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RECOGNITION OF THE FINITE SIMPLE GROUPS $F_4(2^m)$ BY SPECTRUM

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Abstract: The spectrum of a finite group is the set of its element orders. A finite group G is said to be *recognizable by spectrum*, if every finite group with the same spectrum as G is isomorphic to G. The purpose of the paper is to prove that for every natural m the finite simple Chevalley group $F_4(2^m)$ is recognizable by spectrum.

Keywords: recognition by spectrum, finite simple group, group of Lie type

Introduction

The *spectrum* $\omega(G)$ of a finite group G is the set of its element orders. In other words, a natural number n is in $\omega(G)$ if and only if there is an element of order n in G. A finite group G is said to be *recognizable by spectrum* (briefly, *recognizable*) if $H \simeq G$ for every finite group H such that $\omega(H) = \omega(G)$. Since a finite group with a nontrivial normal soluble subgroup is not recognizable (see [1, Lemma 1]), each recognizable group is an extension of the direct product M of simple nonabelian groups by some subgroup of $\mathrm{Out}(M)$. Of most interest is the recognition problem for simple and almost simple groups (a group G is *almost simple* if $S \leqslant G \leqslant \mathrm{Aut}(S)$ for some simple nonabelian group S). In the middle of the 1980s Shi found the first examples of recognizable finite simple groups (see [2, 3]). In 1994 Shi and Brandl proved recognizability of the infinite series of simple linear groups $L_2(q)$, $q \neq 9$ (see [4, 5]). The recognition problem is solved at present for all groups with prime divisors at most 11 (see [6]) and several infinite series of recognizable finite simple and almost simple groups are obtained. The list of groups is available in [6] for which the recognition problem is solved.

The purpose of this article is to prove the following

Theorem. *For every natural number m the group $G = F_4(2^m)$ is recognizable by spectrum.*

Remark. Recognizability of $F_4(2)$ is proved in [7]. So we may assume $m > 1$ while proving the theorem.

§ 1. Preliminaries

The set $\omega(H)$ of a finite group H is closed under divisibility and uniquely determined by the set $\mu(H)$ of those elements in $\omega(H)$ that are maximal under the divisibility relation. Moreover, the set $\omega(H)$ determines the Gruenberg–Kegel graph $GK(H)$ whose vertices are all prime divisors of the order of H and two primes p and q are adjacent if H has an element of order $p{\cdot}q$. Denote by $s(H)$ the number of connected components of $GK(H)$ and by $\pi_i(H)$, $i = 1, \dots, s(H)$, the ith connected component of $GK(H)$. If H has even order then put $2 \in \pi_1(H)$. Denote by $\mu_i(H)$ ($\omega_i(H)$) the set of numbers $n \in \mu(H)$ ($n \in \omega(H)$) such that every prime divisor of n belongs to π_i.

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For the group $G = F_4(q)$ with $q = 2^m$, $m \in \mathbb{N}$, it was shown in [8] that $s(G) = 3$. So we may use the following two lemmas.

Lemma 1.1 (a corollary of the Gruenberg–Kegel theorem). *Let H be a finite group with $s(H) > 2$. Then there exists a simple nonabelian group S such that $S \leq \overline{H} = H/K \leqslant \mathrm{Aut}(S)$ for some nilpotent normal $\pi_1(H)$-subgroup K of H and the group $\overline{H}/S$ is a $\pi_1(H)$-subgroup. Moreover, the graph $GK(L)$ is disconnected, $s(S) \geq s(H)$, and for every i, $2 \leq i \leq s(H)$, there is j, $2 \leq j \leq s(S)$, such that $\omega_i(H) = \omega_j(S)$.*

Proof. See [9].

Lemma 1.2. *Let S be a finite simple nonabelian group nonisomorphic to the alternating group A_6 and $s(S) > 2$. Then S is quasirecognizable, that is, every finite group H with $\omega(H) = \omega(S)$ contains a composition factor isomorphic to S.*

Proof. See [10].

