)$.

Lemma 2.2. [5] *Let q and m be natural numbers, $q > 1$, $m \geqslant 3$, and let $k_m(q)$ be the greatest primitive divisor of $q^m - 1$. Then*

$$k_m(q) = \frac{\phi_m(q)}{(\phi_{m_{r'}}(q), r)},$$

where r is the greatest prime divisor of m.

Usually the number q is fixed (for example, by the choice of a group of Lie type G), and we write R_m, r_m, and k_m instead of $R_m(q)$, $r_m(q)$, and $k_m(q)$ respectively. According to our definitions, if $i \neq j$, then $\pi(R_i) \cap \pi(R_j) = \varnothing$, and so $(k_i, k_j) = 1$.

Lemma 2.3. [6, Lemma 6(iii)] *Let q, k, l be natural numbers. Then*

(a) $(q^k-1, q^l-1) = q^{(k,l)}-1$;

(b) $(q^k+1, q^l+1) = \begin{cases} q^{(k,l)}+1, & \textit{if both } \frac{k}{(k,l)} \textit{ and } \frac{l}{(k,l)} \textit{ are odd,} \\ (2, q+1), & \textit{otherwise;} \end{cases}$

(c) $(q^k-1, q^l+1) = \begin{cases} q^{(k,l)}+1, & \textit{if } \frac{k}{(k,l)} \textit{ is even and } \frac{l}{(k,l)} \textit{ is odd,} \\ (2, q+1), & \textit{otherwise.} \end{cases}$

In particular, for every $q \geqslant 2$, $k \geqslant 1$ the inequality $(q^k-1, q^k+1) \leqslant 2$ holds.

We recall also the following statements [1, statements (1) and (4)]. Given $q = p^\alpha$, where p is a prime, and odd prime $c \neq p$ we have:

$$c \text{ divides } q^x - 1 \text{ if and only if } e(c,q) \text{ divides } x; \tag{1}$$

$$\text{if } c \text{ divides } q^x - \epsilon, \text{ where } \epsilon \in \{+1, -1\}, \text{ then } \eta(e(c,q)) \text{ divides } x. \tag{2}$$

The function $\eta(n)$ is defined in Proposition 2.4.

In the proofs of Propositions 2.4, 2.5, and 2.7 by ϵ, ϵ_i we denote elements from the set $\{+1, 1\}$. For groups of Lie type our notation agrees with that of [1]. We write $A_n^\varepsilon(q)$, $D_n^\varepsilon(q)$, and $E_6^\varepsilon(q)$, where $\varepsilon \in \{+,-\}$, and $A_n^+(q) = A_n(q)$, $A_n^-(q) = {}^2A_n(q)$, $D_n^+(q) = D_n(q)$, $D_n^-(q) = {}^2D_n(q)$, $E_6^+(q) = E_6(q)$, $E_6^-(q) = {}^2E_6(q)$. In [1, Proposition 2.2], considering unitary groups, we define the function

$$\nu(m) = \begin{cases} m & \text{if } m \equiv 0 (\text{mod } 4), \\ \frac{m}{2} & \text{if } m \equiv 2 (\text{mod } 4), \\ 2m & \text{if } m \equiv 1 (\text{mod } 2). \end{cases} \tag{3}$$

Clearly $\nu(m)$ is a bijection from $\mathbb{N}$ onto $\mathbb{N}$ and $\nu^{-1}(m) = \nu(m)$. In most cases it is natural to consider linear and unitary groups together. So we define

$$\nu_\varepsilon(m) = \begin{cases} m & \text{if } \varepsilon = +, \\ \nu(m) & \text{if } \varepsilon = -. \end{cases} \tag{4}$$

Proposition 2.4. *Let G be one of simple groups of Lie type $B_n(q)$ or $C_n(q)$ over a field of characteristic p. Define*

$$\eta(m) = \begin{cases} m & \textit{if } m \textit{ is odd,} \\ \frac{m}{2} & \textit{otherwise.} \end{cases}$$

Let r, s be odd primes with $r, s \in \pi(G) \setminus \{p\}$. Put $k = e(r,q)$ and $l = e(s,q)$, and suppose that $1 \leqslant \eta(k) \leqslant \eta(l)$. Then r and s are non-adjacent if and only if $\eta(k) + \eta(l) > n$, and k, l satisfy the following condition:

$$\frac{l}{k} \textit{ is not an odd natural number} \tag{5}$$

Proof. We prove the "if" part first. Assume that $\eta(k) + \eta(l) \leqslant n$, then there exists a maximal torus T of order $\frac{1}{(2,q-1)}(q^{\eta(k)} + (-1)^k)(q^{\eta(l)} + (-1)^l)(q-1)^{n-\eta(k)-\eta(l)}$ of G (see [1, Lemma 1.2(2)], for example). Both r, s divide $|T|$, hence r, s are adjacent in G. If $\frac{l}{k}$ is an odd integer, then either both k, l are odd and Lemma 2.3(a) implies that $q^{\eta(k)} + (-1)^k = q^k - 1$ divides $q^{\eta(l)} + (-1)^l = q^l - 1$, or both k, l are even and Lemma 2.3(b) implies that $q^{\eta(k)} + (-1)^k = q^{k/2} + 1$ divides $q^{\eta(l)} + (-1)^l = q^{l/2} + 1$. Again both r, s divide $|T|$, where T is a maximal torus of order $\frac{1}{(2,q-1)}(q^{\eta(l)} + (-1)^l)(q-1)^{n-\eta(l)}$ of G (the existence of such torus follows from [1, Lemma 1.2(2)]), so r, s are adjacent.

Now we prove the "only if" part. Assume by contradiction that $\eta(k)+\eta(l) > n$ and l/k is not an odd natural number, but r, s are adjacent. Then G contains an element g of order rs. The element g is semisimple, since $(rs, p) = 1$, hence g is contained in a maximal torus T of G. By [1, Lemma 1.2(2)] it follows that $|T| = \frac{1}{(2,q-1)}(q^{n_1} - \epsilon_1)(q^{n_2} - \epsilon_2)\ldots(q^{n_k} - \epsilon_k)$, where $n_1 + n_2 + \ldots + n_k = n$. Up to renumberring, we may assume that r divides $(q^{n_1} - \epsilon_1)$, while s divides either $(q^{n_1} - \epsilon_1)$ or $(q^{n_2} - \epsilon_2)$. Assume first that s divides $(q^{n_2} - \epsilon_2)$. Then (2) implies that $\eta(k)$ divides n_1 and $\eta(l)$ divides n_2, so $n_1 + n_2 \geqslant \eta(k) + \eta(l) > n$, a contradiction.

Now assume that both r, s divide $(q^{n_1} - \epsilon_1)$. Again (2) implies that both $\eta(k), \eta(l)$ divide n_1. Now $\eta(k) + \eta(l) > n$ and $\eta(k) \leqslant \eta(l)$, so $\eta(l) = n_1$. Assume first that l is odd. Then $l = \eta(l) = n_1$ and s divides $q^l - 1$. Since s is odd, Lemma 2.3 imples that s does not divide $q^l + 1$, hence $q^{n_1} - \epsilon_1 = q^{n_1} - 1$. Since r divides $q^{n_1} - 1$, by using (1) we obtain that k divides $n_1 = l$, hence k is odd. Therefore $\frac{l}{k}$ is an odd integer, a contradiction with (5). Now assume that l is even. Then $l/2 = \eta(l) = n_1$ and s divides $q^l - 1$. In view of (1), s does not divide $q^{l/2} - 1$, hence s divides $q^{l/2} + 1$ and $q^{n_1} - \epsilon_1 = q^{n_1} + 1$. Now (2) implies that $\eta(k)$ divides n_1, hence k divides $2n_1 = l$. By Lemma 2.3(c) we obtain that r does not divide $q^{l/2} - 1$, hence k does not divide $l/2$ and $\frac{l}{k}$ is an odd integer, a contradiction with (5). □

Proposition 2.5. *Let $G = D_n^\varepsilon(q)$ be a finite simple group of Lie type over a field of characteristic p, and let the function $\eta(m)$ be defined as in Proposition 2.4. Suppose r, s are odd primes and $r, s \in \pi(D_n^\varepsilon(q)) \setminus \{p\}$. Put $k = e(r, q)$, $l = e(s, q)$, and $1 \leqslant \eta(k) \leqslant \eta(l)$. Then r and s are non-adjacent if and only if $2 \cdot \eta(k) + 2 \cdot \eta(l) > 2n - (1 - \varepsilon(-1)^{k+l})$, k and l satisfy (5), and, if $\varepsilon = +$, then the chain of equalities*:

$$n = l = 2\eta(l) = 2\eta(k) = 2k \qquad (6)$$

is not true.

Proof. The following inclusions are known $\widetilde{B}_{n-1}(q) \leq \widetilde{D}_n^\varepsilon(q) \leq \widetilde{B}_n(q)$ (see [7, Table 2]), where $\widetilde{B}_{n-1}(q)$, $\widetilde{D}_n^\varepsilon(q)$, $\widetilde{B}_n(q)$ are central extensions of corresponding simple groups and $n \geqslant 4$. Since the Schur multiplier for each of simple groups $B_{n-1}(q)$, $D_n^\varepsilon(q)$, $B_n(q)$ has order equal to 1, 2, or 4, it is clear that two odd prime divisors of the order of a simple group isomorphic to $B_n(q)$ or $D_n^\varepsilon(q)$ are adjacent if and only if they are adjacent in every central extension of the group. Hence if two odd prime divisors of $|D_n^\varepsilon(q)|$ are adjacent in $GK(B_{n-1}(q))$, then they are adjacent in $GK(D_n^\varepsilon(q))$ and if two odd prime divisors of $|D_n^\varepsilon(q)|$ are non-adjacent in $GK(B_n(q))$, then they are non-adjacent in $GK(D_n^\varepsilon(q))$. There can be the following cases:

(i) $\eta(k)+\eta(l)\leqslant n-1$;

(ii) $\eta(k)+\eta(l)\geqslant n$, l/k is an odd number and $\eta(l)\leqslant n-1$;

(iii) $\eta(k)+\eta(l)=n$ and $\frac{l}{k}$ is not an odd natural number;

(iv) $\eta(l)=n$ and $\frac{l}{k}$ is an odd natural number;

(v) $\eta(k)+\eta(l)>n$ and $\frac{l}{k}$ is not an odd natural number.

By Lemma 2.4 in cases (i), (ii) primes r,s are adjacent in $GK(B_{n-1}(q))$, while in case (v) primes r,s are non-adjacent in $GK(B_n(q))$. In view of above notes it follows that we need to consider (iii) and (iv).

Assume first that $\eta(k)+\eta(l)=n$ and $\frac{l}{k}$ is not an odd natural number, i. e., case (iii) holds. Since $(rs,p)=1$, the primes r,s are adjacent in $GK(D_n^{\varepsilon}(q))$ if and only if there exists a maximal torus T of G of order divisible by rs. In view of [1, Lemma 1.2(3)] the order $|T|$ is equal to $\frac{1}{(4,q^n-\varepsilon 1)}(q^{n_1}-\epsilon_1)\cdot\ldots\cdot(q^{n_m}-\epsilon_m)$, where $n_1+\ldots+n_m=n$ and $\epsilon_1\cdot\ldots\cdot\epsilon_m=\varepsilon 1$. Up to renumberring, we may assume that r divides $q^{n_1}-\epsilon_1$, while s divides either $q^{n_1}-\epsilon_1$, or $q^{n_2}-\epsilon_2$.

If s divides $q^{n_1}-\epsilon_1$, then (2) implies that both $\eta(k)$, $\eta(l)$ divide n_1. As in the proof of Proposition 2.4 we derive that r,s are adjacent if and only if $\frac{l}{k}$ is an odd integer.

Assume now that s divides $q^{n_2}-\epsilon_2$. Then (2) implies that $\eta(k)$ divides n_1 and $\eta(l)$ divides n_2. Hence we obtain the following inequalities $n\geqslant n_1+n_2\geqslant\eta(k)+\eta(l)=n$, so $\eta(k)=n_1$, $\eta(l)=n_2$, and $q^{n_1}-\epsilon_1=q^{\eta(k)}+(-1)^k$, $q^{n_2}-\epsilon_2=q^{\eta(l)}+(-1)^l$. If $\varepsilon=-$, then a maximal torus T of order $\frac{1}{(4,q^n+1)}(q^{\eta(k)}+(-1)^k)(q^{\eta(l)}+(-1)^l)$ of G exists if and only if k,l have the distinct parity, i. e., if and only if $2n-(1-\varepsilon(-1)^{k+l})=2n-(1+(-1)^{k+l})=2n$. Hence in this case r,s are non-adjacent if and only if the inequality $2\cdot\eta(k)+2\cdot\eta(l)>2n-(1-\varepsilon(-1)^{k+l})$ holds. If $\varepsilon=+$ and $n_1\neq n_2$, then a maximal torus T of order $\frac{1}{(4,q^n-1)}(q^{\eta(k)}+(-1)^k)(q^{\eta(l)}+(-1)^l)$ of G exists if and only if k,l have the same parity, i. e., if and only if $2n-(1-\varepsilon(-1)^{k+l})=2n-(1-(-1)^{k+l})=2n$. Hence in this case r,s are non-adjacent if and only if the inequality $2\cdot\eta(k)+2\cdot\eta(l)>2n-(1-\varepsilon(-1)^{k+l})$ holds. If $n_1=n_2=n/2$ and $\frac{l}{k}$ is an odd integer, then, r,s are adjacent. Assume that $n_1=n_2=n/2$ and $\frac{l}{k}$ is not an odd integer. The condition $\frac{l}{k}$ is not an odd integer implies that $l\neq k$, so the chain of equalities (6) holds. In this case there exists a maximal torus T of order $\frac{1}{(4,q^n-1)}(q^n-1)=\frac{1}{(4,q^n-1)}(q^{n/2}-1)(q^{n/2}+1)$ of G, so condition (6) is not satisfied and r,s are adjacent.

Now assume that $\eta(l)=n$ and $\frac{l}{k}$ is an odd natural number, i. e., case (iv) holds. In this case there exists a maximal torus T of order $\frac{1}{(4,q^n-\varepsilon 1)}(q^n+(-1)^l)$ of G (if such a torus does not exist then s does not divide $|G|$). The fact that $\frac{l}{k}$ is an odd prime implies that r divides $|T|$, so r,s are adjacent. $\square$

Now we consider simple exceptional groups of Lie type. Note that the orders of maximal tori of simple exceptional groups were listed in [1, Lemma 1.3]. However, for groups $E_7(q)$, $E_8(q)$, and Ree groups ${}^2F_4(2^{2n+1})$ (items (4), (5), and (9) of the lemma respectively), the list of orders of tori is incorrect. The following lemma corrects this.

