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MINIMAL PERMUTATION REPRESENTATIONS OF FINITE SIMPLE EXCEPTIONAL GROUPS OF TYPES E_6, E_7, AND E_8

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A minimal permutation representation of a group is its faithful permutation representation of least degree. We will find degrees and point stabilizers, as well as ranks, subdegrees, and double stabilizers, for groups of types E_6, E_7, and E_8. This brings to a close the study of minimal permutation representations of finite simple Chevalley groups.

This paper continues [1], where minimal permutation representations of finite simple exceptional groups of types G_2 and F_4 were described. Our present goal is to obtain a similar description for minimal permutation representations of groups E_6, E_7, and E_8. We follow the notation and terminology developed in [1], in which the reader can find all necessary preliminary information and a complete list of references.

1. GROUP $E_6(q)$

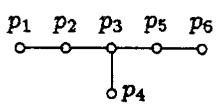
A. Algebra E_6. The rank of E_6 equals 6. Obviously, E_6 is a subalgebra of E_8. If $\mathcal{K}$ is a Cartan subalgebra in E_8, then $\mathcal{K}_{\mathbb{R}}$ is an Euclidean space of dimension 8. Let $e_1, \ldots, e_8$ be an orthonormal basis of $\mathcal{K}_{\mathbb{R}}$. Then a system Π of simple roots for E_6 (as an algebra of E_8) is defined as follows: $p_1 = e_3 - e_4$, $p_2 = e_4 - e_5$, $p_3 = e_5 - e_6$, $p_4 = e_6 - e_7$, $p_5 = e_6 + e_7$, $p_6 = -\frac{1}{2}\sum_{i=1}^{8} e_i$.

The system of positive roots is

$$\Phi^+ = \left\{ \begin{array}{l} e_i \pm e_j,\ i<j,\ i=3,\ldots,6,\ j=4,\ldots,7; \\ -\frac{1}{2}\sum_{i=1}^{8} \varepsilon_i e_i,\ \varepsilon_i = \pm 1,\ \varepsilon_1=\varepsilon_2=\varepsilon_8=1,\ \prod_{i=1}^{8}\varepsilon_i = 1 \end{array} \right\},\quad |\Phi^+| = 36.$$

The matrix A has the form $\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$.

The Dynkin diagram is



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B. Group $E_6(q)$ and its parabolic subgroups of least index. The order of a field K equals $q = p^s$, where p is a prime. If 3 does not divide $q-1$, then the group $G = E_6(q)$ coincides with the universal group $\bar{G} = \bar{E}_6(q)$. Otherwise it is isomorphic to the factor group $\bar{G}$ w.r.t. the center $\bar{Z}$, which is a cyclic subgroup of order 3. Using Lemma 3 of [1], it is easy to verify that

$$\bar{Z} = \{\bar{h}_{p_1}(\mu)\cdot\bar{h}_{p_3}(\mu^2)\cdot\bar{h}_{p_5}(\mu)\cdot\bar{h}_{p_6}(\mu^2) \mid \mu\in K^*,\ \mu^3 = 1\}.$$

From the main result stated in [2] and Proposition 1 of [1], it follows that a subgroup of least index in G should be parabolic. Proposition 1 of [1] allows us to compute orders of maximal parabolic subgroups in G. Comparing these orders, we see that there are, up to conjugation, two subgroups of least index in G: $P_1 = P_{\Pi\setminus\{p_1\}}$ and $P_6 = P_{\Pi\setminus\{p_6\}}$ (for definitions, see [1]), which are conjugate in $\operatorname{Aut} G$.

Elements $\bar{x}_r(t)$ of $\bar{G}$, as well as $\bar{n}_r(t)$ and $\bar{h}_r(t)$, were defined in Lemma 3 in [1]. Let S be a subgroup of G generated by the elements $x_{r_1}(t),\dots,x_{r_k}(t_k)$. Denote by $\bar{S}$ a subgroup of $\bar{G}$ generated by $\bar{x}_{r_1}(t_1),\dots,\bar{x}_{r_k}(t_k)$.

The group $\bar{P}_1$, like $\bar{P}_6$, includes a subgroup $\bar{H}$, and hence also $\bar{Z}$. Thus

$$|\bar{G}| = q^{36}(q^{12}-1)(q^9-1)(q^8-1)(q^6-1)(q^5-1)(q^2-1),$$

$$|\bar{P}_1| = |\bar{P}_6| = q^{36}(q^8-1)(q^6-1)(q^5-1)(q^4-1)(q^2-1)(q-1),$$

$$|G| = |\bar{G}|/d',\ \ |P_1| = |P_6| = |\bar{P}_1|/d',\ \ \text{where } d' = (3, q-1),$$

$$n = |G:P_1| = |G:P_6| = |\bar{G}:\bar{P}_1| = (q^9-1)(q^8+q^4+1)/(q-1).$$

First, we describe a structure of the subgroup $\bar{P}_1$ in the universal group $\bar{G}$, using the Levi decomposition; see [1, Lemma 5]. Obviously, $\bar{P}_1 = \bar{U}_1\cdot\bar{L}_1$ (U_1 coincides with $\bar{U}_1$). For every element $r\in\Phi^+\setminus\Phi_1^+$, we have $r = p_1 + s_1$, where $s_1\in\Phi_1^+$. Therefore, U_1 is an elementary Abelian p-group of order q^{16}.

