MINIMAL PERMUTATION REPRESENTATIONS OF FINITE SIMPLE EXCEPTIONAL GROUPS OF TYPES E₆, E₇, AND E₈

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A minimal permutation representation of a group is its faithful permutation representation of least degree. We will find degrees and point stabilizers, as well as ranks, subdegrees, and double stabilizers, for groups of types E₆, E₇, and E₈. This brings to a close the study of minimal permutation representations of finite simple Chevalley groups.

This paper continues [1], where minimal permutation representations of finite simple exceptional groups of types G₂ and F₄ were described. Our present goal is to obtain a similar description for minimal permutation representations of groups E₆, E₇, and E₈. We follow the notation and terminology developed in [1], in which the reader can find all necessary preliminary information and a complete list of references.

1. GROUP E₆(q)

A. Algebra E₆. The rank of E₆ equals 6. Obviously, E₆ is a subalgebra of E₈. If K is a Cartan subalgebra in E₈, then K is an Euclidean space of dimension 8. Let e₁, ..., e₈ be an orthonormal basis of K. Then a system II of simple roots for E₆ (as an algebra of E₈) is defined as follows: p₁ = e₃ - e₄, p₂ = e₄ - e₅, p₃ = e₅ - e₆, p₄ = e₆ - e₇, p₅ = e₆ + e₇, p₆ = -½ ∑ᵢ=₁⁸ eᵢ.

The system of positive roots is

\[ \Phi^+ = \left\{ e_i \pm e_j, \ i < j, \ i = 3, \ldots, 6, \ j = 4, \ldots, 7; \right\} \]

\[ -\frac{1}{2} \sum_{i=1}^{8} e_i \epsilon_i, \ e_1 = \pm 1, \ e_2 = e_3 = 1, \ \prod_{i=1}^{8} e_i = 1 \]

\[ |\Phi^+| = 36. \]

The matrix A has the form

\[
\begin{pmatrix}
2 & -1 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & -1 & 0 \\
0 & 0 & -1 & 2 & 0 & 0 \\
0 & 0 & -1 & 0 & 2 & -1 \\
0 & 0 & 0 & 0 & -1 & 2
\end{pmatrix}
\]

The Dynkin diagram is

\[ \text{P}_1 - \text{P}_2 \text{P}_3 \text{P}_5 \text{P}_6 \]

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B. Group \(E_6(q)\) and its parabolic subgroups of least index. The order of a field \(K\) equals \(q = p^r\), where \(p\) is a prime. If 3 does not divide \(q - 1\), then the group \(G = E_6(q)\) coincides with the universal group \(\tilde{G} = E_6(q)\). Otherwise it is isomorphic to the factor group \(\tilde{G}\) w.r.t. the center \(\tilde{Z}\), which is a cyclic subgroup of order 3. Using Lemma 3 of [1], it is easy to verify that

\[
\tilde{G} = \{h_{p_1}(\mu) \cdot h_{p_2}(\mu^2) \cdot h_{p_3}(\mu) \cdot h_{p_4}(\mu^2) | \mu \in K^*, \mu^2 = 1\}.
\]

From the main result stated in [2] and Proposition 1 of [1], it follows that a subgroup of least index in \(G\) should be parabolic. Proposition 1 of [1] allows us to compute orders of maximal parabolic subgroups in \(G\). Comparing these orders, we see that there are, up to conjugation, two subgroups of least index in \(G\): \(P_1 = P_{\Pi_1(p_1)}\) and \(P_6 = P_{\Pi_1(p_6)}\) (for definitions, see [1]), which are conjugate in \(\text{Aut} G\).

Elements \(z_\tau(t)\) of \(\tilde{G}\), as well as \(\bar{s}_\tau(t)\) and \(h_\tau(t)\), were defined in Lemma 3 of [1]. Let \(S\) be a subgroup of \(G\) generated by the elements \(z_\tau(t), \ldots, z_\tau(t_k)\). Denote by \(\tilde{S}\) a subgroup of \(\tilde{G}\) generated by \(z_\tau(t_1), \ldots, z_\tau(t_k)\).