Lemma 2.6. (see [8]) *Let $\overline{G}$ be a connected simple exceptional algebraic group of adjoint type and let $G = O^{p'}(\overline{G}_\sigma)$ be a finite simple exceptional group of Lie type.*

1. *For every maximal torus T of $G = E_7(q)$, the number $m = (2, q-1)|T|$ is equal to one of the following: $(q+1)^{n_1}(q-1)^{n_2}$, $n_1 + n_2 = 7$; $(q^2+1)^{n_1}(q+1)^{n_2}(q-1)^{n_3}$, $1 \leqslant n_1 \leqslant 2$, $2n_1+n_2+n_3 = 7$, and $m \neq (q^2+1)(q\pm1)^5$; $(q^3+1)^{n_1}(q^3-1)^{n_2}(q^2+1)^{n_3}(q+1)^{n_4}(q-1)^{n_5}$, $1 \leqslant n_1+n_2 \leqslant 2$, $3n_1+3n_2+2n_3+n_4+n_5 = 7$, and $m \neq (q^3+\epsilon1)(q-\epsilon1)^4$, $m \neq (q^3\pm1)(q^2+1)^2$, $m \neq (q^3+\epsilon1)(q^2+1)(q+\epsilon1)^2$; $(q^4+1)(q^2\pm1)(q\pm1)$; $(q^5\pm1)(q^2-1)$; $(q^5+\epsilon1)(q+\epsilon1)^2$; $q^7\pm1$; $(q-\epsilon1)\cdot(q^2+\epsilon q+1)^3$; $(q^5-\epsilon1)\cdot(q^2+\epsilon q+1)$; $(q^3\pm1)\cdot(q^4-q^2+1)$; $(q-\epsilon1)\cdot(q^6+\epsilon q^3+1)$; $(q^3-\epsilon1)\cdot(q^2-\epsilon q+1)^2$, where $\epsilon = \pm$. Moreover, for every number m given above there exists a torus T with $(2, q-1)|T| = m$.*

2. *Every maximal torus T of $G = E_8(q)$ has one of the following orders: $(q+1)^{n_1}(q-1)^{n_2}$, $n_1 + n_2 = 8$; $(q^2+1)^{n_1}(q+1)^{n_2}(q-1)^{n_3}$, $1 \leqslant n_1 \leqslant 4$, $2n_1+n_2+n_3 = 8$, and $|T| \neq (q^2+1)^3(q\pm1)^2$, $|T| \neq (q^2+1)(q\pm1)^6$; $(q^3+1)^{n_1}(q^3-1)^{n_2}(q^2+1)^{n_3}(q+1)^{n_4}(q-1)^{n_5}$, $1 \leqslant n_1+n_2 \leqslant 2$, $3n_1+3n_2+2n_3+n_4+n_5 = 8$, and $|T| \neq (q^3\pm1)^2(q^2+1)$, $|T| \neq (q^3+\epsilon1)(q-\epsilon1)^5$, $|T| \neq (q^3+\epsilon1)(q^2+1)(q+\epsilon1)^3$, $|T| \neq (q^3+\epsilon1)(q^2+1)^2(q-\epsilon1)$; q^8-1; $(q^4+1)^2$; $(q^4+1)(q^2\pm1)(q\pm1)^2$; $(q^4+1)(q^2-1)^2$; $(q^4+1)(q^3+\epsilon1)(q-\epsilon1)$; $(q^5+\epsilon1)(q+\epsilon1)^3$; $(q^5\pm1)(q+\epsilon1)^2(q-\epsilon1)$; $(q^5+\epsilon1)(q^2+1)(q-\epsilon1)$; $(q^5+\epsilon1)(q^3+\epsilon1)$; $(q^6+1)(q^2\pm1)$; $(q^7\pm1)(q\pm1)$; $(q-\epsilon1)\cdot(q^2+\epsilon q+1)^3\cdot(q\pm1)$; $(q^5-\epsilon1)\cdot(q^2+\epsilon q+1)\cdot(q+\epsilon1)$; $(q^3\pm1)\cdot(q^4-q^2+1)\cdot(q\pm1)$; $(q-\epsilon1)\cdot(q^6+\epsilon q^3+1)\cdot(q\pm1)$; $(q^3-\epsilon1)\cdot(q^2-\epsilon q+1)^2\cdot(q\pm1)$; q^8-q^4+1; $q^8+q^7-q^5-q^4-q^3+q+1$; $q^8-q^6+q^4-q^2+1$; $(q^4-q^2+1)^2$; $(q^6+\epsilon q^3+1)(q^2+\epsilon q+1)$; $q^8-q^7+q^5-q^4+q^3-q+1$; $(q^4+\epsilon q^3+q^2+\epsilon q+1)^2$; $(q^4-q^2+1)(q^2\pm q+1)^2$; $(q^2-q+1)^2\cdot(q^2+q+1)^2$; $(q^2\pm q+1)^4$, where $\epsilon = \pm$. Moreover, for every number given above there exists a torus of corresponding order.*

3. *Every maximal torus T of $G = {}^2F_4(2^{2n+1})$ with $n \geqslant 1$ has one of the following orders: $q^2+\epsilon q\sqrt{2q}+q+\epsilon\sqrt{2q}+1$; $q^2-\epsilon q\sqrt{2q}+\epsilon\sqrt{2q}-1$; q^2-q+1; $(q\pm\sqrt{2q}+1)^2$; $(q-1)(q\pm\sqrt{2q}+1)$; $(q\pm1)^2$; $q^2\pm1$; where $q = 2^{2n+1}$ and $\epsilon = \pm$. Moreover, for every number given above there exists a torus of corresponding order.*

Proposition 2.7. *Let G be a finite simple exceptional group of Lie type over a field of order q and characteristic p. Suppose that r, s are odd primes, and assume that $r, s \in \pi(G) \setminus \{p\}$. Put $k = e(r, q)$, $l = e(s, q)$, and assume that $1 \leqslant k \leqslant l$. Then r and s are non-adjacent if and only if $k \neq l$ and one of the following holds*:

1. *$G = G_2(q)$ and either $r \neq 3$ and $l \in \{3, 6\}$ or $r = 3$ and $l = 9 - 3k$.*

2. *$G = F_4(q)$ and either $l \in \{8, 12\}$, or $l = 6$ and $k \in \{3, 4\}$, or $l = 4$ and $k = 3$.*

3. *$G = E_6(q)$ and either $l = 4$ and $k = 3$, or $l = 5$ and $k \geqslant 3$, or $l = 6$ and $k = 5$, or $l = 8$, $k \geqslant 3$, or $l = 8$, $r = 3$, and $(q-1)_3 = 3$, or $l = 9$, or $l = 12$ and $k \neq 3$.*

4. *$G = {}^2E_6(q)$ and either $l = 6$ and $k = 4$, or $l = 8$, $k \geqslant 3$, or $l = 8$, $r = 3$, and $(q+1)_3 = 3$, or $l = 10$ and $k \geqslant 3$, or $l = 12$ and $k \neq 6$, or $l = 18$.*

5. *$G = E_7(q)$ and either $l = 5$ and $k = 4$, or $l = 6$ and $k = 5$, or $l \in \{14, 18\}$ and $k \neq 2$, or $l \in \{7, 9\}$ and $k \geqslant 2$, or $l = 8$ and $k \geqslant 3, k \neq 4$, or $l = 10$ and $k \geqslant 3, k \neq 6$, or $l = 12$ and $k \geqslant 4, k \neq 6$.*

6. $G = E_8(q)$ *and either* $l = 6$ *and* $k = 5$, *or* $l \in \{7, 14\}$ *and* $k \geqslant 3$, *or* $l = 9$ *and* $k \geqslant 4$, *or* $l \in \{8, 12\}$ *and* $k \geqslant 5, k \neq 6$, *or* $l = 10$ *and* $k \geqslant 3, k \neq 4, 6$, *or* $l = 18$ *and* $k \neq 1, 2, 6$, *or* $l = 20$ *and* $r \cdot k \neq 20$, *or* $l \in \{15, 24, 30\}$.

7. $G = {}^3D_4(q)$ *and either* $l = 6$ *and* $k = 3$, *or* $l = 12$.

Proof. Recal that k_m is the greatest primitive divisor of $q^m - 1$, while R_m is the set of all prime primitive divisors of $q^m - 1$. The orders of maximal tori in exceptional groups are given in [1, Lemma 1.3] and Lemma 2.6, for example.

1. Since $|G_2(q)| = q^6(q^2 - 1)(q^6 - 1)$, the numbers k, l are in the set $\{1, 2, 3, 6\}$. If $\{k, l\} \subseteq \{1, 2\}$, then the existence of a maximal torus of order $q^2 - 1 = (2, q - 1) \cdot k_1 \cdot k_2$ implies the existence of an element of order rs, i. e., r and s are adjacent in $GK(G)$. If $l = 3$ (resp. $l = 6$), then an element g of order s is contained in a unique, up to conjugation, maximal torus of order $q^2 + q + 1 = (3, q - 1)k_3$ (resp. $q^2 - q + 1 = (3, q + 1)k_6$). In this case r, s are non-adjacent if and only if r does not divide $|T|$, whence statement 1 of the lemma follows.

2. Since $|F_4(q)| = q^{24}(q^2 - 1)(q^6 - 1)(q^8 - 1)(q^{12} - 1)$, the numbers k, l are in the set $\{1, 2, 3, 4, 6, 8, 12\}$. If $l \leqslant 3$, then the existence of maximal torus of order $(q^3 - 1)(q + 1) = (2, q - 1) \cdot (3, q - 1)k_1 \cdot k_2 \cdot k_3$ implies that for every $k \leqslant 3$ the primes r, s are adjacent. If $l = 4$, then an element of order s of G lies in a maximal torus T of order equals to either $(q - \epsilon)^2(q^2 + 1)$, or $(q^2 - \epsilon)(q^2 + 1)$. In particular, for every such maximal torus T the inclusion $\pi(T) \subseteq R_1 \cup R_2 \cup R_4$ holds. Moreover there exists a maximal torus of order $q^4 - 1 = (2, q - 1)^2 \cdot k_1 \cdot k_2 \cdot k_4$. So in this case r, s are non-adjacent if and only if r does not divide $k_1 \cdot k_2 \cdot k_4$, i. e., if and only if $k = 3$. If $l = 6$, then each element of order s of G is in a maximal torus T of order equals to either $(q^3 + 1)(q - \epsilon) = (3, q + 1) \cdot k_6 \cdot (q + 1) \cdot (q - \epsilon)$, or $(q^2 - q + 1)^2 = (3, q + 1)^2 \cdot k_6^2$. In particular, for T the inclusion $\pi(T) \subseteq R_1 \cup R_2 \cup R_6$ holds. Moreover there exists a maximal torus of order $(q^3 + 1)(q - 1) = (2, q - 1)(3, q + 1)k_1 \cdot k_2 \cdot k_6$. Thus r, s are non-adjacent if and only if $k \in \{3, 4\}$. If, finally, $l = 8$ (resp. $l = 12$), then every element of order s of G lies in a maximal torus of order $(2, q - 1)k_8$ (resp. k_{12}). Thus r, s are non-adjacent if and only if $k \neq 8$ (resp. $k \neq 12$).

3. Since $|E_6(q)| = \frac{1}{(3,q-1)} q^{36}(q^2 - 1)(q^5 - 1)(q^6 - 1)(q^8 - 1)(q^9 - 1)(q^{12} - 1)$, the numbers k, l are in the set $\{1, 2, 3, 4, 5, 6, 8, 9, 12\}$. If $l \leqslant 3$, then the existence in G of a maximal torus T of order $\frac{1}{(3,q-1)}(q^3 - 1)(q^2 - 1)(q - 1) = (2, q - 1) \cdot k_3 \cdot k_2 \cdot k_1 \cdot (q - 1)^2$ implies that r, s are adjacent. If $l = 4$, then each element of order s of G is in a maximal torus of order equals either $\frac{1}{(3,q-1)}(q^4 - 1)(q - \epsilon_1)(q - \epsilon_2) = \frac{1}{(3,q-1)} \cdot (2, q - 1)^2 \cdot k_1 \cdot k_2 \cdot k_4 \cdot (q - \epsilon_1) \cdot (q - \epsilon_2)$, or $\frac{1}{(3,q-1)}(q^3 + 1)(q^2 + 1)(q - 1) = \frac{1}{(3,q-1)} \cdot (2, q - 1)^2 \cdot (3, q + 1) \cdot k_6 \cdot k_4 \cdot k_2 \cdot k_1$, or $\frac{1}{(3,q-1)}(q^2 + 1)^2(q - 1)^2 = \frac{1}{(3,q-1)} \cdot (2, q - 1)^2 \cdot k_4^2 \cdot (q - 1)^2$. Thus r, s are non-adjacent if and only if $k = 3$. If $l = 5$, then each element of order s of G is in a maximal torus of order $\frac{1}{(3,q-1)}(q^5 - 1)(q - \epsilon) = \frac{1}{(3,q-1)}(5, q - 1)k_5(q - 1)(q - \epsilon)$. Thus r, s are non-adjacent if and only if $k \in \{3, 4\}$. If $l = 6$, then every element of order s of G is in a maximal torus of order equals either $\frac{1}{(3,q-1)}(q^3 + 1)(q^2 + q + 1)(q - \epsilon) = (3, q + 1) \cdot k_6 \cdot k_3 \cdot (q + 1) \cdot (q - \epsilon)$, or $\frac{1}{(3,q-1)}(q^3 + 1)(q^2 + 1)(q - 1) = \frac{1}{(3,q-1)} \cdot (3, q + 1) \cdot k_6 \cdot (2, q - 1) \cdot k_1 \cdot k_2 \cdot k_4$, or $\frac{1}{(3,q-1)}(q^3 + 1)(q^2 - 1)(q - 1) = \frac{1}{(3,q-1)} \cdot (3, q + 1) \cdot k_6 \cdot (2, q - 1)^2 \cdot k_1^2 \cdot k_2^2$, or $\frac{1}{(3,q-1)}(q^2 + q + 1)(q^2 - q + 1)^2 = (3, q + 1)^2 \cdot k_6^2 \cdot k_3$. Thus r, s are non-adjacent if and only if $k = 5$. If $l = 8$, then each element of order s of G is in a maximal torus of order $\frac{1}{(3,q-1)}(q^4 + 1)(q^2 - 1) = \frac{1}{(3,q-1)} \cdot (2, q - 1)^2 \cdot k_8 \cdot k_2 \cdot k_1$. Hence r, s

are non-adjacent if and only if either $k \geqslant 3$ and $k \neq 8$, or $r = 3$ and $(q-1)_3 = 3$. If $l = 9$, then each element of order s of G is in a maximal torus of order $\frac{1}{(3,q-1)}(q^6+q^3+1) = k_9$. Hence r, s are non-adjacent if and only if $k \neq 9$. If, finally, $l = 12$, then every element of order s of G is in maximal torus of order $\frac{1}{(3,q-1)}(q^4-q^2+1)(q^2+q+1) = k_{12} \cdot k_3$. So r, s are non-adjacent if and only if $k \neq 3, 12$.

4. Since $|{}^2E_6(q)| = \frac{1}{(3,q+1)}q^{36}(q^2-1)(q^5+1)(q^6-1)(q^8-1)(q^9+1)(q^{12}-1)$, the numbers k, l are in the set $\{1,2,3,4,6,8,10,12,18\}$. If $l \leqslant 4$, the existence in G of maximal tori of orders $\frac{1}{(3,q+1)}(q^3-1)(q^2+1)(q+1) = \frac{1}{(3,q+1)} \cdot (2,q-1) \cdot k_1 \cdot k_2 \cdot k_3 \cdot k_4$, $\frac{1}{(3,q+1)}(q^2+1)^2(q+1)^2 = \frac{1}{(3,q+1)} \cdot (2,q-1)^2 \cdot k_4^2 \cdot (q+1)^2$, and $\frac{1}{(3,q+1)}(q^3-1)(q^2-1)(q+1) = \frac{1}{(3,q+1)} \cdot (3,q-1) \cdot (2,q-1)^2 \cdot k_3 \cdot k_1^2 \cdot k_2^2$ implies that r, s are adjacent. If $l = 6$, then each element of order s of G is contained in a maximal torus of order equals either $\frac{1}{(3,q+1)}(q^3+1)^2 = (3,q+1) \cdot (q+1)^2 \cdot k_6^2$, or $\frac{1}{(3,q+1)}(q^3+1)(q+1)(q-\epsilon_1)(q-\epsilon_2) = k_6(q+1)^2(q-\epsilon_1)(q-\epsilon_2)$, or $\frac{1}{(3,q+1)}(q^2-q+1)(q^3-\epsilon)(q-1) = (3,q+1) \cdot k_6 \cdot (q^3-\epsilon) \cdot (q-1)$, or $\frac{1}{(3,q+1)}(q^2-q+1)(q^2+q+1)^2 = k_6 \cdot (3,q-1)^2 \cdot k_3^2$, or $\frac{1}{(3,q+1)}(q^4-q^2+1)(q^2-q+1) = k_{12} \cdot k_6$. Thus r, s are non-adjacent if and only if $k = 4$. If $l = 8$, then every element of order s of G is in a maximal torus of order $\frac{1}{(3,q+1)}(q^4+1)(q^2-1) = \frac{1}{(3,q+1)} \cdot (2,q-1)^2 \cdot k_8 \cdot k_2 \cdot k_1$. So r, s are non-adjacent if and only if either $k \geqslant 3$ and $k \neq 8$, or $r = 3$ and $(q+1)_3 = 3$. If $l = 10$, then each element of order s of G is in a maximal torus of order equals $\frac{1}{(3,q+1)}(q^5+1)(q-\epsilon) = \frac{1}{(3,q+1)} \cdot k_{10} \cdot (q+1) \cdot (q-\epsilon)$. Hence r, s are non-adjacent if and only if $k \geqslant 3$, $k \neq 10$. If $l = 12$, then every element of order s of G is contained in a maximal torus of order $\frac{1}{(3,q+1)}(q^4-q^2+1)(q^2-q+1) = k_{12} \cdot k_6$. Therefore r, s are non-adjacent if and only if $k \neq 6, 12$. If, finally $l = 18$, then each element of order s of G is contained in a maximal torus of order $\frac{1}{(3,q+1)}(q^6-q^3+1) = k_{18}$. Hence r, s are non-adjacent if and only if $k \neq 18$.