Let $\bar{H}_1 = \langle\bar{h}_{p_i}(\lambda)\mid i = 2,\dots,6;\ \lambda\in K^*\rangle$ and $\bar{L}_1' = \langle\bar{x}_r(t)\mid r\in\Phi_1,\ t\in K\rangle$. The group $\bar{L}_1'$ is isomorphic to the universal Chevalley group $\bar{D}_5(q)$, and $\bar{L}_1 = \bar{L}_1\cdot\langle h\rangle$, where $h\in\bar{H}\setminus\bar{H}_1$. Let $h_0 = \bar{h}_{p_1}(\lambda^4)\bar{h}_{p_2}(\lambda^5)\bar{h}_{p_3}(\lambda^6)\bar{h}_{p_4}(\lambda^3)\bar{h}_{p_5}(\lambda^4)\bar{h}_{p_6}(\lambda^2)$, where λ generates K^*; $h_0(\mu) = \bar{h}_{p_2}(\mu)\bar{h}_{p_3}(\mu^2)\bar{h}_{p_4}(\mu^3)\bar{h}_{p_6}(\mu^2)$, where $|\mu| = 4$ for $(4, q-1) = 4$, or $h_0(\mu) = \bar{h}_{p_2}(\mu)\bar{h}_{p_4}(\mu)$, where $|\mu| = 2$ for $(q-1, 4) = 2$, and $h_0(\mu) = 1$ for even q. The element h_0 centralizes $\bar{L}_1'$, and $\langle h_0\rangle\cap\bar{L}_1' = \langle h_0(\mu)\rangle$. Hence, the group $\bar{L}_1$ is isomorphic to an extension of the central product of groups $\bar{L}_1'$ and $\langle h_0\rangle$ over the subgroup $\langle h_0(\mu)\rangle$ by a cyclic group of order $e = (q-1, 4)$, that is, $\bar{L}_1 \simeq e\cdot(D_5(q)\times(q-1)/e)\cdot e$. Since the center of $\bar{G}$ lies in $\langle h_0\rangle$, we obtain $P_1 \simeq p^{16s} : (e\cdot(D_5(q)\times(q-1)/e')\cdot e)$, where $e' = ed'$ and $d' = (q-1, 3)$.

C. Representation of G on cosets w.r.t. P_1. Our goal is to define double stabilizers of the representation of G on the cosets w.r.t. P_1, that is, groups of the form $P_1\cap P_1^x$. Therefore, we need to choose appropriate elements x in a way that these do not map into W_1 under the natural homomorphism $\varphi\colon N\to W$. Since $H\le P_1$, the action of an element $n\in N$ on P_1 is determined by the action of its image $w\in W$ on Φ, and so below an element n will be identified with its image w. We need to adopt the following notation: $\Phi_{1,2} = \Phi_1\cap\Phi_2$ and $\Phi_{1,6} = \Phi_1\cap\Phi_6$. The action of w_{p_1} on Φ is shown in the following:

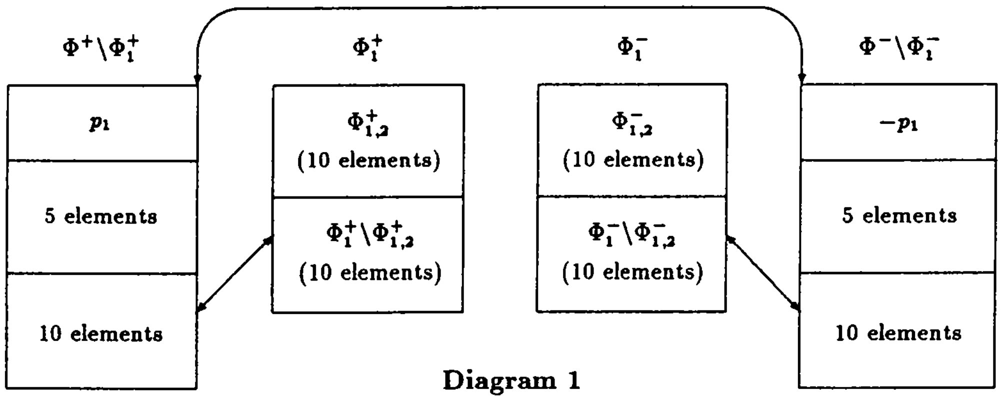


Diagram 1

An arrow pointing from set X to set Y says that $w_{p_1}(X) = Y$; the absence of an arrow outgoing from Z indicates that $w_{p_1}(Z) = Z$.

We start by determining the structure of a double stabiliser $\bar{M}_2 = \bar{P}_1 \cap \bar{P}_1^{w_{p_1}} = (\bar{U}_1 \cap \bar{P}_1^{w_{p_1}}) : (\bar{L}_1 \cap \bar{P}_1^{w_{p_1}})$. From Diagram 1, it follows that $\bar{U}_1 \cap \bar{P}_1^{w_{p_1}} \simeq p^{15s}$. Let $\bar{L}'_{1,2} = \langle \bar{x}_r(t) |\ r \in \Phi_{1,2},\ t \in K \rangle \simeq \bar{A}_4(q) \simeq SL_5(q)$, $\bar{U}_{1,2} = \langle \bar{x}_r(t) |\ r \in \Phi_1^+ \backslash \Phi_{1,2}^+,\ t \in K \rangle \simeq U_{1,2}$. The group $U_{1,2}$ is Abelian, as is U_1. Therefore, $U_{1,2} \simeq p^{10s}$. The center of $\bar{L}'_{1,2}$ is generated by the element

$$z(\mu) = \bar{h}_{p_3}(\mu)\bar{h}_{p_4}(\mu^3)\bar{h}_{p_5}(\mu^4)\bar{h}_{p_6}(\mu^2),$$