The group \(\tilde{P}_1\), like \(\tilde{P}_6\), includes a subgroup \(\tilde{H}\), and hence also \(\tilde{Z}\). Thus

\[
|\tilde{G}| = q^{36}(q^{12} - 1)(q^9 - 1)(q^8 - 1)(q^6 - 1)(q^5 - 1)(q^4 - 1)(q^2 - 1),
\]

\[
|\tilde{P}_1| = |\tilde{P}_6| = q^{36}(q^8 - 1)(q^6 - 1)(q^5 - 1)(q^4 - 1)(q^2 - 1)(q - 1),
\]

\[
|G| = |\tilde{G}|/d', |P_1| = |P_6| = |\tilde{P}_1|/d', \text{ where } d' = (3, q - 1),
\]

\[
n = |G : P_1| = |G : P_6| = |\tilde{G} : \tilde{P}_1| = (q^9 - 1)(q^8 + q^4 + 1)/(q - 1).
\]

First, we describe a structure of the subgroup \(\tilde{P}_1\) in the universal group \(\tilde{G}\), using the Levi decomposition; see [1, Lemma 5]. Obviously, \(\tilde{P}_1 = \tilde{U}_1 \cdot \tilde{L}_1\) (\(U_1\) coincides with \(\tilde{U}_1\)). For every element \(r \in \Phi^+ \setminus \Phi_1^+\), we have \(r = s_1 + s_1\), where \(s_1 \in \Phi_1^+\). Therefore, \(U_1\) is an elementary Abelian \(p\)-group of order \(q^{16}\).

Let \(\tilde{H}_1 = \langle h_{p_i}(\lambda) | i = 2, \ldots, 6; \lambda \in K^* \rangle\) and \(\tilde{L}_1 = \langle z_\tau(t) | r \in \Phi_1, t \in K \rangle\). The group \(\tilde{L}_1\) is isomorphic to the universal Chevalley group \(D_8(q)\), and \(\tilde{L}_1 = \tilde{L}_1 \cdot \langle h \rangle\), where \(h \in \tilde{H} \setminus \tilde{H}_1\). Let \(h_0 = h_{p_1}(\lambda^4)h_{p_2}(\lambda^4)h_{p_3}(\lambda^4)h_{p_4}(\lambda^4)h_{p_5}(\lambda^4)\), where \(\lambda\) generates \(K^*\); \(h_0(\mu) = h_{p_2}(\mu)h_{p_3}(\mu)h_{p_4}(\mu)^2\), where \(|\mu| = 4\) for \((4, q - 1) = 4\), or \(h_0(\mu) = h_{p_2}(\mu)h_{p_3}(\mu)\), where \(|\mu| = 2\) for \((q - 1, 4) = 2\), and \(h_0(\mu) = 1\) for even \(q\). The element \(h_0\) centralizes \(\tilde{L}_1\), and \(\langle h_0 \rangle \cap \tilde{L}_1 = \langle h_0(\mu) \rangle\). Hence, the group \(\tilde{L}_1\) is isomorphic to an extension of the central product of groups \(\tilde{L}_1\) and \(\langle h_0(\mu) \rangle\) over the subgroup \(\langle h_0(\mu) \rangle\) by a cyclic group of order \(e = (q - 1, 4)\), that is, \(\tilde{L}_1 \simeq e \cdot (D_8(q) \times (q - 1)/e)' \cdot e\). Since the center of \(\tilde{G}\) lies in \(\langle h_0 \rangle\), we obtain \(P_1 \simeq p_1^{16e} : (e \cdot (D_8(q) \times (q - 1)/e') \cdot e)\), where \(e' = ed'\) and \(d' = (q - 1, 3)\).

C. Representation of \(G\) on cosets w.r.t. \(P_1\). Our goal is to define double stabilizers of the representation of \(G\) on the cosets w.r.t. \(P_1\), that is, groups of the form \(P_1 \cap P_1^e\). Therefore, we need to choose appropriate elements \(x\) in a way that these do not map into \(W_1\) under the natural homomorphism \(\varphi: N \to W\). Since \(H \leq P_1\), the action of an element \(n \in N\) on \(P_1\) is determined by the action of its image \(w \in W\) on \(\Phi\), and so below an element \(n\) will be identified with its image \(w\). We need to adopt the following notation:

\[
\Phi_1^+ = \Phi_1 \cap \Phi_2^+ \quad \text{and} \quad \Phi_1^+ = \Phi_1 \cap \Phi_6.
\]

The action of \(w_{P_1}\) on \(\Phi\) is shown in the following:
An arrow pointing from set \(X\) to set \(Y\) says that \(w_{p_1}(X) = Y\); the absence of an arrow outgoing from \(Z\) indicates that \(w_{p_1}(Z) = Z\).