5. Since $|E_7(q)| = \frac{1}{(2,q-1)}q^{63}(q^2-1)(q^6-1)(q^8-1)(q^{10}-1)(q^{12}-1)(q^{14}-1)(q^{18}-1)$, the numbers k, l are in $\{1,2,3,4,5,6,7,8,9,10,12,14,18\}$. There exist maximal tori of G of orders equal $\frac{1}{(2,q-1)}(q^5-1)(q^2+q+1) = \frac{1}{(2,q-1)} \cdot (3,q-1) \cdot (5,q-1) \cdot (q-1) \cdot k_5 \cdot k_3$, $\frac{1}{(2,q-1)}(q^4-1)(q^3-1) = (2,q-1) \cdot k_4 \cdot k_1 \cdot k_2 \cdot k_3 \cdot (3,q-1) \cdot (q-1)$ and $\frac{1}{(2,q-1)}(q^5-1)(q^2-1) = k_5 \cdot k_2 \cdot k_1 \cdot (5,q-1) \cdot (q-1)$, so for $l \leqslant 5$ and $(k,l) \neq (4,5)$ the numbers r, s are adjacent. Since for $l = 5$ every element of order s of G is contained in a maximal torus of order either $\frac{1}{(2,q-1)}(q^5-1)(q-1)(q-\epsilon)$, or $\frac{1}{(2,q-1)}(q^5-1)(q^2+q+1) = \frac{1}{(2,q-1)} \cdot (3,q-1) \cdot (5,q-1) \cdot (q-1) \cdot k_5 \cdot k_3$, we obtain that r, s are non-adjacent if $(k,l) \neq (4,5)$. If $l = 6$, then the existence of maximal tori of G of orders equal $\frac{1}{(2,q-1)}(q^3+1)(q^4-1) = (2,q-1) \cdot (3,q+1) \cdot k_1 \cdot k_2 \cdot k_4 \cdot k_6 \cdot (q+1)$ and $\frac{1}{(2,q-1)}(q^6-1)(q-1) = (3,q^2-1) \cdot k_6 \cdot k_3 \cdot k_2 \cdot k_1 \cdot (q-1)$ implies that for $k \leqslant 4$ and $k = 6$ the numbers r, s are adjacent. Every element of order s is in a maximal torus of order equals either $\frac{1}{(2,q-1)}(q^3+1)(q^2+1)(q-\epsilon_1)(q-\epsilon_2) = (3,q+1) \cdot (q+1) \cdot k_6 \cdot k_4 \cdot (q-\epsilon_1) \cdot (q-\epsilon_2)$ with $(\epsilon_1, \epsilon_2) \neq (-1,-1)$, or $\frac{1}{(2,q-1)}(q^3+1)(q-\epsilon_1)(q-\epsilon_2)(q-\epsilon_3)(q-\epsilon_4) = \frac{1}{(2,q-1)} \cdot (3,q+1) \cdot k_6 \cdot (q+1)(q-\epsilon_1)(q-\epsilon_2)(q-\epsilon_3)(q-\epsilon_4)$, or $\frac{1}{(2,q-1)}(q^3+1)(q^3-\epsilon_1)(q-\epsilon_2)$, or $\frac{1}{(2,q-1)}(q^2-q+1)^3(q+1) = \frac{1}{(2,q-1)} \cdot (3,q+1)^3 \cdot k_6^3 \cdot (q+1)$, or $\frac{1}{(2,q-1)}(q^5+1)(q^2-q+1) = \frac{1}{(2,q-1)} \cdot (3,q+1) \cdot (5,q+1) \cdot k_6 \cdot k_{10} \cdot (q+1)$, or $\frac{1}{(2,q-1)}(q^3-1)(q^2-q+1)^2 = \frac{1}{(2,q-1)} \cdot (q-1) \cdot (3,q-1) \cdot k_3 \cdot (3,q+1)^2 \cdot k_6^2$. Since for $k = 5$ the prime r does not divide these numbers, we obtain that r, s are non-adjacent if and

only if $k = 5$. If $l = 7$, then each element of order s of G is in a maximal torus of order $\frac{1}{(2,q-1)}(q^7 - 1) = \frac{1}{(2,q-1)} \cdot (7, q-1) \cdot k_7 \cdot (q-1)$. Hence r, s are non-adjacent if and only if $k \neq 1, 7$. If $l = 8$, then every element of order s of G is in a maximal torus of order equals $\frac{1}{(2,q-1)}(q^4+1)(q^2-\epsilon_1)(q-\epsilon_2) = k_8 \cdot (q^2-\epsilon_1)(q-\epsilon_2)$. Hence r, s are non-adjacent if and only if $k \geqslant 3, k \neq 4$. If $l = 9$, then an element of order s of G is contained in a maximal torus of order $\frac{1}{(2,q-1)}(q-1)(q^6+q^3+1) = \frac{1}{(2,q-1)} \cdot (q-1) \cdot (3, q-1) \cdot k_9$. Therefore r, s are non-adjacent if and only if $k \neq 1, 9$. If $l = 10$, then an element of order s of G is contained in a maximal torus of order equals either $\frac{1}{(2,q-1)}(q^5+1)(q-1)(q-\epsilon) = (2, q-1) \cdot (5, q+1) \cdot k_{10} \cdot k_2 \cdot k_1 \cdot (q-\epsilon)$, or $\frac{1}{(2,q-1)}(q^5+1)(q^2-q+1) = \frac{1}{(2,q-1)} \cdot (5, q+1) \cdot (q+1) \cdot k_{10} \cdot (3, q+1) \cdot k_6$. So r, s are non-adjacent if and only if $k \geqslant 3$ and $k \neq 6$. If $l = 12$, then each element of order s is contained in a maximal torus of order equals $\frac{1}{(2,q-1)}(q^3-\epsilon)(q^4-q^2+1) = \frac{1}{(2,q-1)} \cdot (q^3-\epsilon) \cdot k_{12}$. Hence r, s are non-adjacent if and only if $k \geqslant 4$ and $k \neq 6, 12$. If $l = 14$, then an element of order s of G is contained in a maximal torus of order $\frac{1}{(2,q-1)}(q^7+1) = \frac{1}{(2,q-1)} \cdot (7, q+1) \cdot k_{14} \cdot (q+1)$. Therefore r, s are non-adjacent if and only if $k \neq 2, 14$. If, finally, $l = 18$, then an element of order s of G is contained in a maximal torus of order $\frac{1}{(2,q-1)}(q+1)(q^6-q^3+1) = \frac{1}{(2,q-1)} \cdot (3, q+1) \cdot (q+1) \cdot k_{18}$. Therefore r, s are non-adjacent if and only if $k \neq 2, 18$.

6. Since $|E_8(q)| = q^{120}(q^2-1)(q^8-1)(q^{12}-1)(q^{14}-1)(q^{18}-1)(q^{20}-1)(q^{24}-1)(q^{30}-1)$, the numbers k, l are in the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 18, 20, 24, 30\}$. Since G contains maximal tori of orders $(q^3-\epsilon_1)(q^4-1)(q-\epsilon_2)$, $(q^5-1)(q^2+1)(q+1) = (5, q-1) \cdot k_5 \cdot (2, q-1)^2 \cdot k_4 \cdot k_2 \cdot k_1$, and $(q^5-1)(q^3-1) = (3, q-1) \cdot (5, q-1) \cdot k_5 \cdot k_3 \cdot (q-1)^2$, for $l \leqslant 6$ primes r, s are adjacent if $(k, l) \neq (5, 6)$. If $k = 5$, then every element of order r of G is contained in a maximal torus of order equals either $(q^5-1)(q^3-1) = (3, q-1) \cdot (5, q-1) \cdot k_5 \cdot k_3 \cdot (q-1)^2$, or $(q^5-1)(q^2+1)(q+1) = (5, q-1) \cdot k_5 \cdot (2, q-1)^2 \cdot k_4 \cdot k_2 \cdot k_1$, or $(q^5-1)(q^2-1)(q-\epsilon) = (5, q-1) \cdot k_5 \cdot (2, q-1) \cdot k_2 \cdot k_1 \cdot (q-1) \cdot (q-\epsilon)$, or $(q^5-1)(q-1)^3$, or $(q^4+q^3+q^2+q+1)^2$, and all these orders are not divisible by s for $l = 6$. It follows that if $(k, l) = (5, 6)$, then r, s are non-adjacent. If $l = 7$, then every element of order s of G is contained in a maximal torus of order $(q^7-1)(q-\epsilon) = (7, q-1) \cdot k_7 \cdot (q-1)(q-\epsilon)$. So r, s are non-adjacent if and only if $k \geqslant 3$ and $k \neq 7$. If $l = 8$, then an element of order s of G is contained in a maximal torus of order equals either $(q^4+1)(q^4-\epsilon) = (2, q-1) \cdot k_8 \cdot (q^4-\epsilon)$, or $(q^4+1)(q^3-\epsilon_1)(q-\epsilon_2) = (2, q-1) \cdot k_8 \cdot (q^3-\epsilon_1)(q-\epsilon_2)$ with $(\epsilon_1, \epsilon_2) \neq (-1, -1)$, or $(q^4+1)(q^2-1)^2 = (2, q-1) \cdot k_8 \cdot (q^2-1)^2$, or $(q^4+1)(q^2-\epsilon_1)(q-\epsilon_2)^2 = (2, q-1) \cdot k_8 \cdot (q^2-\epsilon_1) \cdot (q-\epsilon_2)^2$. Hence r, s are non-adjacent if and only if $k = 5, 7$. If $l = 9$, then an element of order s of G is contained in a maximal torus of order equals either $(q^6+q^3+1)(q-1)(q-\epsilon) = (3, q-1) \cdot k_9 \cdot (q-1) \cdot (q-\epsilon)$, or $(q^6+q^3+1)(q^2+q+1) = (3, q-1)^2 \cdot k_9 \cdot k_3$. Hence r, s are non-adjacent if and only if $k \geqslant 4$ and $k \neq 9$. If $l = 10$ then every element of order s of G is contained in a maximal torus of order either $(q^5+1)(q^2-\epsilon_1)(q-\epsilon_2) = (5, q+1) \cdot k_{10} \cdot (q+1)(q^2-\epsilon_1)(q-\epsilon_2)$ with $(\epsilon_1, \epsilon_2) \neq (-1, -1)$, or $(q^5+1)(q^3+1) = (5, q+1) \cdot k_{10} \cdot (q+1)^2 \cdot (3, q+1) \cdot k_6$, or $(q^5+1)(q^2-q+1)(q-1) = (5, q+1) \cdot k_{10} \cdot (3, q+1) \cdot k_6 \cdot (2, q-1) \cdot k_1 \cdot k_2$, or $(q^5+1)(q+1)^3 = (5, q+1) \cdot k_{10} \cdot (q+1)(q+1)^3$, or $(q^4-q^3+q^2-q+1)^2 = ((5, q+1) \cdot k_{10})^2$. Hence r, s are non-adjacent if and only if $k \geqslant 3$ and $k \neq 4, 6, 10$. If $l = 12$, then each element of order s of G is contained in a maximal torus of order equals either $(q^4-q^2+1)(q^2+1)(q^2-\epsilon) = (2, q-1) \cdot k_{12} \cdot k_4 \cdot (q^2-\epsilon)$, or $(q^4-q^2+1)(q^2+1)(q-\epsilon)^2 = (2, q-1) \cdot k_{12} \cdot k_4 \cdot (q-\epsilon)^2$, or $(q^4-q^2+1)(q^3-\epsilon_1)(q-\epsilon_2) = k_{12} \cdot (q^3-\epsilon_1) \cdot (q-\epsilon_2)$, or $(q^4-q^2+1)(q^2+q+1)^2 = (3, q-1)^2 \cdot k_{12} \cdot k_3^2$, or $(q^4-q^2+1)(q^2-q+1)^2 =$

$(3, q+1)^2 \cdot k_{12} \cdot k_6^2$. Hence r, s are non-adjacent if and only if $k \geqslant 5$ and $k \neq 6, 12$. If $l = 14$, then an element of order s of G is contained in a maximal torus of order $(q^7+1)(q-\epsilon) = (7, q+1) \cdot k_{14} \cdot (q+1) \cdot (q-\epsilon)$. Therefore r, s are non-adjacent if and only if $k \geqslant 3$ and $k \neq 14$. If $l = 15, 24, 30$, then each element of order s of G is contained in a maximal torus of order k_l. So r, s are non-adjacent if and only if $k \neq l$. If $l = 18$, then an element of order s of G is contained in a maximal torus of order equals either $(q^6 - q^3 + 1)(q+1)(q-\epsilon) = (3, q+1) \cdot k_{18} \cdot (q+1) \cdot (q-\epsilon)$, or $(q^6 - q^3 + 1)(q^2 - q + 1) = (3, q+1)^2 \cdot k_{18} \cdot k_6$. Hence r, s are non-adjacent if and only if $k \geqslant 3$ and $k \neq 6, 18$. If $l = 20$, then every element of order s of G is contained in a maximal torus of order $q^8 - q^6 + q^4 - q^2 + 1 = (5, q^2+1) \cdot k_{20}$. So r, s are non-adjacent if and only if $r \cdot k \neq 20$ (i. e., $r \neq 5$ or $k \neq 4$) and $k \neq 20$.

7. Since $|{}^3D_4(q)| = q^{12}(q^2-1)(q^6-1)(q^8+q^4+1)$, the numbers k, l are in the set $\{1, 2, 3, 6, 12\}$. Since G contains maximal tori of orders $(q^3 - \epsilon_1)(q - \epsilon_2)$, then for $l \leqslant 3$ primes r, s are adjacent. If $l = 6$, then each element of order s of G is in a maximal torus of order $(q^3+1)(q-\epsilon) = (3, q+1) \cdot k_6 \cdot (q+1) \cdot (q-\epsilon)$. Hence r, s are non-adjacent if and only if $k = 3$. If $l = 12$, then and element of order s of G is contained in a maximal torus of order $q^4 - q^2 + 1 = k_{12}$ and r, s are non-adjacent if and only if $k \neq 12$. □

Now we consider simple Suzuki and Ree groups.

Lemma 2.8. *Let n be a natural number.*

1. *Let* $m_1(B, n) = 2^{2n+1} - 1$,
   $m_2(B, n) = 2^{2n+1} - 2^{n+1} + 1$,
   $m_3(B, n) = 2^{2n+1} + 2^{n+1} + 1$.
   Then $(m_i(B, n), m_j(B, n)) = 1$ *if* $i \neq j$.
2. *Let* $m_1(G, n) = 3^{2n+1} - 1$,
   $m_2(G, n) = 3^{2n+1} + 1$,
   $m_3(G, n) = 3^{2n+1} - 3^{n+1} + 1$,
   $m_4(G, n) = 3^{2n+1} + 3^{n+1} + 1$.
   Then $(m_1(G, n), m_2(G, n)) = 2$ *and* $(m_i(G, n), m_j(G, n)) = 1$ *otherwise.*
3. *Let* $m_1(F, n) = 2^{2n+1} - 1$,
   $m_2(F, n) = 2^{2n+1} + 1$,
   $m_3(F, n) = 2^{4n+2} + 1$,
   $m_4(F, n) = 2^{4n+2} - 2^{2n+1} + 1$,
   $m_5(F, n) = 2^{4n+2} - 2^{3n+2} + 2^{2n+1} - 2^{n+1} + 1$,
   $m_6(F, n) = 2^{4n+2} + 2^{3n+2} + 2^{2n+1} + 2^{n+1} + 1$.
   Then $(m_2(F, n), m_4(F, n)) = 3$ *and* $(m_i(F, n), m_j(F, n)) = 1$ *otherwise.*

Proof. Items (1) and (2) are repeated items (1) and (2) of [1, Lemma 1.5]. Item (3) is corrected with respect to Lemma 2.6. □

If G is a Suzuki or a Ree group over a field of order q, then denote by $S_i(G)$ the set $\pi(m_i(B, n))$ for $G = {}^2B_2(2^{2n+1})$, the set $\pi(m_i(G, n)) \setminus \{2\}$ for $G = {}^2G_2(3^{2n+1})$, and the set $\pi(m_i(F, n)) \setminus \{3\}$ for $G = {}^2F_4(2^{2n+1})$. If G is fixed, then we put $S_i = S_i(G)$, and denote by s_i any prime from S_i.