where $\mu^5 = 1$. Its order is equal to $f = (q-1, 5)$. The elements h_0 and $\bar{h}_{p_1}(\lambda)$ centralize $\bar{L}'_{1,2}$, and the element $z(\mu)\bar{h}_{p_1}(\mu)$ lies in $\langle h_0 \rangle$. Hence $\bar{L}_1 \cap \bar{P}_1^{w_{p_1}} \simeq p^{10s} : (f \cdot ((\bar{A}_4(q) \times (q-1)/f) \cdot (q-1)/f) \cdot f)$. Now it is easy to describe $M_2 = P_1 \cap P_1^{w_{p_1}}$. Factoring out $\bar{M}_2$ by the center of $\bar{G}$, we obtain $M_2 \simeq p^{15s} : (p^{10s} : (f \cdot ((\bar{A}_4(q) \times (q-1)/f) \times (q-1)/f') \cdot f))$, where $f' = f \cdot d'$ and $d' = (q-1, 3)$. So $n_2 = |P_1 : M_2| = |\bar{P}_1 : \bar{M}_2| = q \cdot (q^8 - 1)(q^3 + 1)/(q - 1)$.

Next, consider the action of $w_0 = w_{e_3 - e_7} \cdot w_{e_3 + e_7} = w_{e_3 + e_7} \cdot w_{e_3 - e_7}$ on Φ. Denote by Φ_0^+ a subset $\{e_3 \pm e_j |\ j = 4, \ldots, 7\}$ of Φ^+. Clearly, $|\Phi_0^+| = 8$. A diagram showing the action of w_0 on Φ is this:

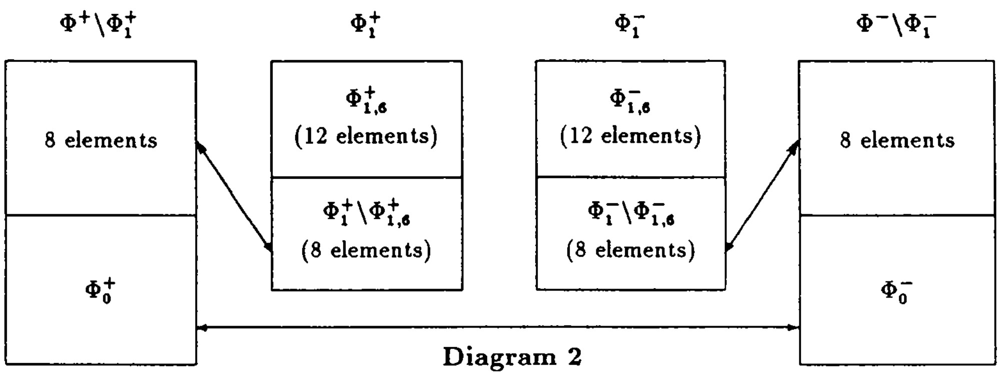


Diagram 2

It follows from the diagram that $\bar{U}_1 \cap \bar{P}_1^{w_0} \simeq p^{8s}$.

We consider the structure of $\bar{L}_1 \cap \bar{P}_1^{w_0}$. Let $\bar{L}'_{1,6} = \langle \bar{x}_r(t) |\ r \in \Phi_{1,6},\ t \in K \rangle$, $\bar{U}_{1,6} = \langle \bar{x}_r(t) |\ r \in \Phi_1^+ \backslash \Phi_{1,6}^+,\ t \in K \rangle$. The group $\bar{U}_{1,6} = U_{1,6}$ is isomorphic to p^{8s}, and $\bar{L}'_{1,6} \simeq \bar{D}_4(q)$. The center of $\bar{L}'_{1,6}$ is described as follows. For $d = (q-1, 2) = 2$, $z(\bar{L}'_{1,6}) = \langle \bar{h}_{p_2}(\mu)\bar{h}_{p_4}(\mu) \rangle \times \langle \bar{h}_{p_4}(\mu)\bar{h}_{p_6}(\mu) \rangle \simeq 2^2$, where μ is an element of order 2 in the multiplicative group of the field K. The element $\bar{h}_{p_2}(\mu)\bar{h}_{p_4}(\mu)$ lies in $\langle h_0 \rangle$. Since h_0 centralizes $\bar{L}'_{1,6}$, we have $\bar{L}_1 \cap \bar{P}_1^{w_0} \simeq p^{8s} : (d \cdot (d \cdot D_4(q) \times (q-1)/d) \cdot d) \cdot (q-1)$.

The center of $\bar{G}$ lies in $\langle h_0 \rangle$. Therefore, $M_3 = P_1 \cap P_1^{w_0} \simeq p^{8s} : (p^{8s} : (d \cdot (d \cdot D_4(q) \times (q-1)/c) \cdot d) \cdot (q-1))$, where $c = d \cdot d'$. The index $|P_1 : M_3| = |\bar{P}_1 : \bar{M}_3| = n_3$ is equal to $q^8(q^5-1)(q^4+1)/(q-1)$. Adding subdegrees $n_1 = 1$, n_2, and n_3 of the representation of G on the cosets w.r.t. P_1, we obtain $n_1 + n_2 + n_3 = n$. Hence, the rank of the representation equals 3.

Remark. A representation of G on the cosets w.r.t. P_6 is similar to the one above.