We start by determining the structure of a double stabiliser \(\tilde{M}_2 = \tilde{P}_1 \cap \tilde{P}_1^{w_{p_1}} = (\tilde{U}_1 \cap \tilde{P}_1^{w_{p_1}}); (\tilde{L}_1 \cap \tilde{P}_1^{w_{p_1}})\). From Diagram 1, it follows that \(\tilde{U}_1 \cap \tilde{P}_1^{w_{p_1}} \simeq p^{15}\). Let \(\tilde{L}_{1,2} = \langle \tilde{z}(t) \rangle \rightarrow \tilde{P}_1, t \in K \rangle \simeq \tilde{A}_4(q) \simeq SL_5(q), \tilde{U}_{1,2} = \langle \tilde{z}(t) \rangle \rightarrow \tilde{P}_1, t \in K \rangle \simeq U_{1,2} \). The group \(U_{1,2}\) is Abelian, as is \(U_1\). Therefore, \(U_{1,2} \simeq p^{10}\).

The center of \(\tilde{L}_{1,2}\) is generated by the element

\[z(\mu) = \tilde{h}_{P_1}(\mu) \tilde{h}_{P_1}(\mu^3) \tilde{h}_{P_1}(\mu^4) \tilde{h}_{P_1}(\mu^2),\]

where \(\mu^5 = 1\). Its order is equal to \(f = (q - 1, 5)\). The elements \(h_0\) and \(\tilde{h}_{P_1}(\lambda)\) centralise \(\tilde{L}_{1,2}\), and the element \(z(\mu)\tilde{h}_{P_1}(\mu)\) lies in \(\langle h_0 \rangle\). Hence \(\tilde{L}_1 \cap \tilde{P}_1^{w_{p_1}} \simeq p^{10}\times (f \cdot (\tilde{A}_4(q) \times (q - 1)/f) \cdot (q - 1)/f) \cdot f)\). Now it is easy to describe \(M_2 = \tilde{P}_1 \cap \tilde{P}_1^{w_{p_1}}\). Factoring out \(\tilde{M}_2\) by the center of \(\tilde{G}_1\), we obtain \(M_2 \simeq p^{15}\times (f \cdot (\tilde{A}_4(q) \times (q - 1)/f) \cdot (q - 1)/f) \cdot f)\), where \(f' = f \cdot d'\) and \(d' = (q - 1, 3)\). So \(n_2 = |P_1 : M_2| = |P_1 : \tilde{P}_1| = q \cdot (q - 1)(q^2 + 1)/(q - 1)\).

Next, consider the action of \(w_0 = w_{e_2 - e_7} \cdot w_{e_8 - e_7} = w_{e_8} - e_7 \cdot w_{e_8}, \) on \(\Phi\). Denote by \(\Phi^+\) a subset \(\{e_3 \pm e_j \mid j = 4, \ldots, 7\}\) of \(\Phi^+\). Clearly, \(|\Phi^+_0| = 8\). A diagram showing the action of \(w_0\) on \(\Phi\) is this:

\[\Phi^+ \setminus \Phi^+_0 \quad \Phi^+_1 \quad \Phi^- \quad \Phi^- \setminus \Phi^-_0\]

\[\Phi^+_1 \quad \Phi^-_1 \quad \Phi^+_0 \quad \Phi^-_0\]

It follows from the diagram that \(\tilde{U}_1 \cap \tilde{P}_1^{w_{p_0}} \simeq p^{8}\).

We consider the structure of \(\tilde{L}_1 \cap \tilde{P}_1^{w_{p_0}}\). Let \(\tilde{L}_{1,6} = \langle \tilde{z}(t) \rangle \rightarrow \tilde{P}_1, t \in K \rangle, \tilde{U}_{1,6} = \langle \tilde{z}(t) \rangle \rightarrow \Phi^+_1 \setminus \Phi^+_1, t \in K \rangle\). The group \(\tilde{U}_{1,6} = U_{1,6}\) is isomorphic to \(p^{8}\), and \(\tilde{L}_{1,6} \simeq \tilde{D}_4(q)\). The center of \(\tilde{L}_{1,6}\) is described as follows. For \(d = (q - 1, 2) = 2\), \(z(\tilde{L}_{1,6}) = (\tilde{h}_{P_1}(\mu) \tilde{h}_{P_1}(\mu)) \times (\tilde{h}_{P_1}(\mu) \tilde{h}_{P_1}(\mu)) \simeq 2^2\), where \(\mu\) is an element of order 2 in the multiplicative group of the field \(K\). The element \(\tilde{h}_{P_1}(\mu) \tilde{h}_{P_1}(\mu)\) lies in \(\langle h_0 \rangle\). Since \(h_0\) centralises \(\tilde{L}_{1,6}\), we have \(\tilde{L}_1 \cap \tilde{P}_1^{w_{p_0}} \simeq p^{8}: (d \cdot (d \cdot D_4(q) \times (q - 1)/d) \cdot d) \cdot (q - 1)\).
The center of \( \hat{G} \) lies in \( \langle h_0 \rangle \). Therefore, \( M_3 = P_1 \cap P_1^{\infty} \simeq p^{8^2} : (p^{8^2} : (d \cdot (d \cdot D_4(q) \times (q-1)/c) \cdot d) \cdot (q-1)) \), where \( c = d \cdot d' \). The index \( |P_1 : M_3| = |\hat{P}_1 : \hat{M}_3| = n_3 \) is equal to \( q^8(q^9 - 1)/(q^6 + 1)/(q - 1) \). Adding subdegrees \( n_1 = 1, n_2, \) and \( n_3 \) of the representation of \( G \) on the cosets w.r.t. \( P_1 \), we obtain \( n_1 + n_2 + n_3 = n \). Hence, the rank of the representation equals 3.