Proposition 2.9. *Let G be a finite simple Suzuki or Ree group over a field of characteristic p, let r, s be odd primes with $r, s \in \pi(G) \setminus \{p\}$. Then r, s are non-adjacent if and only if one of the following holds:*

1. $G = {}^2B_2(2^{2n+1})$, $r \in S_k(G)$, $s \in S_l(G)$ *and* $k \neq l$.
2. $G = {}^2G_2(3^{2n+1})$, $r \in S_k(G)$, $s \in S_l(G)$ *and* $k \neq l$.
3. $G = {}^2F_4(2^{2n+1})$, *either* $r \in S_k(G)$, $s \in S_l(G)$ *and* $k \neq l$, $\{k, l\} \neq \{1, 2\}, \{1, 3\}$; *or* $r = 3$ *and* $s \in S_l(G)$, *where* $l \in \{3, 5, 6\}$.

Proof. Follows from [1, Lemma 1.3], Lemma 2.6, and Lemma 2.8. □

3 Cocliques for groups of Lie type

Let G be a finite simple group of Lie type with the base field of order q and characteristic p. Every $r \in \pi(G) \setminus \{p\}$ is known to be a primitive prime divisor of $q^i - 1$, where i is bounded by some function depending on the Lie rank of G. Given a finite simple group of Lie type G, define a set $I(G)$ as follows. If G is neither a Suzuki, nor a Ree group, then $i \in I(G)$ if and only if $\pi(G) \cap R_i(q) \neq \varnothing$. If G is either a Suzuki or a Ree group, then $i \in I(G)$ if and only if $\pi(G) \cap S_i(G) \neq \varnothing$. Notice that if $\pi(G) \cap R_i(q) \neq \varnothing$ (resp. $\pi(G) \cap S_i(G) \neq \varnothing$), then $R_i(q) \subseteq \pi(G)$ (resp. $S_i(G) \subseteq \pi(G)$). Thus, the following partition of $\pi(G)$ arises:

$$\pi(G) = \{p\} \sqcup \bigsqcup_{i \in I(G)} R_i,$$

or

$$\pi(G) = \{2\} \sqcup \bigsqcup_{i \in I(G)} S_i$$

in case of Suzuki groups, or

$$\pi(G) = \{2\} \sqcup \{3\} \cup \bigsqcup_{i \in I(G)} S_i$$

in case of Ree groups.

As followed from an adjacency criterion, two distinct primes from the same class of the partition are always adjacent. Moreover, in most cases an answer to the question: whether two primes from distinct classes R_i and R_j (or S_i and S_j) of the partition are adjacent, depends only on the choice of the indices i and j. We formalize this inference by the following definitions.

Definition 3.1. Suppose G is a finite simple group of Lie type with the base field of order q and characteristic p, and G is not isomorphic to ${}^2B_2(2^{2m+1})$, ${}^2G_2(3^{2m+1})$, ${}^2F_4(2^{2m+1})$, and $A_2^{\varepsilon}(q)$. Define the set $M(G)$ to be a subset of $I(G)$ such that $i \in M(G)$ if and only if the intersection of R_i and every coclique of maximal size of $GK(G)$ is nonempty.

Definition 3.2. If $G = {}^2B_2(2^{2m+1})$ or ${}^2G_2(3^{2m+1})$, $m \geqslant 1$, then put $M(G) = I(G)$. If $G = {}^2F_4(2^{2m+1})$, $m \geqslant 2$, then put $M(G) = \{2, 3, 4, 5, 6\}$. If $G = {}^2F_4(8)$, then put $M(G) = \{5, 6\}$.

Definition 3.3. Suppose G is a finite simple group of Lie type with the base field of order q and characteristic p, and G is not isomorphic to ${}^2B_2(2^{2m+1})$, ${}^2G_2(3^{2m+1})$, ${}^2F_4(2^{2m+1})$, and $A_2^{\varepsilon}(q)$. A set $\Theta(G)$ consists of all subsets $\theta(G)$ of $\pi(G)$ satisfying the following conditions:

(a) p lies in $\theta(G)$ if and only if p lies in every coclique of maximal size of $GK(G)$;

(b) for every $i \in M(G)$ exactly one prime from R_i lies in $\theta(G)$.

Definition 3.4. Let $G = {}^2B_2(2^{2m+1})$. A set $\Theta(G)$ consists of all subsets $\theta(G)$ of $\pi(G)$ satisfying the following conditions:

(a) $p = 2$ lies in $\theta(G)$;

(b) for every $i \in M(G)$ exactly one prime from S_i lies in $\theta(G)$.

Definition 3.5. Let $G = {}^2G_2(3^{2m+1})$. A set $\Theta(G)$ consists of all subsets $\theta(G)$ of $\pi(G)$ satisfying the following conditions:

(a) $p = 3$ lies in $\theta(G)$;

(b) for every $i \in M(G)$ exactly one prime from S_i lies in $\theta(G)$.

Definition 3.6. Let $G = {}^2F_4(2^{2m+1})$, $m \geqslant 1$. A set $\Theta(G)$ consists of all subsets $\theta(G)$ of $\pi(G)$ satisfying the following condition:

(a) for every $i \in M(G)$ exactly one prime from S_i lies in $\theta(G)$.

Definition 3.7. Let $G = A_2^{\varepsilon}(q)$, and $(q, \varepsilon) \neq (2, -)$. If $q + \varepsilon 1 \neq 2^k$, then put $M(G) = \{\nu_\varepsilon(2), \nu_\varepsilon(3)\}$, and if $q + \varepsilon 1 = 2^k$, then $M(G) = \{\nu_\varepsilon(3)\}$. A set $\Theta(G)$ consists of all subsets $\theta(G)$ of $\pi(G)$ satisfying the following conditions.

(1) p lies in $\theta(G)$ if and only if $q + \varepsilon 1 \neq 2^k$;

(2) if $(q - \varepsilon 1)_3 = 3$, then $3 \in \theta(G)$.

(3) for every $i \in M(G)$ exactly one prime from $R_{\nu_\varepsilon(i)}$ lies in $\theta(G)$, excepting one case: if $2 \in R_{\nu_\varepsilon(2)}$, then 2 does not lie in $\theta(G)$.

Remark. A function ν_ε is defined in (4).

Definition 3.8. Let G be a finite simple group of Lie type. The subset $\theta'(G)$ of $\pi(G)$ is an element of $\Theta'(G)$, if for every $\theta(G) \in \Theta(G)$ the union $\rho(G) = \theta(G) \cup \theta'(G)$ is a coclique of maximal size in $GK(G)$.

Now we describe cocliques of maximal size for groups of Lie type. First we consider classical groups postponing groups $A_1(q)$, $A_2^{\varepsilon}(q)$ to the end of the section.

Proposition 3.9. *If G is one of finite simple groups $A_{n-1}(q)$, ${}^2A_{n-1}(q)$ with the base field of characteristic p and order q, and $n \geqslant 4$, then $t(G)$, and the sets $\Theta(G)$, $\Theta'(G)$ are listed in Table* 2.

Proof. It is obvious that the function ν_ε defined in (4) is a bijection on $\mathbb{N}$, so ν_ε^{-1} is well defined. Moreover, since ν_ε^2 is the identity map, we have $\nu_\varepsilon^{-1} = \nu_\varepsilon$.

Using Lemma 2.1 and an information on the orders of groups $A_{n-1}^{\varepsilon}(q)$, we obtain that a number i lies in $I(G)$, if the following conditions holds:

(a) $\nu_\varepsilon(i) \leqslant n$;

(b) $i \neq 1$ for $q = 2, 3$, and $i \neq 6$ for $q = 2$.

By [1, Propositions 2.1, 2.2, 4.1, and 4.2] two distinct primes from R_i are adjacent for every $i \in I(G)$.

Denote by $N(G)$ the set $\{i \in I(G) \mid n/2 < \nu_\varepsilon(i) \leqslant n\}$ and by χ any set of type $\{r_i \mid i \in N(G)\}$ such that $|\chi \cap R_i| = 1$ for all $i \in N(G)$. Note that $1, 2$ can not lie in $N(G)$, because $n \geqslant 4$. In particular, 2 does not lie in any χ. Let $i \neq j$ and $n/2 < \nu_\varepsilon(i), \nu_\varepsilon(j) \leqslant n$. Then $\nu_\varepsilon(i) + \nu_\varepsilon(j) > n$ and $\nu_\varepsilon(i)$ does not divide $\nu_\varepsilon(j)$. By [1, Propositions 2.1, 2.2], primes r_i and r_j are not adjacent. Thus, every χ forms a coclique of $GK(G)$.

Denote by ξ the set

$$\{p\} \cup \bigcup_{i \in I(G) \setminus N(G)} R_i.$$

By [1, Propositions 2.1, 2.2, 3.1, 4.1, 4.2] every two distinct primes from ξ are adjacent in $GK(G)$. Thus, every coclique of $GK(G)$ contains at most one prime from ξ.

Case 1. Let $n \geqslant 7$.

If $q = 2$ and $G = A_{n-1}(q)$ we assume that $n \geqslant 13$ first, in order to avoid the exceptions arising because of $R_6 = \varnothing$ for $q = 2$.

The conditions on n implies that $|N(G)| \geqslant 4$. By [1, Proposition 3.1], we have that $t(p, G) \leqslant 3$, so p can not lie in any coclique of maximal size. By [1, Propositions 4.1, 4.2], the same assertion is true for any primitive prime divisor r_i, where $\nu_\varepsilon(i) = 1$. Thus, solving the problem, does a prime r lie in a coclique of maximal size of $GK(G)$, we may assume that r is neither a characteristic, nor a divisor of $q - \varepsilon 1$. Hence [1, Propositions 2.1 and 2.2] will be the main technical tools.

Suppose that $n = 2t + 1$ is odd. If $\nu_\varepsilon(i) \leqslant n/2$, then there exist at least two distinct numbers j, k from $N(G)$ such that r_i is adjacent to r_j and r_k. Indeed, if $\nu_\varepsilon(i) < t$, then we take j and k such that $\nu_\varepsilon(j) = t + 1$ and $\nu_\varepsilon(k) = t + 2$, while if $\nu_\varepsilon(i) = t$, then we take j and k such that $\nu_\varepsilon(j) = t + 1$ and $\nu_\varepsilon(k) = 2t$. Thus, $M(G) = N(G)$, every $\theta(G) \in \Theta(G)$ is of type $\{r_i \mid n/2 < \nu_\varepsilon(i) \leqslant n\}$, $\Theta'(G) = \varnothing$, and $t(G) = t + 1 = [(n + 1)/2]$.

Suppose that $n = 2t$ is even. If $\nu_\varepsilon(i) < n/2$, then there exist at least two distinct numbers j, k from $N(G)$ such that r_i is adjacent to r_j and r_k. It is sufficient, to take j and k such that $\nu_\varepsilon(j) = t + 1$, and $\nu_\varepsilon(k) = t + 2$ if $\nu_\varepsilon(i) < t - 1$, or $\nu_\varepsilon(k) = 2t - 2$ if $\nu_\varepsilon(i) = t - 1$. On the other hand, if $\nu_\varepsilon(i) = t = n/2$, then r_i is adjacent to r_j, where $\nu_\varepsilon(j) = 2t = n$, and is non-adjacent to every r_k, where $k \in N(G)$ and $k \neq j$. Thus, $M(G) = N(G) \setminus \{\nu_\varepsilon(n)\}$, every $\theta(G) \in \Theta(G)$ is of type $\{r_i \mid n/2 < \nu_\varepsilon(i) < n\}$, and $\Theta'(G)$ consists of one-element sets of type $\{r_{\nu_\varepsilon(n/2)}\}$ or $\{r_{\nu_\varepsilon(n)}\}$. Hence, $t(G) = t = [(n + 1)/2]$.

It remains to consider the following cases: $q = 2$, $G = A_{n-1}(q)$, and $7 \leqslant n \leqslant 12$. All results (see Table 2) are obtained by arguments similar to that in general case with respect to the fact: $R_6 = \varnothing$, and can be easily verified by using [1, Propositions 2.1, 2.2, 3.1, 4.1, 4.2]. The most interesting case arises, when $n = 8$. In that case $\Theta(G)$ consists of one-element sets $\theta(G)$ of type $\{r_7\}$, while $\Theta'(G)$ consists of two-elements sets $\theta'(G)$ of types $\{p, r_8\}$, $\{r_4, r_5\}$, $\{r_3, r_8\}$, or $\{r_5, r_8\}$.

Case 2. Let $n = 6$.

First, we assume that $q \neq 2$. Then $N(G) = \{\nu_\varepsilon(4), \nu_\varepsilon(5), \nu_\varepsilon(6)\}$, and $|N(G)| = 3$. Therefore, a set of type $\{r_{\nu_\varepsilon(4)}, r_{\nu_\varepsilon(5)}, r_{\nu_\varepsilon(6)}\}$ forms a coclique in $GK(G)$, and $t(G) \geqslant 3$. Arguing as in previous case, we obtain that any prime $r_{\nu_\varepsilon(3)}$ is adjacent to $r_{\nu_\varepsilon(6)}$, and a set of type $\{r_{\nu_\varepsilon(3)}, r_{\nu_\varepsilon(4)}, r_{\nu_\varepsilon(5)}\}$ is a coclique. By [1, Proposition 3.1], we have that a set of type $\{p, r_{\nu_\varepsilon(5)}, r_{\nu_\varepsilon(6)}\}$ is a coclique, and p is adjacent to any prime $r_{\nu_\varepsilon(4)}$. If $\nu_\varepsilon(i) = 2$, then r_i is adjacent to p, and is non-adjacent to r_j if and only if $\nu_\varepsilon(j) = 5$, so $t(r_i, G) = 2$. Let r be a divisor of $q - \varepsilon 1$. If $r \neq 3$ or $(q - \varepsilon 1)_3 \neq 3$, then [1, Propositions 4.1, 4.2] implies that $t(r,G)=2$, while if $r=3$ and $(q-\varepsilon 1)_3=3$, we have that $t(3,G)=3$ and a set of type $\{3, r_{\nu_\varepsilon(5)}, r_{\nu_\varepsilon(6)}\}$ is a coclique in $GK(G)$. Thus, if $q\neq 2$, then $M(G)=\{\nu_\varepsilon(5)\}$, and every $\theta(G)\in\Theta(G)$ is of type $\{r_{\nu_\varepsilon(5)}\}$. Every $\theta'(G)\in\Theta'(G)$ is a two-element set of type $\{p, r_{\nu_\varepsilon(6)}\}$, $\{r_{\nu_\varepsilon(3)}, r_{\nu_\varepsilon(4)}\}$, $\{r_{\nu_\varepsilon(4)}, r_{\nu_\varepsilon(6)}\}$, and if $(q-\varepsilon 1)_3=3$ is also of type $\{3, r_{\nu_\varepsilon(6)}\}$.

Let $G=A_5(2)$. Since $R_6=R_1=\varnothing$, we have that every $\theta(G)\in\Theta(G)$ is of type $\{r_3, r_4, r_5\}$, and $\Theta'(G)=\varnothing$.

Let $G={}^2A_5(2)$. Since $R_6=R_{\nu(3)}=\varnothing$ and $(q+1)_3=3$, we have that every $\theta(G)\in\Theta(G)$ is of type $\{r_3, r_{10}\}$, and every $\theta'(G)\in\Theta'(G)$ is a one element set of type $\{p\}$, $\{r_4\}$, or $\{3\}$.

In all cases $t(G)=3$.

Case 3. Let $n=5$.