THEOREM 1. For simple non-Abelian groups $G = E_6(q)$, the parameters n, n_2, n_3, P, M_2, and M_3 of minimal permutation representations are given in the following list:

$n = \frac{(q^9-1)(q^8+q^4+1)}{q-1}$, $n_2 = \frac{(q^8-1)(q^3+1)}{q-1}$, $n_3 = q^8 \cdot \frac{(q^5-1)(q^4+1)}{q-1}$;

$P = p^{16s} : (e \cdot (D_5(q) \times (q-1)/e') \cdot e)$,

$M_2 = p^{15s} : (p^{10s} : (f \cdot ((\bar{A}_4(q) \times (q-1)/f) \times (q-1)/f') \cdot f))$,

$M_3 = p^{8s} : (p^{8s} : (d \cdot (d \cdot D_4(q) \times (q-1)/c) \cdot d) \cdot (q-1))$,

where $d = (q-1, 2)$, $d' = (q-1, 3)$, $e = (q-1, 4)$, $f = (q-1, 5)$, $e' = e \cdot d'$, $f' = f \cdot d'$, $c = d \cdot d'$.

The rank of the representation equals 3.

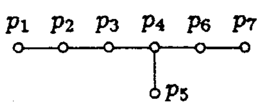
2. GROUP $E_7(q)$

A. Algebra E_7. The rank of E_7 equals 7. To determine a system of simple roots for E_7, we assume that it is embedded in E_8. Let $\mathcal{K}$ be a Cartan subalgebra in E_8 and $\mathcal{K}_{\mathbb{R}}$ be the corresponding Euclidean space with orthonormal basis $e_1, \ldots, e_8$. Then a system Π of simple roots for E_7 (as a subalgebra in E_8) is defined as follows: $p_1 = e_2 - e_3$, $p_2 = e_3 - e_4$, $p_3 = e_4 - e_5$, $p_4 = e_5 - e_6$, $p_5 = e_6 - e_7$, $p_6 = e_6 + e_7$, $p_7 = -\frac{1}{2} \sum_{i=1}^{8} e_i$.

The system of positive roots is

$$\Phi^+ = \left\{ \begin{array}{l} e_i \pm e_j, \ i < j, \ i = 2, \ldots, 6, \ j = 3, \ldots, 7; \\ -e_1 - e_8, \\ -\frac{1}{2} \sum_{i=1}^{8} \varepsilon_i e_i, \ \varepsilon_i = \pm 1, \ \varepsilon_1 = \varepsilon_8 = 1, \ \prod_{i=1}^{8} \varepsilon_i = 1 \end{array} \right\}, \quad |\Phi^+| = 63.$$

The matrix A has the form $\left(\begin{array}{c|ccc} 2 & -1 & 0 \ \ldots & 0 \\ \hline -1 & & & \\ 0 & & A' & \\ \vdots & & & \\ 0 & & & \end{array} \right)$, where A' coincides with a matrix A for E_6. The Dynkin diagram is



B. Group $E_7(q)$ and its parabolic subgroups of least index. If 2 does not divide $q-1$, then the group $G = E_7(q)$ coincides with the universal group $\bar{G} = \bar{E}_7(q)$. Otherwise it is isomorphic to the factor group $\bar{G}$ over the center $\bar{Z}$, which is a cyclic subgroup of order 2. Using Lemma 3 in [1] yields $\bar{Z} = \{\bar{h}_{p_1}(\mu)\bar{h}_{p_3}(\mu)\bar{h}_{p_5}(\mu) \mid \mu \in K^*, \ \mu^2 = 1\}$.

From the main result in [2] and Proposition 1 of [1], it follows that a subgroup of least index in G should be parabolic. Proposition 1 of [1] allows us to compute orders of maximal parabolic subgroups in G. Comparing these orders yields that, up to conjugation, the subgroup of least index in G is P_1. The corresponding subgroup $\bar{P}_1$ of $\bar{G}$, obviously, contains the center $\bar{Z}$. Thus

$|\bar{G}| = q^{63}(q^{18}-1)(q^{14}-1)(q^{12}-1)(q^{10}-1)(q^{8}-1)(q^{6}-1)(q^{2}-1)$,

$|\bar{P}_1| = q^{63}(q^{12}-1)(q^{9}-1)(q^{8}-1)(q^{6}-1)(q^{5}-1)(q^{2}-1)(q-1)$,

$|G| = |\bar{G}|/d$, $|P_1| = |\bar{P}_1|/d$, where $d=(q-1,2)$, $n = |G:P_1| = |\bar{G}:\bar{P}_1| = (q^{14}-1)(q^9+1)(q^5+1)/(q-1)$.

First, we describe the structure of the group $\bar{P}_1 = \bar{U}_1 \cdot \bar{L}_1$. For every element $r \in \Phi^+\backslash\Phi_1^+$, we have $r = p_1 + s_1$, where $s_1 \in \Phi_1^+$. Therefore, $\bar{U}_1 \simeq U_1 \simeq p^{27s}$. Let $\bar{H}_1 = \langle \bar{h}_{p_i}(\lambda) | \ i = 2,\dots,7;\ \lambda \in K^* \rangle$ and $\bar{L}_1' = \langle \bar{x}_r(t) | \ r \in \Phi_1,\ t \in K\rangle$. The group $\bar{L}_1'$ is isomorphic to the universal Chevalley group $\bar{E}_6(q)$, and $\bar{L}_1 = \bar{L}_1' \cdot \langle h\rangle$, where $h \in \bar{H}\backslash\bar{H}_1$. Let $h_0 = \bar{h}_{p_1}(\lambda^3)\bar{h}_{p_2}(\lambda^4)\bar{h}_{p_3}(\lambda^5)\bar{h}_{p_4}(\lambda^6)\bar{h}_{p_5}(\lambda^3)\bar{h}_{p_6}(\lambda^4)\bar{h}_{p_7}(\lambda^2)$ and $h_0(\mu) = \bar{h}_{p_2}(\mu)\bar{h}_{p_3}(\mu^2)\bar{h}_{p_6}(\mu)\bar{h}_{p_7}(\mu^2)$, where λ generates K^*, and $\mu^3 = 1$. Then $\bar{L}_1$ is isomorphic to an extension of the central product of groups $\bar{L}_1'$ and $\langle h_0\rangle$ over the subgroup $\langle h_0(\mu)\rangle$ by a cyclic group of order d', that is, $\bar{L}_1' \simeq d'\cdot(E_6(q)\times(q-1)/d')\cdot d'$, where $d' = (q-1,3)$.