Remark. A representation of \( G \) on the cosets w.r.t. \( P_6 \) is similar to the one above.

THEOREM 1. For simple non-Abelian groups \( G = E_6(q) \), the parameters \( n, n_2, n_3, P, M_2, \) and \( M_3 \) of minimal permutation representations are given in the following list:

\[
\begin{align*}
\lambda & = (q^2 - 1)(q^4 - 1)(q^6 - 1),
\lambda_2 = (q^8 - 1)(q^8 - 1),
\lambda_3 = q^8(q^9 - 1)/(q^6 + 1)/(q - 1),
\mu & = (e \cdot (D_4(q) \times (q - 1)/e') \cdot e),
\mu_2 = (q^{10} : (f \cdot ((A_4(q) \times (q - 1)/f') \times (q - 1)/f') \cdot f)),
\mu_3 = p^{16^2} : (d \cdot (d \cdot D_4(q) \times (q - 1)/d') \cdot d) \cdot (q - 1)),
\end{align*}
\]

where \( d = (q - 1, 2), d' = (q - 1, 3), e = (q - 1, 4), f = (q - 1, 5), e' = e \cdot d', f' = f \cdot d', c = d \cdot d' \).

The rank of the representation equals 3.

2. GROUP \( E_7(q) \)

A. Algebra \( E_7 \). The rank of \( E_7 \) equals 7. To determine a system of simple roots for \( E_7 \), we assume that it is embedded in \( E_8 \). Let \( K \) be a Cartan subalgebra in \( E_8 \) and \( K_B \) be the corresponding Euclidean space with orthonormal basis \( e_1, \ldots, e_8 \). Then a system \( \Pi \) of simple roots for \( E_7 \) (as a subalgebra in \( E_8 \)) is defined as follows: \( p_1 = e_2 - e_3, p_2 = e_3 - e_4, p_3 = e_4 - e_5, p_4 = e_5 - e_6, p_5 = e_6 - e_7, p_6 = e_6 + e_7, p_7 = -\frac{1}{2} \sum_{i=1}^{8} e_i \).

The system of positive roots is

\[
\Phi^+ = \left\{ e_i \pm e_j, \quad i < j, \quad i = 2, \ldots, 6, \quad j = 3, \ldots, 7; \right. \\
\left. -e_1 - e_8, \\
-\frac{1}{2} \sum_{i=1}^{8} e_i e_i, \quad e_i = \pm 1, \quad e_1 = e_8 = 1, \quad \prod_{i=1}^{8} e_i = 1 \right\}, \quad |\Phi^+| = 63.
\]

The matrix \( A \) has the form

\[
\begin{pmatrix}
2 & -1 & 0 & \cdots & 0 \\
-1 & 0 & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}
\]

where \( A' \) coincides with a matrix \( A \) for \( E_6 \). The Dynkin diagram is

\[
\begin{array}{cccccccc}
P_1 & P_2 & P_3 & P_4 & P_5 & P_6 & P_7 \\
\end{array}
\]