Lemma 1.3. *Let H be a finite group, $K \lhd H$, and let H/K be a Frobenius group with kernel F and cyclic complement C. If $(|F|, |K|) = 1$ and F does not lie in $KC_H(K)/K$ then $p|C| \in \omega(H)$ for some prime divisor p of $|K|$.*

Proof. See [11, Lemma 1].

Lemma 1.4. *Let $L = G_2(q)$, where $q = p^n$ and p is a prime. Then L contains the Frobenius subgroup FC whose kernel F is an elementary abelian p-group of order q^2 and whose complement $C = \langle c \rangle$ is a cyclic group of order $q^2 - 1$.*

Proof. Let Φ be a root system, let Φ^+ be a positive system, and let $\Pi = \{\alpha_1, \alpha_2\}$ be a fundamental root system of Lie algebra G_2 where the root α_2 is longer than α_1; that is, $(\alpha_2, \alpha_2) = 3(\alpha_1, \alpha_1)$. Denote by $x_\alpha(t)$, where $\alpha \in \Phi$, $t \in \mathbf{F}_q$, the root element of L; by X_α, the corresponding root subgroup; by $H = \langle h_{\alpha_i}(u) \mid i = 1, 2, u \in \mathbf{F}_q^* \rangle$, the Cartan subgroup of L; and by $U = \langle X_\alpha \mid \alpha \in \Phi^+ \rangle$, the maximal unipotent subgroup corresponding to Φ^+. The subgroup U is a Sylow p-subgroup of L. Up to conjugation there exist two maximal parabolic subgroups in L. Following [12], where such subgroups are described in detail, we denote these groups by P_1 and P_2. Of interest to us is the group P_1. It admits the Levi decomposition: $P_1 = U_1 : L_1$, where $U_1 = \langle X_\alpha \mid \alpha \in \Phi^+ \setminus \{\alpha_2\} \rangle$ is a unipotent subgroup of order q^5, $L_1 = \langle H, X_{\alpha_2}, X_{-\alpha_2} \rangle$ is a subgroup of order $q(q^2 - 1)(q - 1)$, $U_1 \cap L_1 = 1$, and $P_1 = N_L(U_1)$ is the normalizer of U_1 in L.

Denote by F the subgroup of U_1 generated by the root subgroups $X_{3\alpha_1+\alpha_2}$ and $X_{3\alpha_1+2\alpha_2}$. In view of the Chevalley commutator formula [13, Theorem 5.2.2], the elements $x_{3\alpha_1+\alpha_2}(t)$ and $x_{3\alpha_1+2\alpha_2}(u)$ commute for all $t, u \in \mathbf{F}_q$. Thus, F is an elementary abelian p-group of order q^2.

The Cartan subgroup H normalizes every root subgroup. Furthermore, using the Chevalley commutator formula it is easy to verify that $X_\alpha^g \subseteq F$, where $\alpha = 3\alpha_1 + \alpha_2$ or $3\alpha_1 + 2\alpha_2$, and g runs through the set of the elements of the type $x_{\pm\alpha_2}(t)$, $t \in \mathbf{F}_q$. Therefore, the subgroup L_1 normalizes F.

Consider F as a two-dimensional vector space V over the field of order q and choose the elements $x_{3\alpha_1+\alpha_2}(1)$ and $x_{3\alpha_1+2\alpha_2}(1)$ as a basic vectors v_1 and v_2 of V. Since L_1 normalizes F, there is a natural homomorphism ψ from L_1 to $GL(V)$. The images of the elements $x_{\alpha_2}(t)$, $x_{-\alpha_2}(t)$ and $h_{\alpha_1}(\lambda)$ with $t \in \mathbf{F}_q$, $\lambda \in \mathbf{F}_q^*$, of L_1 under ψ are as follows:

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \quad \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}.$$

Since these matrices generate the group $GL_2(q)$, the map ψ is an epimorphism. But $|L_1| = |GL_2(q)|$, and so $L_1 \simeq GL_2(q)$.