Since the center of $\bar{G}$ is a subgroup of $\langle h_0\rangle$, we obtain $P_1 \simeq p^{27s} : (d'\cdot(E_6(q)\times(q-1)/c)\cdot d')$, where $c = d\cdot d'$, $d = (2, q-1)$.

C. Representation of G on cosets w.r.t. P_1. The element w_{p_1} acts on Φ as is shown in Diagram 1 [for the group $E_6(q)$], but the orders of Φ^+, Φ_1^+, and $\Phi_{1,2}^+$ are, of course, greater in this case. Namely, $|\Phi^+| = 63$, $|\Phi_1^+| = 36$, and $|\Phi_{1,2}^+| = 20$. Thus $\bar{U}_1 \cap \bar{P}_1^{w_{p_1}} \simeq p^{26s}$.

Let $\bar{L}_{1,2}' = \langle \bar{x}_r(t) | \ r \in \Phi_{1,2},\ t\in K\rangle \simeq \bar{D}_5(q)$ and $\bar{U}_{1,2} = \langle \bar{x}_r(t) | \ r \in \Phi_1^+\backslash\Phi_{1,2}^+,\ t \in K\rangle \simeq U_{1,2}$. Since the equality $r = p_2 + s_2$, where $s_2 \in \Phi_{1,2}^+$, holds for every element $r \in \Phi_1^+\backslash\Phi_{1,2}^+$, the group $U_{1,2}$ is Abelian. It is isomorphic to p^{16s}.

The elements h_0 and $\bar{h}_{p_1}(\lambda)$, where λ generates K^*, centralize $\bar{L}_{1,2}'$, and the element $z(\mu)\cdot\bar{h}_{p_1}(\mu)$, where $z(\mu) \in Z(\bar{L}_{1,2}')$ and $\mu^4 = 1$, lies in $\langle h_0\rangle$. Hence $\bar{L}_1 \cap \bar{P}_1^{w_{p_1}} \simeq p^{16s} : (e\cdot((\bar{D}_5(q)\times(q-1)/e)\times(q-1)/e)\cdot e)$, where $e = (q-1,4)$. Factoring out M_2 by the center of $\bar{G}$, we find

$$M_2 = P_1 \cap P_1^{w_{p_1}} \simeq p^{26s} : (p^{16s} : (c'\cdot((\bar{D}_5(q)\times(q-1)/e)\times(q-1)/e)\cdot e)),$$

where $c' = e/d$. Therefore, $|P_1 : M_2| = |\bar{P}_1 : \bar{M}_2| = n_2 = q\cdot(q^9-1)(q^8+q^4+1)/(q-1)$.

Let $w_0 = w_{e_2-e_7}w_{e_2+e_7} = w_{e_2+e_7}w_{e_2-e_7}$. Denote by Φ_0^+ a subset $\{e_2 \pm e_j | \ j = 3,\dots,7\}$ of Φ^+, and by u an element $-e_1 - e_8$ of Φ^+. A diagram showing the action of w_0 on Φ is this:

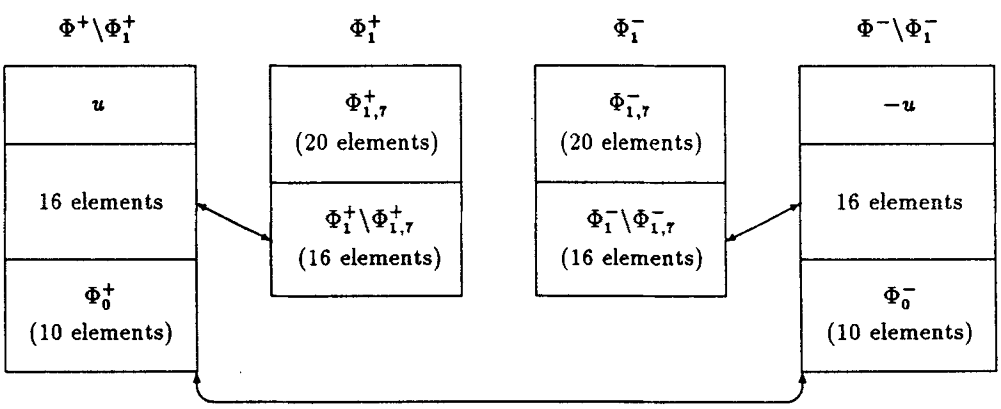


Diagram 3

It follows from the diagram that $\bar U_1 \cap \bar P_1^{w_0} \simeq p^{17s}$.

Let $\bar L'_{1,7} = \langle \bar x_r(t) |\ r \in \Phi_{1,7},\ t \in K\rangle$, $\bar U_{1,7} = \langle \bar x_r(t) |\ r \in \Phi_1^+ \backslash \Phi_{1,7}^+,\ t \in K\rangle \simeq U_{1,7}$. The group $U_{1,7}$ is isomorphic to p^{16s}, and $\bar L'_{1,7}$ is isomorphic to $\bar D_5(q)$. Following essentially the same argument as was used for M_2, we see that $M_3 = P_1 \cap P_1^{w_0} \simeq p^{17s}\colon (p^{16s}\colon (c' \cdot ((\bar D_5(q) \times (q-1)/e) \times (q-1)/e) \cdot e)$, where $c' = e/d$, $e = (q-1, 4)$, $d = (2, q-1)$. Thus $n_3 = |P_1 : M_3| = |\bar P_1 : \bar M_3| = q^{10}(q^9-1)(q^8+q^4+1)/(q-1)$.