B. Group \( E_7(q) \) and its parabolic subgroups of least index. If \( 2 \) does not divide \( q - 1 \), then the group \( G = E_7(q) \) coincides with the universal group \( \hat{G} = E_7(q) \). Otherwise it is isomorphic to the factor group \( \hat{G} \) over the center \( \hat{Z} \), which is a cyclic subgroup of order 2. Using Lemma 3 in [1] yields \( \hat{Z} = \{ \mu \hat{h}_{p_1}(\mu)\hat{h}_{p_2}(\mu)\hat{h}_{p_5}(\mu) | \mu \in K^*, \mu^2 = 1 \} \).
From the main result in [2] and Proposition 1 of [1], it follows that a subgroup of least index in $G$ should be parabolic. Proposition 1 of [1] allows us to compute orders of maximal parabolic subgroups in $G$. Comparing these orders yields that, up to conjugation, the subgroup of least index in $G$ is $P_1$. The corresponding subgroup $P_1$ of $G$, obviously, contains the center $Z$. Thus

$$|G| = q^{63}(q^{18} - 1)(q^{14} - 1)(q^{10} - 1)(q^8 - 1)(q^6 - 1)(q^2 - 1),$$

$$|P_1| = q^{63}(q^{18} - 1)(q^{10} - 1)(q^8 - 1)(q^6 - 1)(q^2 - 1)(q - 1),$$

$$|G| / |P_1| = d, \text{ where } d = (q - 1, 2),$$

$$n = |G : P_1| = |G : P_1| = (q^{14} - 1)(q^9 + 1)(q^5 + 1)/(q - 1).$$

First, we describe the structure of the group $P_1 = U_1 \cdot L_1$. For every element $r \in \Phi^+ \setminus \Phi^+_1$, we have $r = p_1 + s_1$, where $s_1 \in \Phi^+_1$. Therefore, $U_1 \simeq U_1 \simeq p^{27q}$. Let $\tilde{H}_1 = \langle \tilde{h}_{p_1}(\lambda) | \ i = 2, \ldots, 7; \ \lambda \in K^* \rangle$ and $L'_1 = \langle \tilde{x}_r(t) | r \in \Phi_1, t \in K \rangle$. The group $L'_1$ is isomorphic to the universal Chevalley group $E_6(q)$, and $L_1 = L_1 \cdot (h)$, where $h \in \tilde{H} \setminus \tilde{H}_1$. Let $h_0 = \tilde{h}_{p_1}(\lambda^3)\tilde{h}_{p_2}(\lambda^4)\tilde{h}_{p_6}(\lambda^5)\tilde{h}_{p_9}(\lambda^6)\tilde{h}_{p_{14}}(\lambda^7)\tilde{h}_{p_1}(\lambda^2)$ and $h_0(\mu) = h_{p_4}(\mu)h_{p_5}(\mu^2)h_{p_6}(\mu)h_{p_7}(\mu^2)$, where $\lambda$ generates $K^*$, and $\mu^3 = 1$. Then $\tilde{L}_1$ is isomorphic to an extension of the central product of groups $L'_1$ and $(h_0)$ over the subgroup $(h_0(\mu))$ by a cyclic group of order $d'$, that is, $\tilde{L}_1 \simeq d' \cdot (E_6(q) \times (q - 1)/d') \cdot d'$, where $d' = (q - 1, 3)$.

Since the center of $\tilde{G}$ is a subgroup of $(h_0)$, we obtain $P_1 \simeq p^{27q} : (d' \cdot (E_6(q) \times (q - 1)/c) \cdot d')$, where $c = d \cdot d', d = (2, q - 1)$.

C. Representation of $G$ on cosets w.r.t. $P_1$. The element $w_{p_1}$ acts on $\Phi$ as is shown in Diagram 1 [for the group $E_6(q)$], but the orders of $\Phi^+$, $\Phi^+_1$, and $\Phi^+_1 \setminus \Phi^+_1$ are, of course, greater in this case. Namely, $|\Phi^+| = 63$, $|\Phi^+_1| = 36$, and $|\Phi^+_1 \setminus \Phi^+_1| = 20$. Thus $U_1 \cap P_1^{w_{p_1}} \simeq p^{26q}$.

Let $L'_1,2 = \langle \tilde{x}_r(t) | r \in \Phi_1, t \in K \rangle \simeq D_5(q)$ and $U_1,2 = \langle \tilde{x}_r(t) | r \in \Phi_1 \setminus \Phi_1, t \in K \rangle \simeq U_1,2$. Since the equality $r = p_2 + s_2$, where $s_2 \in \Phi^+_1 \setminus \Phi^+_1$, holds for every element $r \in \Phi^+_1 \setminus \Phi^+_1$, the group $U_1,2$ is Abelian. It is isomorphic to $P_6^q$.