We have $N(G)=\{\nu_\varepsilon(4), \nu_\varepsilon(5)\}$, and $|N(G)|=2$, so $t(G)\leqslant 3$. Assume now that $G\neq {}^2A_4(2)$. Then $R_{\nu_\varepsilon(3)}$ is always nonempty, and a set of type $\{r_{\nu_\varepsilon(3)}, r_{\nu_\varepsilon(4)}, r_{\nu_\varepsilon(5)}\}$ is a coclique in $GK(G)$. By [1, Proposition 3.1], we have that a set of type $\{p, r_{\nu_\varepsilon(4)}, r_{\nu_\varepsilon(5)}\}$ is also a coclique. A prime $r_{\nu_\varepsilon(2)}$ is adjacent to p, and is non-adjacent to r_j if and only if $\nu_\varepsilon(j)=5$. Let r be a divisor of $q-\varepsilon 1$. If $r\neq 5$ or $(q-\varepsilon 1)_5\neq 5$, then [1, Propositions 4.1, 4.2] implies that $t(r,G)=2$, while if $r=5$ and $(q-\varepsilon 1)_5=5$, we have that $t(5,G)=3$ and a set of type $\{5, r_{\nu_\varepsilon(4)}, r_{\nu_\varepsilon(5)}\}$ is a coclique in $GK(G)$. Thus, if $G\neq {}^2A_4(2)$, then $M(G)=N(G)$, every $\theta(G)\in\Theta(G)$ is of type $\{r_{\nu_\varepsilon(4)}, r_{\nu_\varepsilon(5)}\}$. Every $\theta'(G)\in\Theta'(G)$ is a one-element set of type $\{p\}$ or $\{r_{\nu_\varepsilon(3)}\}$, and if $(q-\varepsilon 1)_5=5$ is also of type $\{5\}$.

Let $G={}^2A_4(2)$. Since $R_6=R_{\nu(3)}=\varnothing$ and $(q+1)_5=1$, we have that every $\theta(G)\in\Theta(G)$ is of type $\{p, r_4, r_{10}\}$, and $\Theta'(G)=\varnothing$.

In all cases $t(G)=3$.

Case 4. Let $n=4$.

First, we assume that $G\neq {}^2A_3(2)$. Then $N(G)=\{\nu_\varepsilon(3), \nu_\varepsilon(4)\}$, and $|N(G)|=2$, so $t(G)\leqslant 3$. By [1, Proposition 3.1], we have that a set of type $\{p, r_{\nu_\varepsilon(3)}, r_{\nu_\varepsilon(4)}\}$ is a coclique in $GK(G)$. A prime $r_{\nu_\varepsilon(2)}$ is adjacent to p and any prime $r_{\nu_\varepsilon(4)}$. If r is an odd prime divisor of $q-\varepsilon 1$, then [1, Propositions 4.1, 4.2] implies that $t(r,G)=2$. The same assertion is true for $r=2$ if and only if $(q-\varepsilon 1)_2\neq 4$, while if $(q-\varepsilon 1)_2=4$ then $\{2, r_{\nu_\varepsilon(3)}, r_{\nu_\varepsilon(4)}\}$ is a coclique. Therefore, if $(q-\varepsilon 1)_2\neq 4$, then every $\theta(G)\in\Theta(G)$ is of type $\{p, r_{\nu_\varepsilon(3)}, r_{\nu_\varepsilon(4)}\}$, and $\Theta'(G)=\varnothing$. But if $(q-\varepsilon 1)_2=4$, then $M(G)=N(G)$, every $\theta(G)\in\Theta(G)$ is of type $\{r_{\nu_\varepsilon(3)}, r_{\nu_\varepsilon(4)}\}$, and $\Theta'(G)=\{\{2\},\{p\}\}$. Anyway, $t(G)=3$.

Let $G={}^2A_3(2)$. Since $R_6=R_{\nu(3)}=\varnothing$ and $(q+1)_4=1$, we obtain that $\Theta(G)=\{\{r_4\}\}$, and $\Theta'(G)=\{\{p\},\{r_2\}\}$. In this case, $t(G)=2$. □

Proposition 3.10. *If G is one of finite simple groups $B_n(q)$, $C_n(q)$, $D_n(q)$ or ${}^2D_n(q)$ with the base field of characteristic p and order q, then $t(G)$, and the sets $\Theta(G)$, $\Theta'(G)$ are listed in Table* 3.

Proof. Using Lemma 2.1 and an information on the orders of groups under consideration, we obtain that a number i lies in $I(G)$, if the following conditions holds:

(a) $\eta(i)\leqslant n$;

(b) $i\neq 1$ for $q=2,3$, and $i\neq 6$ for $q=2$;

(c) $i\neq 2n$ for $G=D_n(q)$;

(d) $i\neq n$ for $G={}^2D_n(q)$ and n odd.

By [1, Proposition 4.3 and 4.4] and Propositions 2.4 and 2.5 it follows that for every $i \in I(G)$ two distinct primes from R_i are adjacent.

Denote by $N(G)$ the set $\{i \in I(G) \mid n/2 < \eta(i) \leqslant n\}$ and by χ any set of type $\{r_i \mid i \in N(G)\}$ such that $|\chi \cap R_i| = 1$ for all $i \in N(G)$. Let $i \neq j$ and $n/2 < \eta(i), \eta(j) \leqslant n$. We have $\eta(i) + \eta(j) > n$. Suppose that i/j is an odd natural number. Then i and j is of the same parity, so $\eta(i)/\eta(j)$ is also an odd natural number. Since $i \neq j$, we have $\eta(j) > 2\eta(i) > n$, contrary to the choice of j. Thus i/j is not an odd number. By Propositions 2.4 and 2.5, primes r_i and r_j are not adjacent. Thus, every χ forms a coclique of $GK(G)$.

Denote by ξ the set

$$\{p\} \cup \bigcup_{i \in I(G) \setminus N(G)} R_i.$$

By Propositions 2.4, 2.5 and [1, Propositions 3.1, 4.3, 4.4] every two distinct primes from ξ are adjacent in $GK(G)$. Thus, every coclique of $GK(G)$ contains at most one prime from ξ.

Now we determine cocliques of maximal size considering the groups of different types separately. However, by [1, Theorem 7.5], we have $GK(B_n(q)) = GK(C_n(q))$, and so analysis for groups of types B_n and C_n is mutual.

Case 1. Let G be one of the simple groups $B_n(q)$ or $C_n(q)$.

Suppose that $n = 2$. If $q = 2$, then the group G is not simple, so we can assume that $q \geqslant 3$. If $q = 3$, then $I(G) = \{2, 4\}$, and if $q > 3$, then $I(G) = \{1, 2, 4\}$. In both cases, $N(G) = \{4\}$. Since r_4 is non-adjacent to every $r \in \xi$, we have $M(G) = N(G) = \{4\}$, every $\theta(G) \in \Theta(G)$ is a one-element set containing exactly one element r_4 from R_4. Every $\theta'(G) \in \Theta'(G)$ is a one-element set containing exactly one element from ξ. Thus, $t(G) = 2$.

Suppose that $n = 3$. If $q \neq 2$ then $N(G) = \{3, 6\}$, and if $q = 2$ then $N(G) = \{3\}$. The set $\{1, 2, 3, 4, 6\}$ includes $I(G)$, and so $\xi = \{p\} \cup R_1 \cup R_2 \cup R_4$, where $\{p\}$, R_2 and R_4 are always nonempty. The prime p and any prime r_4 are adjacent one to another, and are non-adjacent to every r_i with $i \in N(G)$. On the other hand, for $i \in \{1, 2\}$ and $j \in \{3, 6\}$, primes r_i and r_j are adjacent. Therefore, $M(G) = N(G)$, $\theta(G)$ is of type $\{r_3\}$ for $q = 2$, and is of type $\{r_3, r_6\}$ otherwise. The set $\Theta'(G)$ consists of one-element sets of type $\{p\}$, $\{r_2\}$, and $\{r_4\}$, if $q = 2$, and sets of type $\{p\}$, and $\{r_4\}$ otherwise. Thus, $t(G) = 2$ for $q = 2$, and $t(G) = 3$ otherwise.

Let $n \geqslant 4$. Now we consider four different cases subject to residue of n modulo 4. We write $n = 4t + k$, where $k = 0, 1, 2, 3$, and $t \geqslant 1$. If $q = 2$ we assume that $t > 1$ to avoid exceptional cases that arise because of $R_6 = \varnothing$ for $q = 2$.

Suppose that $n = 4t$. Then

$$N(G) = \{2t + 1, 2t + 3, \ldots, 4t - 1, 4t + 2, 4t + 4, \ldots, 8t\},$$

and so $|N(G)| = 3t$. By adjacency criterion, r_{4t} is non-adjacent to every r_i, where $i \in N(G)$. Therefore, $t(G) \geqslant 3t + 1 \geqslant 4$. By [1, Propositions 3.1, 4.3], we have $t(2, G) \leqslant t(p, G) < 4$, so p and 2 cannot lie in any coclique of maximal size. Furthermore, if $\eta(i) < n/2 = 2t$, then any odd r_i is adjacent to r_{4t}, r_{2t+1}, r_{4t+2}. Therefore, $M(G) = N(G) \cup \{n\}$, every $\theta(G) \in \Theta(G)$ is of type $\{r_i \mid n/2 \leqslant \eta(i) \leqslant n\}$, $\Theta'(G) = \varnothing$, so $t(G) = 3t + 1 = [(3n + 5)/4]$.

Suppose that $n = 4t + 1$. Then

$$N(G) = \{2t + 1, 2t + 3, \dots, 4t + 1, 4t + 2, 4t + 4, \dots, 8t + 2\},$$

so $|N(G)| = 3t+2$ and $t(G) \geqslant 5$. By [1, Propositions 3.1, 4.3], we have $t(2, G) \leqslant t(p, G) < 4$. Therefore, p and 2 cannot lie in any coclique of maximal size. If $\eta(i) < n/2$, then any odd r_i is adjacent to r_{2t+1}, r_{4t+2}, so cannot lie in any coclique of maximal size. Thus, $M(G) = N(G)$, every $\theta(G) \in \Theta(G)$ is of type $\{r_i \mid n/2 < \eta(i) \leqslant n\} = \{r_i \mid n/2 \leqslant \eta(i) \leqslant n\}$, $\Theta'(G) = \varnothing$, and $t(G) = 3t + 2 = [(3n + 5)/4]$.

Suppose that $n = 4t + 2$. Then

$$N(G) = \{2t + 3, 2t + 5, \dots, 4t + 1, 4t + 4, 4t + 6, \dots, 8t + 4\},$$

so $|N(G)| = 3t + 1$ and $t(G) \geqslant 4$. Since $t(2, G) \leqslant t(p, G) < 4$, primes p and 2 cannot lie in any coclique of maximal size. Any primes r_{2t+1} and r_{4t+2} are adjacent one to another and are non-adjacent to every r_i with $i \in N(G)$. If $\eta(i) < n/2$, then r_i is adjacent to r_{4t+4}, r_{4t+2}, and r_{2t+1}. Therefore, $N(G) = M(G)$, every $\theta(G) \in \Theta(G)$ is of type $\{r_i \mid n/2 < \eta(i) \leqslant n\}$, and $\Theta'(G)$ consists of one-element sets of type $\{r_{2t+1}\}$ or $\{r_{4t+2}\}$. Thus, $t(G) = 3t + 2 = [(3n + 5)/4]$.

Suppose that $n = 4t + 3$. Then

$$N(G) = \{2t + 3, 2t + 5, \dots, 4t + 3, 4t + 4, 4t + 6, \dots, 8t + 6\},$$

so $|N(G)| = 3t + 3$ and $t(G) \geqslant 6$. Since $t(2, G) \leqslant t(p, G) < 4$, primes p and 2 cannot lie in a coclique of maximal size. If $\eta(i) < 2t + 1$, then r_i is adjacent to r_{4t+4}, r_{4t+6}, and r_{2t+3}. Assume that $\eta(i) = 2t + 1$. If r_i is adjacent to r_j with $j \in N(G)$, then $j = 4t + 4$. Since there are two distinct numbers $2t + 1$ and $4t + 2$ such that the value of function η of them is equal to $2t + 1$, we have that $\Theta'(G)$ consists of one-element sets of one of three types: $\{r_{4t+4}\}$, $\{r_{2t+1}\}$ or $\{r_{4t+2}\}$. Thus, $M(G) = N(G) \setminus \{4t + 4\}$, every $\theta(G) \in \Theta(G)$ is of type $\{r_i \mid (n + 1)/2 < \eta(i) \leqslant n\}$, and $t(G) = 3t + 3 = [(3n + 5)/4]$.

It remains to consider the cases: $q = 2$ and $n = 4 + k$, where $k = 0, 1, 2, 3$. All results (see Table 3) are obtained by arguments similar to that in general case with respect to the fact: $R_{4t+2} = R_6 = \varnothing$, and can be easily verified by using Proposition 2.4 and [1, Propositions 3.1, 4.3].

Case 2. Let $G = D_n(q)$.

Suppose that $n = 4$. If $q \neq 2$ then $N(G) = \{3, 6\}$, and if $q = 2$ then $N(G) = \{3\}$. The set $\{1, 2, 3, 4, 6\}$ includes $I(G)$, so $\xi = \{p\} \cup R_1 \cup R_2 \cup R_4$, where $\{p\}$, R_2 and R_4 are always nonempty. The prime p and any prime r_4 are adjacent one to another, and are non-adjacent to every r_i with $i \in N(G)$. On the other hand, for $i \in \{1, 2\}$ and $j \in \{3, 6\}$, primes r_i and r_j are adjacent. Therefore, $M(G) = N(G)$, $\theta(G)$ is of type $\{r_3\}$ for $q = 2$, and is of type $\{r_3, r_6\}$ otherwise. The set $\Theta'(G)$ consists of one-element sets of type $\{p\}$, $\{r_2\}$, and $\{r_4\}$, if $q = 2$, and sets of type $\{p\}$, and $\{r_4\}$ otherwise. Thus, $t(G) = 2$ for $q = 2$, and $t(G) = 3$ otherwise.

Let $n > 4$. Now we consider four different cases subject to residue of n modulo 4. We write $n = 4t + k$, where $k = 0, 1, 2, 3$, and $t \geqslant 1$. If $q = 2$ we assume that $t > 1$ to avoid exceptional cases that arise because of $R_6 = \varnothing$ for $q = 2$.

Suppose that $n = 4t > 4$. Then

$$N(G) = \{2t+1, 2t+3, \dots, 4t-1, 4t+2, 4t+4, \dots, 8t-2\},$$

so $|N(G)| = 3t - 1 > 4$. By [1, Propositions 3.1, 4.4], we have $t(2, G) \leqslant t(p, G) < 4$, so p and 2 cannot lie in any coclique of maximal size. By adjacency criterion, r_{4t} is non-adjacent to every r_i, where $i \in N(G)$. On the other hand, any prime r_{4t-2} is adjacent to r_{4t}, r_{4t+2}, any prime r_{2t-1} is adjacent to r_{4t}, r_{2t+1}, and if $\eta(i) < 2t-1$, then r_i is adjacent to at least three primes from every χ. Therefore, $M(G) = N(G) \cup \{n\}$, every $\theta(G) \in \Theta(G)$ is of type $\{r_i \mid n/2 \leqslant \eta(i) \leqslant n, i \neq 2n\}$, $\Theta'(G) = \varnothing$, and $t(G) = 3t = [(3n+1)/4]$.

Suppose that $n = 4t + 1$. Then

$$N(G) = \{2t+1, 2t+3, \dots, 4t+1, 4t+2, 4t+4, \dots, 8t\},$$

so $|N(G)| = 3t + 1 \geqslant 4$. By [1, Propositions 3.1, 4.4], we have $t(2, G) \leqslant t(p, G) < 4$. Therefore, p and 2 cannot lie in any coclique of maximal size. If $\eta(i) < 2t$, then any prime r_i is adjacent to r_{4t+2}, r_{2t+1}. Assume that $i = 4t$, then r_i is adjacent to r_j, where $j \in N(G)$, if and only if $j = 4t + 2$. Thus, $M(G) = N(G) \setminus \{n + 1\}$, every $\theta(G) \in \Theta(G)$ is of type $\{r_i \mid n/2 < \eta(i) \leqslant n, i \neq n+1, 2n\}$, and $\Theta'(G)$ consists of one-element sets of type $\{r_{4t}\}$ or $\{r_{4t+2}\}$. Therefore, $t(G) = 3t + 1 = [(3n + 1)/4]$.