Consider the action of an element $w_1 = w_0 \cdot w_4$ on Φ, where w_u is a reflection corresponding to the element u, defined above. A diagram depicting that action is the following:

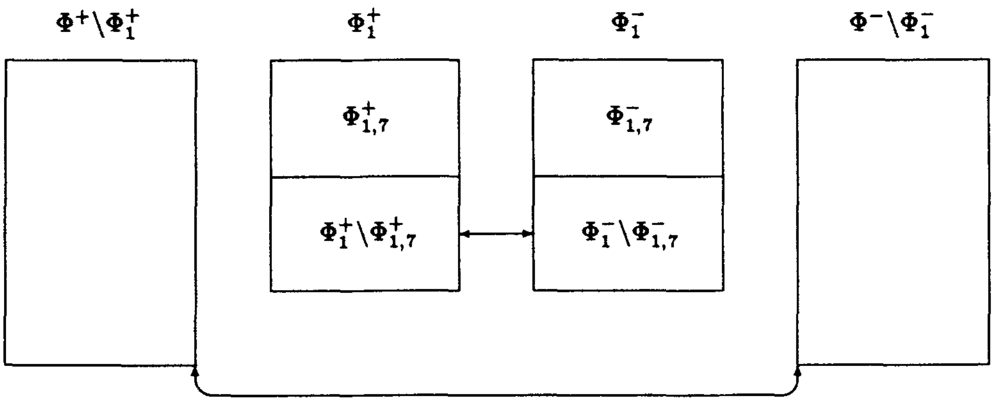


Diagram 4

Obviously, $M_4 = P_1 \cap P_1^{w_1} = L_1$. Therefore, $|P_1 : M_4| = q^{27s}$. We have $1 + |P_1 : M_2| + |P_1 : M_3| + |P_1 : M_4| = |G : P_1|$. Hence the rank of the representation equals 4.

THEOREM 2. For simple non-Abelian groups $G = E_7(q)$, the parameters n, n_2, n_3, n_4, P, M_2, M_3, and M_4 of minimal permutation-representations are given in the following list:

$n = \frac{(q^{14}-1)(q^9+1)(q^5+1)}{q-1}$, $n_2 = q \cdot \frac{(q^9-1)(q^8+q^4+1)}{q-1}$, $n_3 = q^{10} \cdot \frac{(q^9-1)(q^8+q^4+1)}{q-1}$, $n_4 = q^{27}$,

$P = p^{27s} : (d' \cdot (E_6(q) \times (q-1)/c) \cdot d')$,

$M_2 = p^{26s} : (p^{16s} : (c' \cdot ((\bar D_5(q) \times (q-1)/e) \times (q-1)/e) \cdot e))$,

$M_3 = p^{17s} : (p^{16s} : (c' \cdot ((\bar D_5(q) \times (q-1)/e) \times (q-1)/e) \cdot e))$,

$M_4 = d' \cdot (E_6(q) \times (q-1)/c) \cdot d'$,

where $d = (q-1, 2)$, $d' = (q-1, 3)$, $e = (q-1, 4)$, $c' = e/d$, $c = d \cdot d'$.

The rank of the representation equals 4.

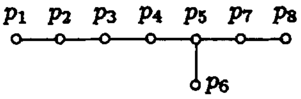
3. GROUP $E_8(q)$

A. Algebra E_8. The rank of E_8 equals 8. If $e_1, \ldots, e_8$ is an orthonormal basis of the Euclidean space $\mathcal{K}_{\mathbb{R}}$, where $\mathcal{K}$ is a Cartan subalgebra of E_8, then a system Π of simple roots for E_8 is defined as follows: $p_1 = e_1 - e_2$, $p_2 = e_2 - e_3$, $p_3 = e_3 - e_4$, $p_4 = e_4 - e_5$, $p_5 = e_5 - e_6$, $p_6 = e_6 - e_7$, $p_7 = e_6 + e_7$, $p_8 = -\frac{1}{2}\sum_{i=1}^{8} e_i$.

The system of positive roots is

$$\Phi^+ = \left\{ \begin{array}{l} e_i \pm e_j,\ i < j,\ i = 1, \ldots, 6,\ j = 2, \ldots, 7; \\ - \pm e_i - e_8,\ i = 1, \ldots, 7; \\ -\frac{1}{2}\sum_{i=1}^{8} \varepsilon_i e_i,\ \varepsilon_i = \pm 1,\ \varepsilon_8 = 1,\ \prod_{i=1}^{8} \varepsilon_i = 1 \end{array} \right\},\quad |\Phi^+| = 120.$$

The matrix A has the form $\left(\begin{array}{c|ccc} 2 & -1 & 0 \;\ldots & 0 \\ \hline -1 & & & \\ 0 & & A' & \\ \vdots & & & \\ 0 & & & \end{array}\right)$, where A' coincides with a matrix A for E_7. The Dynkin diagram is



B. Group $E_8(q)$ and its parabolic subgroups of least index. The universal group $\bar{E}_8(q)$ coincides with the adjoint group $G = E_8(q)$. From the main result in [2] and Proposition 1 of [1], it follows that a subgroup of least index in G should be parabolic. Proposition 1 of [1] allows us to compute orders of maximal parabolic subgroups of G. Comparing these orders, we see that, up to conjugation, the subgroup of least index in G is P_1. Thus

$|G| = q^{120}(q^{30}-1)(q^{24}-1)(q^{20}-1)(q^{18}-1)(q^{14}-1)(q^{12}-1)(q^8-1)(q^2-1)$,

$|P_1| = q^{120}(q^{18}-1)(q^{14}-1)(q^{12}-1)(q^{10}-1)(q^8-1)(q^6-1)(q^2-1)(q-1)$,

$|G : P_1| = (q^{30}-1)(q^{12}+1)(q^{10}+1)(q^6+1)/(q-1)$.