The elements $h_0$ and $\tilde{h}_{p_1}(\lambda)$, where $\lambda$ generates $K^*$, centralize $L'_1,2$, and the element $z(\mu) \cdot \tilde{h}_{p_1}(\mu)$, where $z(\mu) \in Z(L'_1,2)$ and $\mu^4 = 1$, lies in $(h_0)$. Hence $\tilde{L}_1 \cap P_1^{w_{p_1}} \simeq p^{16q} : (e \cdot ((D_5(q) \times (q - 1)/e) \times (q - 1)/e) \cdot e)$, where $e = (q - 1, 4)$. Factoring out $M_2$ by the center of $\tilde{G}$, we find

$$M_2 = P_1 \cap P_1^{w_{p_1}} \simeq p^{26q} : (p^{16q} : (c' \cdot ((D_5(q) \times (q - 1)/e) \times (q - 1)/e) \cdot e)), \text{ where } c' = e/d.$$
It follows from the diagram that $U_1 \cap P_{17} \simeq p^{17}$. Let $\bar{L}_{1,7} = \langle \bar{e}_r(t) \mid r \in \Phi_1, t \in K \rangle$, $\bar{U}_{1,7} = \langle \bar{e}_r(t) \mid r \in \Phi_1^+ \Phi_1^-, t \in K \rangle \simeq U_{1,7}$. The group $U_{1,7}$ is isomorphic to $p^{16s}$, and $\bar{L}_{1,7}$ is isomorphic to $\bar{D}_s(q)$. Following essentially the same argument as was used for $M_2$, we see that $M_3 = P_1 \cap P_{17} \simeq p^{17s} : (p^{16s} \cdot (c' \cdot ((D_5(q) \times (q - 1)/c) \times (q - 1)/c) \cdot c')$, where $c' = e/d$, $e = (q - 1, 4)$, $d = (2, q - 1)$. Thus $n_3 = |P_1 : M_3| = |\bar{P}_1 : \bar{M}_3| = q^{10}(q^9 - 1)(q^8 + q^4 + 1)/(q - 1)$.

Consider the action of an element $u_1 = u_0 \cdot w_4$ on $\Phi$, where $w_u$ is a reflection corresponding to the element $u$, defined above. A diagram depicting that action is the following:

![Diagram 4]

Obviously, $M_4 = P_1 \cap P_{17}^R = L_1$. Therefore, $|P_1 : M_4| = q^{27s}$. We have $1 + |P_1 : M_2| + |P_1 : M_3| + |P_1 : M_4| = |G : P_1|$. Hence the rank of the representation equals 4.

**THEOREM 2.** For simple non-Abelian groups $G = E_7(q)$, the parameters $n, n_2, n_3, n_4, P, M_2, M_3,$ and $M_4$ of minimal permutation representations are given in the following list:

- $n = (q^{14-1})(q^8+1)(q^8+1), n_2 = q \cdot (q^{14-1})(q^8+1)(q^8+1), n_3 = q^{10}, (q^{14-1})(q^8+1)(q^8+1), n_4 = q^{27},$
- $P = p^{27s} : (d' \cdot (E_6(q) \times (q - 1)/c) \cdot d'),$
- $M_2 = p^{26s} : (p^{16s} : (c' \cdot ((D_5(q) \times (q - 1)/c) \times (q - 1)/c) \cdot e)),$
- $M_3 = p^{17s} : (p^{16s} : (c' \cdot ((D_5(q) \times (q - 1)/c) \times (q - 1)/c) \cdot e)),$
- $M_4 = d' \cdot (E_6(q) \times (q - 1)/c) \cdot d',$

where $d = (q - 1, 2), d' = (q - 1, 3), e = (q - 1, 4), c' = e/d, c = d \cdot d'$.

The rank of the representation equals 4.

### 3. GROUP $E_8(q)$

**A. Algebra $E_8$.** The rank of $E_8$ equals 8. If $e_1, \ldots, e_8$ is an orthonormal basis of the Euclidean space $K_8$, where $K$ is a Cartan subalgebra of $E_8$, then a system II of simple roots for $E_8$ is defined as follows:

$p_1 = e_1 - e_2, p_2 = e_2 - e_3, p_3 = e_3 - e_4, p_4 = e_4 - e_5, p_5 = e_5 - e_6, p_6 = e_6 - e_7, p_7 = e_7 - e_8, p_8 = -\frac{1}{2} \sum_{i=1}^{8} e_i.$

The system of positive roots is

$$\Phi^+ = \left\{ e_i \pm e_j, i < j, i = 1, \ldots, 6, j = 2, \ldots, 7; \right\} \cdot \left\{ -e_i \pm e_j, i = 1, \ldots, 7; \right\} \cdot \left\{ -\frac{1}{2} \sum_{i=1}^{8} e_i, e_i = \pm 1, e_8 = 1, \prod_{i=1}^{8} e_i = 1 \right\}, \quad |\Phi^+| = 120.
The matrix $A$ has the form

\[
\begin{pmatrix}
2 & -1 & 0 & \cdots & 0 \\
-1 & \ddots & \ddots & \ddots & \ddots \\
0 & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & \ddots & A' \\
\end{pmatrix}
\]

where $A'$ coincides with a matrix $A$ for $E_7$.