Suppose that $n = 4t + 2$. Then

$$N(G) = \{2t+3, 2t+5, \dots, 4t+1, 4t+4, 4t+6, \dots, 8t+2\},$$

so $|N(G)| = 3t \geqslant 3$. Any primes r_{2t+1} and r_{4t+2} are adjacent one to another and are non-adjacent to every r_i with $i \in N(G)$. Hence $t(G) \geqslant 4$. Since $t(2, G) \leqslant t(p, G) < 4$, primes p and 2 cannot lie in any coclique of maximal size. If $\eta(i) < n/2$, then r_i is adjacent to r_{2t+1}, r_{4t+2}, r_{4t+4}. Therefore, $N(G) = M(G)$, every $\theta(G) \in \Theta(G)$ is of type $\{r_i \mid n/2 < \eta(i) \leqslant n, i \neq 2n\}$, and $\Theta'(G)$ consists of one-element sets of type $\{r_{2t+1}\}$ or $\{r_{4t+2}\}$. Thus, $t(G) = 3t + 1 = [(3n + 1)/4]$.

Suppose that $n = 4t + 3$. Then

$$N(G) = \{2t+3, 2t+5, \dots, 4t+3, 4t+4, 4t+6, \dots, 8t+4\},$$

so $|N(G)| = 3t + 2 \geqslant 5$. Since $t(2, G) \leqslant t(p, G) < 4$, primes p and 2 cannot lie in a coclique of maximal size. By adjacency criterion, r_{2t+1} is non-adjacent to every r_i, where $i \in N(G)$. On the other hand, if $\eta(i) < 2t + 1$ or $i = 4t + 2$, then r_i is adjacent to r_{4t+4}, r_{2t+1}. Therefore, $M(G) = N(G) \cup \{(n - 1)/2\}$, every $\theta(G) \in \Theta(G)$ is of type $\{r_i \mid (n-1)/2 \leqslant \eta(i) \leqslant n, i \neq 2n, n-1\}$, $\Theta'(G) = \varnothing$, and $t(G) = 3t + 3 = (3n + 3)/4$. In case $n = 4t + 3$, any coclique of maximal size does not contain primes of type r_{4t+2}, and so the group $D_7(2)$ is considered as well.

It remains to consider the cases: $q = 2$ and $n = 4 + k$, where $k = 1, 2$. Both results (see Table 3) are obtained by arguments similar to that in general case with respect to the fact: $R_6 = \varnothing$, and can be easily verified by using Proposition 2.5 and [1, Propositions 3.1, 4.4].

Case 3. Let $G = {}^2D_n(q)$.

Suppose that $n = 4$. If $q \neq 2$ then $N(G) = \{3, 6, 8\}$, and if $q = 2$ then $N(G) = \{3, 8\}$. The set $\{1, 2, 3, 4, 6, 8\}$ includes $I(G)$, and so $\xi = \{p\} \cup R_1 \cup R_2 \cup R_4$, where $\{p\}$, R_2, and R_4

are always nonempty. The prime p and any prime r_4 are adjacent one to another, and are non-adjacent to every r_i with $i \in N(G)$. On the other hand, for $i \in \{1, 2\}$ and $j \in \{3, 6\}$, primes r_i and r_j are adjacent. Therefore, $M(G) = N(G)$, $\theta(G)$ is of type $\{r_3, r_8\}$ for $q = 2$, and is of type $\{r_3, r_6, r_8\}$ otherwise. The set $\Theta'(G)$ consists of one-element sets of type $\{p\}$, and $\{r_4\}$. Thus, $t(G) = 3$ for $q = 2$, and $t(G) = 4$ otherwise.

Let $n > 4$. Now we consider four different cases subject to residue of n modulo 4. We write $n = 4t + k$, where $k = 0, 1, 2, 3$, and $t \geqslant 1$. If $q = 2$ we assume that $t > 1$ to avoid exceptional cases that arise because of $R_6 = \varnothing$ for $q = 2$.

Suppose that $n = 4t > 4$. Then

$$N(G) = \{2t + 1, 2t + 3, \dots, 4t - 1, 4t + 2, 4t + 4, \dots, 8t\},$$

so $|N(G)| = 3t > 4$. By [1, Propositions 3.1, 4.4], we have $t(2, G) \leqslant t(p, G) \leqslant 4$, so p and 2 cannot lie in any coclique of maximal size. By adjacency criterion, r_{4t} is non-adjacent to every r_i, where $i \in N(G)$. On the other hand, any prime r_{2t-1} is adjacent to r_{4t}, r_{4t+2}, any prime r_{4t-2} is adjacent to r_{4t}, r_{2t+1}, and if $\eta(i) < 2t - 1$, then r_i is adjacent to at least three primes from every χ. Therefore, $M(G) = N(G) \cup \{n\}$, every $\theta(G) \in \Theta(G)$ is of type $\{r_i \mid n/2 \leqslant \eta(i) \leqslant n\}$, $\Theta'(G) = \varnothing$, and $t(G) = 3t + 1 = [(3n + 4)/4]$.

Suppose that $n = 4t + 1$. Then

$$N(G) = \{2t + 1, 2t + 3, \dots, 4t - 1, 4t + 2, 4t + 4, \dots, 8t + 2\},$$

so $|N(G)| = 3t + 1 \geqslant 4$. By [1, Propositions 3.1, 4.4], we have $t(2, G) \leqslant t(p, G) < 4$. Therefore, p and 2 cannot lie in any coclique of maximal size. If $\eta(i) < 2t$, then any prime r_i is adjacent to r_{4t+2}, r_{2t+1}. Assume that $i = 4t$, then r_i is adjacent to r_j, where $j \in N(G)$, if and only if $j = 2t+1$. Thus, $M(G) = N(G) \setminus \{(n+1)/2\}$, every $\theta(G) \in \Theta(G)$ is of type $\{r_i \mid n/2 < \eta(i) \leqslant n, i \neq (n + 1)/2, n\}$, and $\Theta'(G)$ consists of one-element sets of type $\{r_{4t}\}$ or $\{r_{2t+1}\}$. Therefore, $t(G) = 3t + 1 = [(3n + 4)/4]$.

Suppose that $n = 4t + 2$. Then

$$N(G) = \{2t + 3, 2t + 5, \dots, 4t + 1, 4t + 4, 4t + 6, \dots, 8t + 4\},$$

so $|N(G)| = 3t + 1 \geqslant 4$. Any primes r_{2t+1}, r_{4t} and r_{4t+2} are adjacent one to another and are non-adjacent to every r_i with $i \in N(G)$. Hence $t(G) > 4$. Since $t(2, G) \leqslant t(p, G) \leqslant 4$, primes p and 2 cannot lie in any coclique of maximal size. If $\eta(i) < 2t$, then r_i is adjacent to r_{2t+1}, r_{4t+2}, r_{4t}, r_{4t+4}. Therefore, $N(G) = M(G)$, every $\theta(G) \in \Theta(G)$ is of type $\{r_i \mid n/2 < \eta(i) \leqslant n\}$, and $\Theta'(G)$ consists of one-element sets of type $\{r_{2t+1}\}$, $\{r_{4t}\}$ or $\{r_{4t+2}\}$. Thus, $t(G) = 3t + 2 = [(3n + 4)/4]$.

Suppose that $n = 4t + 3$. Then

$$N(G) = \{2t + 3, 2t + 5, \dots, 4t + 1, 4t + 4, 4t + 6, \dots, 8t + 6\},$$

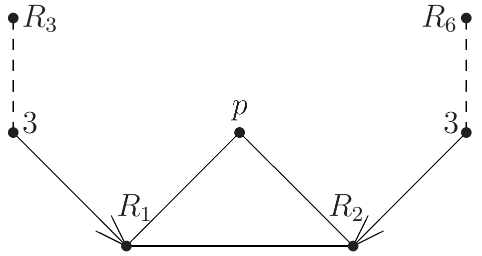
so $|N(G)| = 3t + 2 \geqslant 5$. Since $t(2, G) \leqslant t(p, G) < 4$, primes p and 2 cannot lie in a coclique of maximal size. By adjacency criterion, r_{4t+2} is non-adjacent to every r_i, where $i \in N(G)$. On the other hand, if $\eta(i) < 2t + 1$ or $i = 2t + 1$, then r_i is adjacent to r_{4t+4}, r_{4t+2}. Therefore, $M(G) = N(G) \cup \{n - 1\}$, every $\theta(G) \in \Theta(G)$ is of type $\{r_i \mid (n-1)/2 \leqslant \eta(i) \leqslant n, i \neq n, (n-1)/2\}$, $\Theta'(G) = \varnothing$, and $t(G) = 3t+3 = [(3n+4)/4]$.

It remains to consider the cases: $q = 2$ and $n = 4+k$, where $k = 1, 2, 3$. All results (see Table 3) are obtained by arguments similar to that in general case with respect to the fact: $R_{4t+2} = R_6 = \varnothing$, and can be easily verified by using Proposition 2.5 and [1, Propositions 3.1, 4.4]. $\square$

Proposition 3.11. *If G is an finite simple exceptional group of Lie type over a field of characteristic p, then $t(G)$, and the sets $\Theta(G)$, $\Theta'(G)$ are listed in Table 4.*

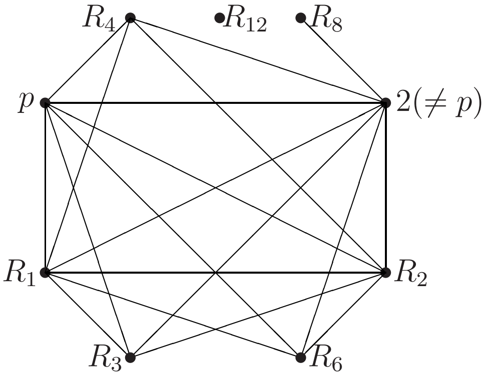
Proof. We consider all types of exceptional groups of Lie type separately. Following [9], we use the compact form of the prime graph $GK(G)$. By the compact form we mean a graph whose vertices are labeled with marks R_i. A vertex labeled R_i represents the clique of $GK(G)$ such that every vertex in this clique labeled by a prime from R_i. An edge connecting R_i and R_j represents the set of edges of $GK(G)$ that connect each vertex in R_i with each vertex in R_j. If an edge occurs under some condition, we draw such edge by a dotted line and write corresponding occurence condition. The technical tools for determining the compact form of the prime graph $GK(G)$ for an exceptional group of Lie type G are Propositions 2.7 and 2.9, and also [1, Propositions 3.2, 3.3, and 4.5]. Notice that the compact form of $GK(G)$ can be considered as a graphical form of the adjacency criterion in $GK(G)$.

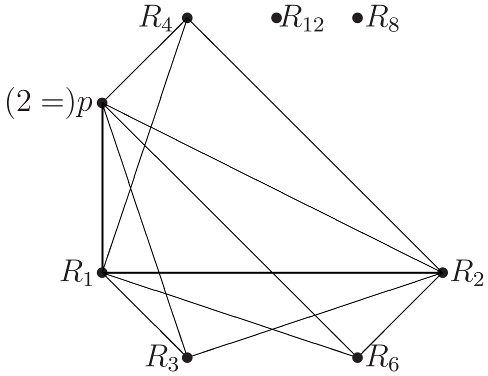
The compact form for $GK(G_2(q))$



Let $G = G_2(q)$. In the compact form for $GK(G_2(q))$ the vector from 3 to R_1 (resp. R_2) and the dotted edge $(3, R_3)$ (resp.$(3, R_6)$) mean that R_1 (resp. R_2) and R_3 (resp. R_6) are not connected, but if $3 \in R_1$, i. e., $q \equiv 1 \pmod 3$ (resp. $3 \in R_2$, i. e., $q \equiv -1 \pmod 3$), then there exists an edge between 3 and R_3 (resp. R_6). If $R_1 = \varnothing$, then one need to remove vertex R_1 with all corresponding edges. From the compact form of $GK(G)$ it is evident, that $\Theta(G) = \{\{r_3, r_6\} \mid r_i \in R_i\}$, while $\Theta'(G) = \{\{p\}, \{r_1\}, \{r_2\} \mid r_i \in R_i \setminus \{3\}\}$.

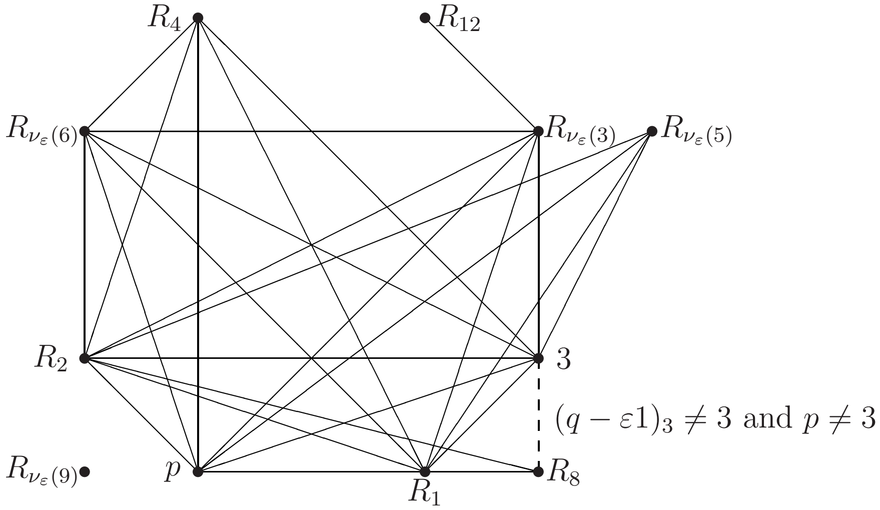
The compact form for $GK(F_4(q))$





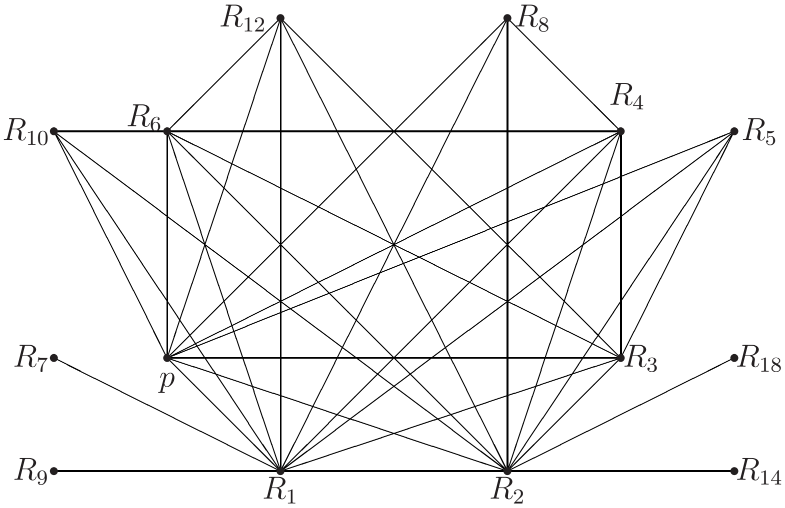
Let $G = F_4(q)$. It is evident from the compact form for $GK(F_4(q))$ that $\{2, p, R_1, R_2, R_3\}$ is a clique, while the remaining vertices in the compact form are pairwise non-adjacent. Since R_3 in non-adjacent to R_4, R_6, R_8, R_{12} and the remaining vertices from the set $\{2, p, R_1, R_2\}$ are adjacent to at least two vertices from the set $\{R_4, R_6, R_8, R_{12}\}$, we obtain that $\Theta(G) = \{\{r_3, r_4, r_6, r_8, r_{12}\} \mid r_i \in R_i\}$ if $R_6 \neq \varnothing$ and $\Theta(G) = \{\{r_3, r_4, r_8, r_{12}\} \mid r_i \in R_i\}$ if $R_6 = \varnothing$ (i. e., if $q = 2$). In both cases $\Theta'(G) = \varnothing$.