In view of the Levi decomposition, $P_1 = U_1 \cdot L_1$. For all but one $r \in \Phi^+\backslash\Phi_1^+$, we have $r = p_1 + s_1$, where $s_1 \in \Phi_1^+$. The excepted element is $u = e_1 - e_8 = 2p_1 + 3p_2 + 4p_3 + 5p_4 + 6p_5 + 3p_6 + 4p_7 + 2p_8$. For every r (except u of course) of $\Phi^+\backslash\Phi_1^+$, there exists an element $r' \in \Phi^+\backslash\Phi_1^+$ such that $r + r' = u$. Therefore, Lemma 1 in [1] (Chevalley commutator formula) implies that U_1 is isomorphic to $p^s \cdot p^{56s}$.

Let $L_1' = \langle X_r \mid r \in \Phi_1\rangle \simeq \bar{E}_7(q)$. The element $h_u = h_u(\lambda) = h_{p_1}(\lambda^2)h_{p_2}(\lambda^3)h_{p_3}(\lambda^4)h_{p_4}(\lambda^5)h_{p_5}(\lambda^6) \times h_{p_6}(\lambda^3)h_{p_7}(\lambda^4)h_{p_8}(\lambda^2)$, where λ generates K^*, centralizes L_1' since $(u, p_i) = 0$ for every $i = 2, \ldots, 8$. Furthermore, the center of the subgroup L_1' is in $\langle h_u\rangle$. Hence $L_1 \simeq d{\cdot}(E_7(q) \times (q-1)/d){\cdot}d$, where $d = (q-1, 2)$.

C. Representation of G on cosets w.r.t. P_1. The element w_{p_1} acts on Φ as is shown in Diagram 1 (see Sec. 1.C). Therefore, $U_1 \cap P_1^{w_{p_1}} \simeq p^s \cdot p^{54s} \times p^s$. Let $L_{1,2}' = \langle X_r \mid r \in \Phi_{1,2}\rangle \simeq \bar{E}_6(q)$ and $U_{1,2} = \langle X_r \mid r \in \Phi_1^+\backslash\Phi_{1,2}^+\rangle$. It is clear that $U_{1,2} \simeq p^{27s}$.

The elements h_u and $h_{p_1}(\lambda)$, where λ generates K^*, centralize $L_{1,2}'$ and so does the element $h_u(\lambda)h_{p_1}(\lambda^{-2})$. Furthermore, $Z(L_{1,2}') \le \langle h_u h_{p_1}(\lambda^{-2})\rangle$. Hence $L_1 \cap P_1^{w_{p_1}} \simeq p^{27s} : (d' \cdot (E_6(q) \times (q-1)/d') \times (q-1)) \cdot d'$.

Thus $M_2 = P_1 \cap P_1^{w_{p_1}} \simeq (p^s \cdot p^{54s} \times p^s) : (p^{27s} : (d' \cdot (E_6(q) \times (q-1)/d') \times (q-1)) \cdot d')$, where $d' = (q-1, 3)$, and $|P_1 : M_2| = q \cdot (q^{14}-1)(q^9+1)(q^5+1)/(q-1)$.

We know that $(u, p_i) = 0$ for every $i = 2, \ldots, 8$. Therefore, w_u acts on Φ as follows: $w_u(\Phi_1) = \Phi_1$, $w_u(\Phi^+\backslash\Phi_1^+) = \Phi^-\backslash\Phi_1^-$, $w_u(\Phi^-\backslash\Phi_1^-) = \Phi^+\backslash\Phi_1^+$. Hence $P_1 \cap P_1^{w_u} = M_3 = L_1$. Thus $|P_1 : M_3| = q^{57s}$.

Consider an element $w_0 = w_u \cdot w_{p_1}$. The following diagram reflects the action of w_0 on Φ:

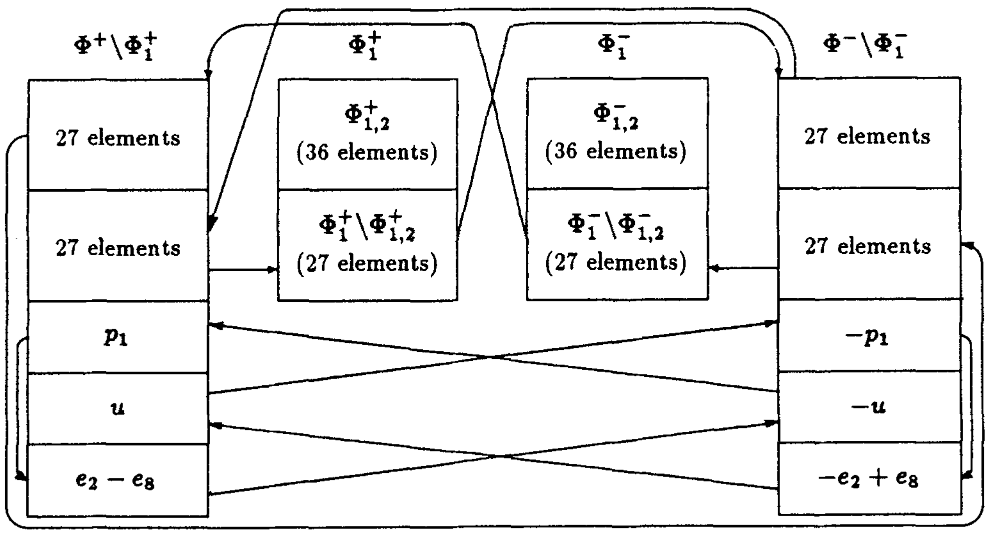


Diagram 5

In this way $U_1 \cap P_1^{w_0} \simeq p^{28s}$ and $L_1 \cap P_1^{w_0} = L_{1,2} = L_1 \cap P_1^{w_{p_1}}$. Therefore, $M_4 \simeq p^{28s} : (p^{27s} : (d' \cdot (E_6(q) \times (q-1)/d') \times (q-1)) \cdot d')$. The index of M_4 in P_1 equals $q^{29} \cdot (q^{14}-1)(q^9+1)(q^5+1)/(q-1)$.

Now consider an element $w_1 = w_{e_1-e_7} w_{e_1+e_7}$, whose action on Φ is shown in the following diagram:

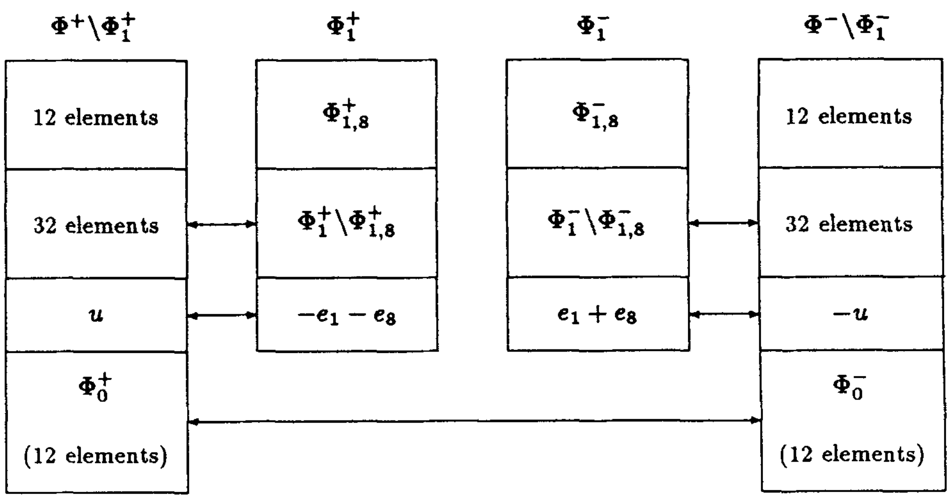


Diagram 6

Here Φ_0^+ denotes the set $\{e_1 \pm e_j \mid j = 2, \ldots, 7\}$. Thus $U_1 \cap P_1^{w_1} \simeq p^s \cdot p^{32s} \times p^{12s}$. Let $L'_{1,8} = \langle X_r \mid r \in \Phi_{1,8}\rangle \simeq \bar{D}_6(q)$ and $U_{1,8} = \langle X_r \mid r \in \Phi_1^+\backslash\Phi_{1,8}^+\rangle \simeq p^{33s}$. Since h_u centralizes $L'_{1,8}$, we obtain $M_5 = P_1 \cap P_1^{w_1} \simeq (p^s \cdot p^{32s} \times p^{12s}) : (p^{33s} : (d \cdot (d \cdot D_6(q) \times (q-1)/d) \cdot d) \cdot (q-1))$, where $d = (q-1, 2)$, and $|P_1 : M_5| = q^{12}(q^{14}-1)(q^{12}+q^6+1)(q^8+q^4+1)/(q-1)$.

We have $|G : P_1| = 1 + \sum_{i=2}^{5} |P_1 : M_i|$. Hence, the rank of G's representation on the cosets w.r.t. P_1 equals 5.

THEOREM 3. For simple non-Abelian groups $G = E_8(q)$, the parameters n, n_2, n_3, n_4, n_5, P, M_2, M_3, M_4, and M_5 of minimal permutation representations are given in the following list:

$n = \frac{(q^{30}-1)(q^{12}+1)(q^{10}+1)(q^{6}+1)}{q-1}$, $n_2 = q \cdot \frac{(q^{14}-1)(q^{9}+1)(q^{5}+1)}{q-1}$, $n_3 = q^{57}$, $n_4 = q^{29} \cdot \frac{(q^{14}-1)(q^{9}+1)(q^{5}+1)}{q-1}$,
$n_5 = q^{12} \cdot \frac{(q^{14}-1)(q^{12}+q^{6}+1)(q^{8}+q^{4}+1)}{q-1}$;

$P = (p^{s} \cdot p^{56s}) : (d \cdot (E_7(q) \times (q-1)/d) \cdot d)$,

$M_2 = (p^{s} \cdot p^{54s} \times p^{s}) : (p^{27s} : (d' \cdot (E_6(q) \times (q-1)/d') \times (q-1)) \cdot d')$,

$M_3 = d \cdot (E_7(q) \times (q-1)/d) \cdot d$,

$M_4 = p^{28s} : (p^{27s} : (d' \cdot (E_6(q) \times (q-1)/d') \times (q-1)) \cdot d')$,

$M_5 = (p^{s} \cdot p^{32s} \times p^{12s}) : (p^{33s} : (d \cdot (d \cdot D_6(q) \times (q-1)/d) \cdot d) \cdot (q-1))$,

where $d = (q-1, 2)$, $d' = (q-1, 3)$.

The rank of the representation equals 5.

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