Dynkin diagram is

\[
P_1 \rightarrow P_2 \rightarrow P_3 \rightarrow P_4 \rightarrow P_5 \rightarrow P_7 \rightarrow P_8
\]

B. Group $E_8(q)$ and its parabolic subgroups of least index. The universal group $\widetilde{E}_8(q)$ coincides with the adjoint group $G = E_8(q)$. From the main result in [2] and Proposition 1 of [1], it follows that a subgroup of least index in $G$ should be parabolic. Proposition 1 of [1] allows us to compute orders of maximal parabolic subgroups of $G$. Comparing these orders, we see that, up to conjugation, the subgroup of least index in $G$ is $P_1$. Thus

\[
|G| = q^{120}(q^{30} - 1)(q^{24} - 1)(q^{20} - 1)(q^{16} - 1)(q^{12} - 1)(q^{8} - 1)(q^2 - 1),
\]

\[
|P_1| = q^{120}(q^{18} - 1)(q^{14} - 1)(q^{10} - 1)(q^8 - 1)(q^6 - 1)(q^2 - 1)(q - 1),
\]

\[
|G : P_1| = (q^{30} - 1)(q^{12} + 1)(q^{10} + 1)(q^8 + 1)/(q - 1).
\]

In view of the Levi decomposition, $P_1 = U_1 \cdot L_1$. For all but one $r \in \Phi^+ \setminus \Phi^+_1$, we have $r = p_1 + s_1$, where $s_1 \in \Phi^+_1$. The excepted element is $u = e_1 - e_8 = 2p_1 + 3p_2 + 4p_3 + 5p_4 + 6p_5 + 7p_6 + 2p_7 + 2p_8$. For every $r$ (except $u$ of course) of $\Phi^+ \setminus \Phi^+_1$, there exists an element $r' \in \Phi^+ \setminus \Phi^+_1$ such that $r + r' = u$. Therefore, Lemma 1 in [1] (Chevalley commutator formula) implies that $U_1$ is isomorphic to $p^r \cdot p^{56}$. Let

\[
L'_1 = \langle X_r | r \in \Phi_1 \rangle \simeq E_7(q).
\]

The element $h_u = h_u(\lambda) = h_{p_1}(\lambda^3)h_{p_2}(\lambda^4)h_{p_3}(\lambda^5)h_{p_4}(\lambda^6)h_{p_5}(\lambda^7)$ centralises $L'_1$ since $(u, p_i) = 0$ for every $i = 2, \ldots, 8$. Furthermore, the center of the subgroup $L'_1$ is in $\langle h_u \rangle$. Hence $L_1 \simeq (E_7(q) \times (q - 1)/d) \cdot d$, where $d = (q - 1, 2)$. C. Representation of $G$ on cosets w.r.t. $P_1$. The element $w_{p_i}$ acts on $\Phi$ as is shown in Diagram 1 (see Sec. 1.C). Therefore, $U_1 \cap P_1^{w_{p_1}} \simeq p^r \cdot p^{54} \times p^r$. Let $L_{1, 2} = \langle X_r | r \in \Phi_{1, 2} \rangle \simeq E_6(q)$ and $U_{1, 2} = \langle X_r | r \in \Phi^+ \setminus \Phi^+_{1, 2} \rangle$. It is clear that $U_{1, 2} \simeq p^{27}$. The elements $h_u$ and $h_{p_1}(\lambda)$, where $\lambda$ generates $K^*$, centralize $L_{1, 2}$ and so does the element $h_u(\lambda)h_{p_1}(\lambda^{-2})$. Furthermore, $Z(L_{1, 2}) \leq \langle h_u, h_{p_1}(\lambda^{-2}) \rangle$. Hence $L_1 \cap P_1^{w_{p_1}} \simeq p^{27} \cdot (d' \cdot (E_6(q) \times (q - 1)/(q - 1)) \cdot d')$. Thus $M_2 = P_1 \cap P_1^{w_{p_1}} \simeq (p^{27} \cdot p^{54} \times p^r) \cdot (p^{27} \times (d' \cdot (E_6(q) \times (q - 1)/(q - 1)) \cdot d'))$, where $d' = (q - 1, 3)$, and $|P_1 : M_2| = q \cdot (q^{14} - 1)(q^{9} + 1)(q^5 + 1)/(q - 1)$. We know that $(u, p_i) = 0$ for every $i = 2, \ldots, 8$. Therefore, $w_u$ acts on $\Phi$ as follows: $w_u(\Phi_1) = \Phi_1$, $w_u(\Phi^+ \setminus \Phi_1) = \Phi^- \setminus \Phi_1$, $w_u(\Phi^- \setminus \Phi_1) = \Phi^+ \setminus \Phi_1$. Hence $P_1 \cap P_1^{w_{p_1}} = M_2 = L_1$. Thus $|P_1 : M_2| = q^{27}$. Consider an element $w_0 = w_u \cdot w_{p_1}$. The following diagram reflects the action of $w_0$ on $\Phi$:
Diagram 5