The compact form for $GK(E_6^\varepsilon(q))$



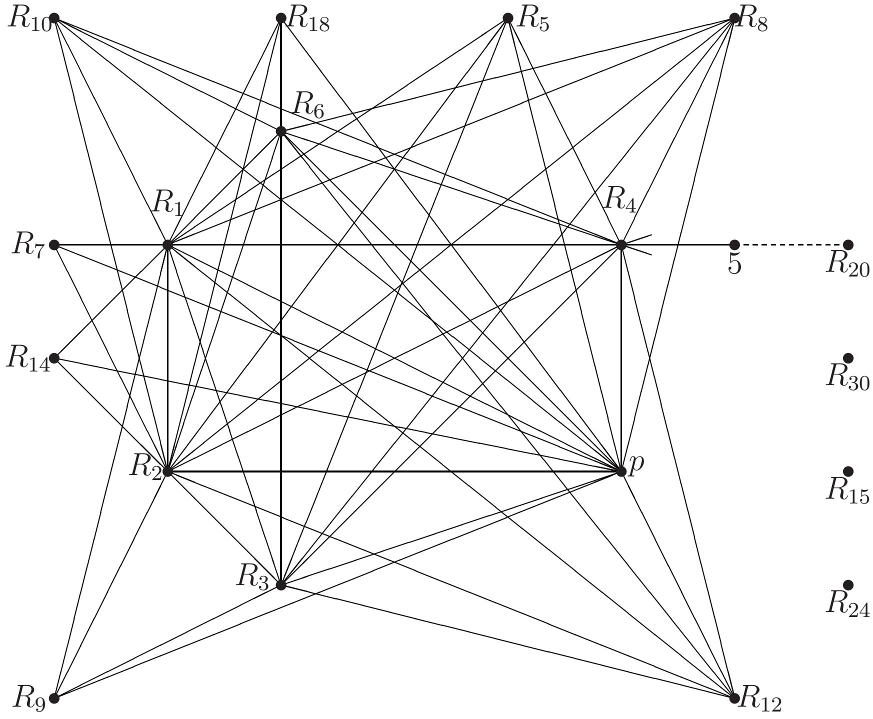
Let $G = E_6^\varepsilon(q)$. In the compact form for $GK(E_6(q))$ the set $\{3, p, R_1, R_2, R_{\nu_\varepsilon(3)}, R_{\nu_\varepsilon(6)}\}$ forms a clique, while the remaining vertices are pairwise non-adjacent. Moreover, $R_{\nu_\varepsilon(3)}$ and $R_{\nu_\varepsilon(6)}$ are the only vertices from $\{3, p, R_1, R_2, R_{\nu_\varepsilon(3)}, R_{\nu_\varepsilon(6)}\}$, which are adjacent to precisely one of the remaining vertices (namely, $R_{\nu_\varepsilon(3)}$ is adjacent to R_{12}, and $R_{\nu_\varepsilon(6)}$ is adjacent to R_4). Thus $\Theta(G) = \{\{r_{\nu_\varepsilon(5)}, r_8, r_{\nu_\varepsilon(9)}\} \mid r_i \in R_i\}$ and $\Theta'(G) = \{\{r_4, r_{\nu_\varepsilon(3)}\}, \{r_{\nu_\varepsilon(6)}, r_{12}\}, \{r_4, r_{12}\} \mid r_i \in R_i\}$. Since $R_6 = \varnothing$ for $q = 2$, we obtain exceptions mentioned in Table 4.

The compact form for $GK(E_7(q))$



Let $G = E_7(q)$. In the compact form for $GK(E_7(q))$ the set $\{p, R_1, R_2, R_3, R_4, R_6\}$ forms a clique, while the remaining vertices are pairwise non-adjacent. Moreover, R_4 is the only vertices from $\{p, R_1, R_2, R_3, R_4, R_6\}$, which are adjacent to precisely one of the remaining vertices (namely, R_4 is adjacent to R_8). Thus $\Theta(G) = \{\{r_5, r_7, r_9, r_{10}, r_{12}, r_{14}, r_{18}\} \mid r_i \in R_i\}$ and $\Theta'(G) = \{\{r_4\}, \{r_8\} \mid r_i \in R_i\}$.

The compact form for $GK(E_8(q))$

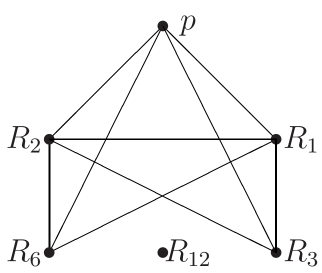


Let $G = E_8(q)$. In the compact form for $GK(E_8(q))$, the vector from 5 to R_4 and the dotted edge $(5, R_{20})$ mean that R_4 and R_{20} are not connected, but if $5 \in R_4$ (i. e., $q^2 \equiv -1 \pmod 5$), then there exists an edge between 5 and R_{20}. Now $\{p, R_1, R_2, R_3, R_4, R_6\}$ forms a clique, while the remaining vertices are pairwise non-adjacent. Notice that each vertex from the clique $\{p, R_1, R_2, R_3, R_4, R_6\}$ is adjacent to at least two vertices from the set of remaining vertices. So

$$\Theta(G) = \{\{r_5, r_7, r_8, r_9, r_{10}, r_{12}, r_{14}, r_{15}, r_{18}, r_{20}, r_{24}, r_{30}\} \mid r_i \in R_i\}$$

and $\Theta'(G) = \varnothing$.

The compact form for $GK(^3D_4(q))$



Let $G = {}^3D_4(q)$. From the compact form for $GK(^3D_4(q))$ we immediately obtain that $\Theta(G) = \{\{r_3, r_6, r_{12}\} \mid r_i \in R_i\}$ and $\Theta'(G) = \varnothing$ if $q \neq 2$. For $q = 2$ the result follows from the compact form for the prime graph $GK(^3D_4(q))$, and the fact that $R_6 = \varnothing$.

Let $G = {}^2B_2(q)$. In this case primes $s_i \in S_i$ and $s_j \in S_j$ are adjacent if and only if $i = j$, while $p = 2$ is non-adjacent to all vertices, and the proposition follows.

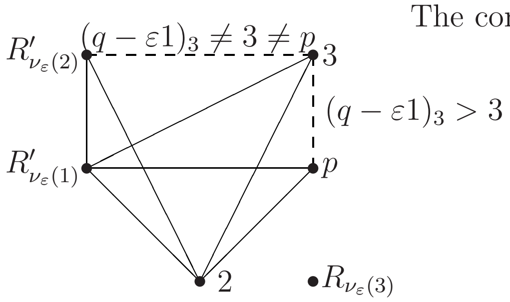
Let $G = {}^2G_2(q)$. In this case odd primes $s_i \in S_i$ and $s_j \in S_j$ are adjacent if and only if $i = j$, while $p = 3$ is non-adjacent to all odd primes. Since 2 is adjacent to s_1, s_2, and p, we obtain the statement of the proposition in this case.

Let $G = {}^2F_4(q)$. If $q > 8$, then any set of type $\{s_2, s_3, s_4, s_5, s_6\}$ forms a coclique in $GK(G)$ by Proposition 2.9. The same proposition together with [1, Proposition 3.3] implies that the set $\{2\} \cup S_1 \cup S_2$ forms a clique in $GK(G)$, any prime s_3 is adjacent to s_1 and 2, and 3 is adjacent to s_2 and s_4. By using this information we obtain the proposition in this case. If $G = {}^2F_4(8)$ then $S_2 = \pi(9) \setminus \{3\} = \varnothing$, whence every $\theta(G) \in \Theta(G)$ is of type $\{s_5, s_6\}$, and every $\theta'(G) \in \Theta'(G)$ is a two-element set of type either $\{s_1, s_4\}$, or $\{3, s_3\}$, or $\{2, s_4\}$, or $\{s_3, s_4\}$. The group $G = {}^2F_4(2)$ is not simple, and its derived subgroup $T = {}^2F_4(2)'$ is the simple Tits group. Using [2], we obtain that the prime graph of the Tits group T contains a unique coclique $\rho(T) = \{3, 5, 13\}$ of maximal size. $\square$

Proposition 3.12. *If $G \simeq A^{\varepsilon}_{n-1}(q)$ is an finite simple groups of Lie type over a field of characteristic p and $n \in \{2, 3\}$, then $t(G)$, and the sets $\Theta(G)$, $\Theta'(G)$ are listed in Table 2.*

Proof. Let $G = A_1(q)$. Then the compact form for $GK(A_1(q))$ is a coclique with the set of vertices $\{R_1, R_2, p\}$. Thus $\Theta(G) = \{\{r_1, r_2, p\} \mid r_i \in R_i\}$ and $\Theta'(G) = \varnothing$.

The compact form for $GK(A^{\varepsilon}_2(q))$



Let $G = A^{\varepsilon}_2(q)$. Set $R'_{\nu_\varepsilon(1)} = R_{\nu_\varepsilon(1)} \setminus \{2, 3\}$, and $R'_{\nu_\varepsilon(2)} = R_{\nu_\varepsilon(2)} \setminus \{2, 3\}$. Assume first that $(q - \varepsilon 1)_3 > 3$. Then the set $\{2, 3, p, R'_{\nu_\varepsilon(1)}\}$ is a clique in the compact form for $GK(A^{\varepsilon}_2(q))$, while $R'_{\nu_\varepsilon(2)}$ and $R_{\nu_\varepsilon(3)}$ are non-adjacent. If $R_{\nu_\varepsilon(2)} \neq \{2\}$ (i. e., $q + \varepsilon 1 \neq 2^k$ and $R'_{\nu_\varepsilon(2)} \neq \varnothing$), then p is the only vertex from the clique $\{2, 3, p, R'_{\nu_\varepsilon(1)}\}$, which is non-adjacent to both $R'_{\nu_\varepsilon(2)}$ and $R_{\nu_\varepsilon(3)}$. Hence $\Theta(G) = \{\{p, r_{\nu_\varepsilon(2)} \neq 2, r_{\nu_\varepsilon(3)}\} \mid r_i \in R_i\}$ and $\Theta'(G) = \varnothing$. If $R_{\nu_\varepsilon(2)} = \{2\}$ (i. e., $q + \varepsilon 1 = 2^k$ and $R'_{\nu_\varepsilon(2)} = \varnothing$), then $\Theta(G) = \{\{r_{\nu_\varepsilon(3)}\} \mid r_{\nu_\varepsilon(3)} \in R_{\nu_\varepsilon(3)}\}$ and $\Theta'(G) = \{\{2\}, \{p\}, \{r_{\nu_\varepsilon(1)}\} \mid r_{\nu_\varepsilon(1)} \in R_{\nu_\varepsilon(1)}\}$.

Now assume that $(q - \varepsilon 1)_3 = 3$. Then the set $\{2, p, R'_{\nu_\varepsilon(1)}\}$ is a clique in the compact form for $GK(A^{\varepsilon}_2(q))$, while 3, $R'_{\nu_\varepsilon(2)}$, and $R_{\nu_\varepsilon(3)}$ are pairwise non-adjacent. Since p is the only vertex from the clique $\{2, p, R'_{\nu_\varepsilon(1)}\}$, which is non-adjacent to 3, $R'_{\nu_\varepsilon(2)}$, and $R_{\nu_\varepsilon(3)}$, we obtain that $\Theta(G) = \{\{3, p, r_{\nu_\varepsilon(2)} \neq 2, r_{\nu_\varepsilon(3)}\} \mid r_i \in R_i\}$ if $R_{\nu_\varepsilon(2)} \neq \{2\}$, and $\Theta(G) = \{\{3, p, r_{\nu_\varepsilon(3)}\} \mid r_{\nu_\varepsilon(3)} \in R_{\nu_\varepsilon(3)}\}$ if $R_{\nu_\varepsilon(2)} = \{2\}$. In both cases $\Theta'(G) = \varnothing$.

Assume at the end that $(q - \varepsilon 1)_3 = 1$, i. e., either $(q + \varepsilon 1)_3 > 1$ and $3 \in R_{\nu_\varepsilon(2)} \neq \{2\}$, or $p = 3$. As above we have that the set $\{2, p, R'_{\nu_\varepsilon(1)}\}$ is a clique in the compact form for $GK(A^{\varepsilon}_2(q))$, while $R'_{\nu_\varepsilon(2)}$ and $R_{\nu_\varepsilon(3)}$ are pairwise non-adjacent. Since p is the only vertex from the clique $\{2, p, R'_{\nu_\varepsilon(1)}\}$, which is non-adjacent to $R'_{\nu_\varepsilon(2)}$ and $R_{\nu_\varepsilon(3)}$, and since either $3 \in R_{\nu_\varepsilon(2)}$ or $p = 3$, we obtain that $\Theta(G) = \{\{p, r_{\nu_\varepsilon(2)} \neq 2, r_{\nu_\varepsilon(3)}\} \mid r_i \in R_i \setminus \{2\}\}$ and $\Theta'(G) = \varnothing$ if $R_{\nu_\varepsilon(2)} \neq \{2\}$, and $\Theta(G) = \{\{r_{\nu_\varepsilon(3)}\} \mid r_{\nu_\varepsilon(3)} \in R_{\nu_\varepsilon(3)}\}$ and $\Theta'(G) = \{\{p\}, \{r_{\nu_\varepsilon(1)}\}, \{2 = r_{\nu_\varepsilon(2)}\} \mid r_{\nu_\varepsilon(1)} \in R_{\nu_\varepsilon(1)}\}$ if $R_{\nu_\varepsilon(2)} = \{2\}$. $\square$

Below we give Tables 2, 3, 4. These tables are organized in the following way. Column 1 represents a group of Lie type G with the base field of order q and characteristic p, Column 2 contains conditions on G, and Column 3 contains value of $t(G)$. In Columns 4 and 5 we list the elements of $\Theta(G)$ and $\Theta'(G)$, that is sets $\theta(G) \in \Theta(G)$ and $\theta'(G) \in \Theta'(G)$, and omit the braces for one-element sets. In particular, the item $\{p, 3, r_2 \neq 2, r_3\}$ in Column 4 means $\Theta(G) = \{\{p, 3, r_2, r_3\} \mid r_2 \in R_2 \setminus \{2\}, r_3 \in R_3\}$ and the item p, r_4 in Column 5 means $\Theta'(G) = \{\{p\}, \{r_4\} \mid r_4 \in R_4\}$.