In this way $U_1 \cap P_1^{2s} \simeq p^{2s}$ and $L_1 \cap P_1^{2s} = L_1.2 = L_1 \cap P_1^{2s}$. Therefore, $M_4 \simeq p^{2s} : (p^{2s} : (d' \cdot (E_6(q) \times (q - 1)/d') \times (q - 1)) \cdot d')$. The index of $M_4$ in $P_1$ equals $q^{29} \cdot (q^{14} - 1)(q^9 + 1)(q^5 + 1)/(q - 1)$.

Now consider an element $w_1 = w_{e_1 - e_8} w_{e_1 + e_8}$, whose action on $\Phi$ is shown in the following diagram:

Diagram 6

Here $\Phi_0^+$ denotes the set $\{e_1 \pm e_j | j = 2, \ldots, 7\}$. Thus $U_1 \cap P_1^{2s} \simeq p^t \cdot p^{2s} \times p^{12s}$. Let $L_{1,8} = \langle X_r | r \in \Phi_{1,8} \rangle \simeq D_6(q)$ and $U_{1,8} = \langle X_r | r \in \Phi_{1,8}^+ \rangle \simeq p^{3s}$. Since $h_0$ centralizes $L_{1,8}$, we obtain $M_5 = P_1 \cap P_1^{2s} \simeq (p^t \cdot p^{2s} \times p^{12s}) : (p^{3s} : (d \cdot (d \cdot D_6(q) \times (q - 1)/d) \cdot d) \cdot (q - 1))$, where $d = (q - 1,2)$, and $|P_1 : M_5| = q^{12}(q^{14} - 1)(q^{12} + q^6 + 1)(q^8 + q^4 + 1)/(q - 1)$.

We have $|G : P_1| = 1 + \sum_{i=2}^5 |P_1 : M_i|$. Hence, the rank of $G$'s representation on the cosets w.r.t. $P_1$ equals 5.

THEOREM 3. For simple non-Abelian groups $G = E_8(q)$, the parameters $n$, $n_2$, $n_3$, $n_4$, $n_5$, $P$, $M_2$, $M_3$, $M_4$, and $M_5$ of minimal permutation representations are given in the following list:
\[ n = (q^{12} - 1)(q^{12} + 1)(q^{14} + 1)(q^{16} + 1), \quad n_2 = q \cdot (q^{12} - 1)(q^{12} + 1)(q^{14} + 1), \quad n_3 = q^{57}, \quad n_4 = q^{29} \cdot (q^{14} - 1)(q^{16} + 1)(q^{18} + 1), \]
\[ n_5 = q^{12} \cdot (q^{14} - 1)(q^{12} + 1)(q^{14} + 1); \]

\[ P = (p^c \cdot p^{56}) : (d \cdot (E_7(q) \times (q - 1)/d) \cdot d), \]
\[ M_2 = (p^c \cdot p^{54} \times p^c) : (p^{27} : (d' \cdot (E_8(q) \times (q - 1)/d') \times (q - 1)) \cdot d'), \]
\[ M_3 = d \cdot (E_7(q) \times (q - 1)/d) \cdot d, \]
\[ M_4 = p^{28} : (p^{27} : (d' \cdot (E_8(q) \times (q - 1)/d') \times (q - 1)) \cdot d'), \]
\[ M_5 = (p^c \cdot p^{52} \times p^{12}) : (p^{32} : (d \cdot (d \cdot D_6(q) \times (q - 1)/d) \cdot d) \cdot (q - 1)), \]

where \( d = (q - 1, 2), \quad d'' = (q - 1, 3). \]

The rank of the representation equals 5.

REFERENCES