Table 2. Cocliques for finite simple linear and unitary groups

G	Conditions	$t(G)$	$\Theta(G)$	$\Theta'(G)$
$A_1(q)$	$q > 3$	3	$\{p, r_1, r_2\}$	$\varnothing$
$A_2(q)$	$(q-1)_3 = 3,\ q+1 \neq 2^k$	4	$\{p, 3, r_2 \neq 2, r_3\}$	$\varnothing$
	$(q-1)_3 = 3,\ q+1 = 2^k$	3	$\{3, p, r_3\}$	$\varnothing$
	$(q-1)_3 \neq 3,\ q+1 \neq 2^k$	3	$\{p, r_2 \neq 2, r_3\}$	$\varnothing$
	$(q-1)_3 \neq 3,\ q+1 = 2^k$	2	r_3	$p, r_1, 2 = r_2$
$A_3(q)$	$(q-1)_2 \neq 4$	3	$\{p, r_3, r_4\}$	$\varnothing$
	$(q-1)_2 = 4$	3	$\{r_3, r_4\}$	$p, 2$
$A_4(q)$	$(q-1)_5 \neq 5$	3	$\{r_4, r_5\}$	p, r_3
	$(q-1)_5 = 5$	3	$\{r_4, r_5\}$	$5, p, r_3$
$A_5(q)$	$q = 2$	3	$\{r_3, r_4, r_5\}$	$\varnothing$
	$q > 2$ and $(q-1)_3 \neq 3$	3	r_5	$\{p, r_6\}$,$\{r_3, r_4\}$, $\{r_4, r_6\}$
	$(q-1)_3 = 3$	3	r_5	$\{p, r_6\}$,$\{r_3, r_4\}$, $\{r_4, r_6\}$, $\{3, r_6\}$
$A_{n-1}(q)$, $n \geqslant 7$	n is odd and $q \neq 2$ for $7 \leqslant n \leqslant 11$	$[\frac{n+1}{2}]$	$\{r_i \mid \frac{n}{2} < i \leqslant n\}$	$\varnothing$
	n is even and $q \neq 2$ for $8 \leqslant n \leqslant 12$	$[\frac{n+1}{2}]$	$\{r_i \mid \frac{n}{2} < i < n\}$	$r_{\frac{n}{2}}$,r_n
	$n = 7,\ q = 2$	3	$\{r_5, r_7\}$	r_3, r_4
	$n = 8,\ q = 2$	3	r_7	$\{p, r_8\}$,$\{r_5, r_8\}$, $\{r_3, r_8\}$,$\{r_4, r_5\}$
	$n = 9,\ q = 2$	4	$\{r_5, r_7, r_8, r_9\}$	$\varnothing$
	$n = 10,\ q = 2$	4	$\{r_7, r_9\}$	$\{r_4, r_{10}\}$,$\{r_8, r_{10}\}$ $\{r_5, r_8\}$
	$n = 11,\ q = 2$	5	$\{r_7, r_8, r_9, r_{11}\}$	r_5, r_{10}
	$n = 12,\ q = 2$	6	$\{r_7, r_8, r_9, r_{10}, r_{11}, r_{12}\}$	$\varnothing$
${}^2A_2(q)$, $q > 2$	$(q+1)_3 = 3,\ q-1 \neq 2^k$	4	$\{p, 3, r_1 \neq 2, r_6\}$	$\varnothing$
	$(q+1)_3 = 3,\ q-1 = 2^k$	3	$\{3, p, r_6\}$	$\varnothing$
	$(q+1)_3 \neq 3,\ q-1 \neq 2^k$	3	$\{p, r_1 \neq 2, r_6\}$	$\varnothing$
	$(q+1)_3 \neq 3,\ q-1 = 2^k > 2$	2	r_6	$p, r_2, 2 = r_1$
	$q = 3$	2	r_6	$p, r_2 = 2$
${}^2A_3(q)$	$(q+1)_2 \neq 4$ and $q \neq 2$	3	$\{p, r_6, r_4\}$	$\varnothing$
	$(q+1)_2 = 4$	3	$\{r_6, r_4\}$	$p, 2$
	$q = 2$	2	r_4	p, r_2
${}^2A_4(q)$	$q = 2$	3	$\{p, r_4, r_{10}\}$	$\varnothing$
	$q > 2$ and $(q+1)_5 \neq 5$	3	$\{r_4, r_{10}\}$	p, r_6
	$(q+1)_5 = 5$	3	$\{r_4, r_{10}\}$	$5, p, r_6$
${}^2A_5(q)$	$q = 2$	3	$\{r_{10}, r_3\}$	$3, p, r_4$
	$(q+1)_3 \neq 3$	3	r_{10}	$\{p, r_3\}$,$\{r_6, r_4\}$, $\{r_4, r_3\}$
	$q > 2$ and $(q+1)_3 = 3$	3	r_{10}	$\{p, r_3\}$,$\{r_6, r_4\}$, $\{r_4, r_3\}$, $\{3, r_3\}$
${}^2A_{n-1}(q)$, $n \geqslant 7$	n is odd	$[\frac{n+1}{2}]$	$\{r_i \mid \frac{n}{2} < \nu(i) \leqslant n\}$	$\varnothing$
	n is even	$[\frac{n+1}{2}]$	$\{r_i \mid \frac{n}{2} < \nu(i) < n\}$	$r_{\nu(\frac{n}{2})}, r_{\nu(n)}$

Table 3. Cocliques for finite simple symplectic and orthogonal groups

G	Conditions	$t(G)$	$\Theta(G)$	$\Theta'(G)$
$B_n(q)$ or $C_n(q)$	$n=2$, $q=3$	2	r_4	p, r_2
	$n=2$, $q>3$	2	r_4	p, r_1, r_2
	$n=3$ and $q=2$	2	r_3	p, r_2, r_4
	$n=3$ and $q>2$	3	$\{r_3, r_6\}$	p, r_4
	$n=4$ and $q=2$	3	$\{r_3, r_4, r_8\}$	$\varnothing$
	$n=5$ and $q=2$	4	$\{r_5, r_8, r_{10}\}$	r_3, r_4
	$n=6$ and $q=2$	5	$\{r_3, r_5, r_8, r_{10}, r_{12}\}$	$\varnothing$
	$n=7$ and $q=2$	6	$\{r_5, r_7, r_{10}, r_{12}, r_{14}\}$	r_3, r_8
	$n>3$, $n\equiv 0,1 \pmod 4$ and $(n,q)\neq(4,2),(5,2)$	$\left[\frac{3n+5}{4}\right]$	$\{r_i \mid \frac{n}{2}\leqslant \eta(i)\leqslant n\}$	$\varnothing$
	$n>3$, $n\equiv 2 \pmod 4$ and $(n,q)\neq(6,2)$	$\left[\frac{3n+5}{4}\right]$	$\{r_i \mid \frac{n}{2}< \eta(i)\leqslant n\}$	$r_{n/2}, r_n$
	$n>3$, $n\equiv 3 \pmod 4$ and $(n,q)\neq(7,2)$	$\left[\frac{3n+5}{4}\right]$	$\{r_i \mid \frac{n+1}{2}< \eta(i)\leqslant n\}$	$r_{(n-1)/2}, r_{n-1}, r_{n+1}$
$D_n(q)$	$n=4$, $q=2$	2	r_3	p, r_2, r_4
	$n=4$ and $q>2$	3	$\{r_3, r_6\}$	p, r_4
	$n=5$ and $q=2$	4	$\{r_3, r_4, r_5, r_8\}$	$\varnothing$
	$n=6$ and $q=2$	4	$\{r_3, r_5, r_8, r_{10}\}$	$\varnothing$
	$n>4$, $n\equiv 0 \pmod 4$	$\left[\frac{3n+1}{4}\right]$	$\{r_i \mid \frac{n}{2}\leqslant \eta(i)\leqslant n, i\neq 2n\}$	$\varnothing$
	$n>4$, $n\equiv 1 \pmod 4$ and $(n,q)\neq(5,2)$	$\left[\frac{3n+1}{4}\right]$	$\{r_i \mid \frac{n}{2}< \eta(i)\leqslant n, i\neq 2n, n+1\}$	r_{n-1}, r_{n+1}
	$n>4$, $n\equiv 2 \pmod 4$ and $(n,q)\neq(6,2)$	$\left[\frac{3n+1}{4}\right]$	$\{r_i \mid \frac{n}{2}< \eta(i)\leqslant n, i\neq 2n\}$	$r_{n/2}, r_n$
	$n>4$, $n\equiv 3 \pmod 4$	$\frac{3n+3}{4}$	$\{r_i \mid \frac{n-1}{2}\leqslant \eta(i)\leqslant n, i\neq 2n, n-1\}$	$\varnothing$
${}^2D_n(q)$	$n=4$, $q=2$	3	$\{r_3, r_8\}$	p, r_4
	$n=4$ and $q>2$	4	$\{r_3, r_6, r_8\}$	p, r_4
	$n=5$ and $q=2$	3	$\{r_8, r_{10}\}$	p, r_3, r_4
	$n=6$ and $q=2$	5	$\{r_5, r_8, r_{10}, r_{12}\}$	r_3, r_4
	$n=7$ and $q=2$	5	$\{r_5, r_{10}, r_{12}, r_{14}\}$	r_3, r_8
	$n>4$, $n\equiv 0 \pmod 4$ and	$\left[\frac{3n+4}{4}\right]$	$\{r_i \mid \frac{n}{2}\leqslant \eta(i)\leqslant n\}$	$\varnothing$
	$n>4$, $n\equiv 1 \pmod 4$ and $(n,q)\neq(5,2)$	$\left[\frac{3n+4}{4}\right]$	$\{r_i \mid \frac{n}{2}< \eta(i)\leqslant n, i\neq n, \frac{n+1}{2}\}$	$r_{(n+1)/2}, r_{n-1}$
	$n>4$, $n\equiv 2 \pmod 4$ and $(n,q)\neq(6,2)$	$\left[\frac{3n+4}{4}\right]$	$\{r_i \mid \frac{n}{2}< \eta(i)\leqslant n\}$	$r_{n/2}, r_{n-2}, r_n$
	$n>4$, $n\equiv 3 \pmod 4$ and $(n,q)\neq(7,2)$	$\left[\frac{3n+4}{4}\right]$	$\{r_i \mid \frac{n-1}{2}\leqslant \eta(i)\leqslant n, i\neq n, \frac{n-1}{2}\}$	$\varnothing$

Table 4. Cocliques for finite simple exceptional groups

G	Conditions	$t(G)$	$\Theta(G)$	$\Theta'(G)$
$G_2(q)$	$q = 3, 4$	3	$\{r_3, r_6\}$	p, r_2
	$q = 8$	3	$\{r_3, r_6\}$	p, r_1
	$q = 3^m > 3$	3	$\{r_3, r_6\}$	p, r_1, r_2
	$q \equiv 1 (\mathrm{mod}\ 3)$ and $q \neq 4$	3	$\{r_3, r_6\}$	$p, r_2, r_1 \neq 3$
	$q \equiv 2 (\mathrm{mod}\ 3)$ and $q \neq 8$	3	$\{r_3, r_6\}$	$p, r_1, r_2 \neq 3$
$F_4(q)$	$q = 2$	4	$\{r_3, r_4, r_8, r_{12}\}$	$\varnothing$
	$q > 2$	5	$\{r_3, r_4, r_6, r_8, r_{12}\}$	$\varnothing$
$E_6(q)$	$q = 2$	5	$\{r_4, r_5, r_8, r_9\}$	r_3, r_{12}
	$q > 2$	5	$\{r_5, r_8, r_9\}$	$\{r_3, r_4\}$, $\{r_4, r_{12}\}$, $\{r_6, r_{12}\}$
${}^2E_6(q)$	$q = 2$	5	$\{r_8, r_{10}, r_{12}, r_{18}\}$	r_3, r_4
	$q > 2$	5	$\{r_8, r_{10}, r_{18}\}$	$\{r_3, r_{12}\}$, $\{r_4, r_6\}$, $\{r_4, r_{12}\}$
$E_7(q)$		8	$\{r_5, r_7, r_9, r_{10}, r_{12}, r_{14}, r_{18}\}$	r_4, r_8
$E_8(q)$		12	$\{r_5, r_7, r_8, r_9, r_{10}, r_{12}, r_{14}, r_{15}, r_{18}, r_{20}, r_{24}, r_{30}\}$	$\varnothing$
${}^3D_4(q)$	$q = 2$	2	r_{12}	p, r_2, r_3
	$q > 2$	3	$\{r_3, r_6, r_{12}\}$	$\varnothing$
${}^2B_2(2^{2n+1})$	$n \geqslant 1$	4	$\{2, s_1, s_2, s_3\}$	$\varnothing$
${}^2G_2(3^{2n+1})$	$n \geqslant 1$	5	$\{3, s_1, s_2, s_3, s_4\}$	$\varnothing$
${}^2F_4(2^{2n+1})$	$n \geqslant 2$,	5	$\{s_2, s_3, s_4, s_5, s_6\}$	$\varnothing$
${}^2F_4(8)$		4	$\{s_5, s_6\}$	$\{3, s_3\}$, $\{s_1, s_4\}$, $\{2, s_4\}$, $\{s_3, s_4\}$
${}^2F_4(2)'$		3	$\{3, 5, 13\}$	$\varnothing$

4 Appendix

In this section we give a list of corrections for [1] which we obtain in the present paper.

Items (4), (5), (9) of Lemma 1.3 should be substituted by items (1), (2), (3) of Lemma 2.6 of the present paper respectively.

Lemma 1.4 should be substituted by Lemma 2.1.

Lemma 1.5 should be substituted be Lemma 2.8.

Proposition 2.3 should be substituted by Proposition 2.4.

Proposition 2.4 should be substituted by Proposition 2.5.

Proposition 2.5 should be substituted by Proposition 2.7.

In Tables 4 and 8 the following corrections are necessary.

The lines

$A_{n-1}(q)$	$n = 3$, $(q-1)_3 = 3$, and $q+1 \neq 2^k$	4	$\{p, 3, r_2, r_3\}$
	$n = 3$, $(q-1)_3 \neq 3$, and $q+1 \neq 2^k$	3	$\{p, r_2, r_3\}$

should be substituted by the lines

$A_{n-1}(q)$	$n = 3$, $(q-1)_3 = 3$, and $q+1 \neq 2^k$	4	$\{p, 3, r_2 \neq 2, r_3\}$
	$n = 3$, $(q-1)_3 \neq 3$, and $q+1 \neq 2^k$	3	$\{p, r_2 \neq 2, r_3\}$

The lines

${}^2A_{n-1}(q)$	$n = 3$, $(q+1)_3 = 3$, and $q-1 \neq 2^k$	4	$\{p, 3, r_1, r_6\}$
	$n = 3$, $(q+1)_3 \neq 3$, and $q-1 \neq 2^k$	3	$\{p, r_1, r_6\}$

should be substituted by the lines

${}^2A_{n-1}(q)$	$n = 3$, $(q+1)_3 = 3$, and $q-1 \neq 2^k$	4	$\{p, 3, r_1 \neq 2, r_6\}$
	$n = 3$, $(q+1)_3 \neq 3$, and $q-1 \neq 2^k$	3	$\{p, r_1 \neq 2, r_6\}$

In Table 4 in the penultimate line corresponding to $D_n(q)$ instead of $n \equiv 1 \pmod 1$, $n > 4$ there should be $n \equiv 1 \pmod 2$, $n > 4$.

In Table 8 the following corrections are necessary.

The line

$D_n(q)$	$n \geqslant 4$, $(n, q) \neq (4, 2), (5, 2), (6, 2)$	$\left[\frac{3n+1}{4}\right]$	$\{r_{2i} \mid \left[\frac{n+1}{2}\right] \leqslant i < n\} \cup$ $\cup \{r_i \mid \left[\frac{n}{2}\right] < i \leqslant n,$ $i \equiv 1 \pmod 2\}$

should be substituted by

$D_n(q)$	$n \geqslant 4$, $n \not\equiv 3 \pmod 4$, $(n, q) \neq (4, 2), (5, 2), (6, 2)$	$\left[\frac{3n+1}{4}\right]$	$\{r_{2i} \mid \left[\frac{n+1}{2}\right] \leqslant i < n\} \cup$ $\cup \{r_i \mid \left[\frac{n}{2}\right] < i \leqslant n,$ $i \equiv 1 \pmod 2\}$
	$n \equiv 3 \pmod 4$	$\frac{3n+3}{4}$	$\{r_{2i} \mid \left[\frac{n+1}{2}\right] \leqslant i < n\} \cup$ $\cup \{r_i \mid \left[\frac{n}{2}\right] \leqslant i \leqslant n,$

In Table 8 the line

${}^2D_n(q)$	$n \geqslant 4$, $n \not\equiv 1 \pmod 4$, $(n,q) \neq (4,2),(6,2),(7,2)$	$\left[\frac{3n+4}{4}\right]$	$\{r_{2i} \mid \left[\frac{n}{2}\right] \leqslant i \leqslant n\} \cup$ $\cup \{r_i \mid \left[\frac{n}{2}\right] < i \leqslant n,$ $i \equiv 1 \pmod 2\}$

should be substituted by the line

${}^2D_n(q)$	$n \geqslant 4$, $n \not\equiv 1 \pmod 4$, $(n,q) \neq (4,2),(6,2),(7,2)$	$\left[\frac{3n+4}{4}\right]$	$\{r_{2i} \mid \left[\frac{n}{2}\right] \leqslant i \leqslant n\} \cup$ $\cup \{r_i \mid \left[\frac{n}{2}\right] < i < n,$ $i \equiv 1 \pmod 2\}$

In Table 9 the following corrections are necessary.
The line

$E_6(q)$	$q = 2$	5	$\{5, 12, 17, 19, 31\}$
	$q > 2$	6	$\{r_4, r_5, r_6, r_8, r_9, r_{12}\}$

should be substituted by the line

$E_6(q)$	none	5	$\{r_4, r_5, r_8, r_9, r_{12}\}$

The line

$E_7(q)$	none	7	$\{r_7, r_8, r_9, r_{10}, r_{12}, r_{14}, r_{18}\}$

should be substituted by the line

$E_7(q)$	none	8	$\{r_5, r_7, r_8, r_9, r_{10}, r_{12}, r_{14}, r_{18}\}$

The line

$E_8(q)$	none	11	$\{r_7, r_8, r_9, r_{10}, r_{12}, r_{14}, r_{15}, r_{18}, r_{20}, r_{24}, r_{30}\}$

should be substituted by the line

$E_8(q)$	none	12	$\{r_5, r_7, r_8, r_9, r_{10}, r_{12}, r_{14}, r_{15}, r_{18}, r_{20}, r_{24}, r_{30}\}$

A revised variant of [1] can be found in
http://arxiv.org/abs/math/0506294.

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