Now the vector space V can be identified with the additive group of the field of order q^2. Then the operator of right multiplication by a primitive field element induces a nonsingular linear transformation φ

of order q^2-1 of the space V which is obviously regular on V. Its preimage c in L_1 shares the same property. Thus, the subgroup FC with $C=\langle c\rangle$ is the desired Frobenius group. The lemma is proved.

REMARK. The proof of the lemma above is a slight modification of the proof of Lemma 2.1. in [14] where the same result was obtained for $G_2(q)$, $q=p^n$, with p an odd prime.

Lemma 1.5. *Suppose that $G=F_4(q)$, $q=2^m$, $m\in\mathbb{N}$, acts on a finite 2-group V. Then for every element $x\in G$ of odd order the group $C_V(x)$ is nontrivial.*

PROOF. There exist an algebraic group $\widetilde{G}$ over the algebraic closure of $\mathbf{F}_q$ and an epimorphism σ of $\widetilde{G}$ such that $G=O^{2'}(\widetilde{G}_\sigma)$, where $\widetilde{G}_\sigma$ is the centralizer of σ in $\widetilde{G}$. Moreover, there exists a maximal σ-stable torus T of G which contains x.

Without loss of generality, we may assume that V is an absolutely irreducible module for G. Since every such module by the Steinberg theorem [15, Theorems 41 and 43] is the tensor product of the $\widetilde{G}$-modules obtained from the so-called basic modules by applying the powers of a Frobenius automorphism, we can suppose that V is basic. The basic modules for $\widetilde{G}$ are essentially determined by Veldkamp in [16]. It follows from Table II of [16] that T centralizes some nonzero subspace in V, so the lemma is proved.

The authors are grateful to Frank Lübeck who explained to one of them how to apply the results by Veldkamp [16] to what is needed in this paper.

The last lemma describes the set $\mu(G)$.

Lemma 1.6. *Let $G=F_4(2^m)$, $m\in\mathbb{N}$ and $m>1$. Then*

$$\mu(G)=\{16, 8(q-1), 8(q+1), 4(q^2-1), 4(q^2+1), 4(q^2-q+1), 4(q^2+q+1),$$
$$2(q-1)(q^2+1), 2(q+1)(q^2+1), 2(q^3-1), 2(q^3+1),$$
$$(q^2-1)(q^2-q+1), (q^2-1)(q^2+q+1), q^4-1, q^4+1, q^4-q^2+1\}.$$

In particular, $\mu_2(G)=\{q^4+1\}$, $\mu_3(G)=\{q^4-q^2+1\}$.

PROOF. The conjugacy classes of the group G were determined by Shinoda in [17]. We use his results to obtain the element orders of G.

2-ELEMENTS. Theorem 2.1 in [17] asserts that in G there are 35 2-element conjugacy classes (including the identity element). Their representatives are given in the same theorem. Using the Chevalley commutator formula we obtain the following:

x_0 is the identity element;
$x_1,\dots,x_4$ are elements of order 2;
$x_5,\dots,x_{19}$ are elements of order 4;
$x_{20},\dots,x_{30}$ are elements of order 8;
$x_{31},\dots,x_{34}$ are elements of order 16.

$2'$-ELEMENTS. It is well known that in G each $2'$-element lies in some maximal torus. According to [17], G contains 25 maximal tori:

$H(1)\simeq Z_{q-1}\times Z_{q-1}\times Z_{q-1}\times Z_{q-1}$;
$H(2)\simeq Z_{q-1}\times Z_{q-1}\times Z_{q^2-1}$;
$H(3)\simeq Z_{q-1}\times Z_{q-1}\times Z_{q-1}\times Z_{q+1}$;
$H(4)\simeq Z_{q-1}\times Z_{q-1}\times Z_{q+1}\times Z_{q+1}$;
$H(5)\simeq Z_{q^2-1}\times Z_{q-1}\times Z_{q+1}$;
$H(i)\simeq Z_{q-1}\times Z_{q^3-1}$, where $i=6,\dots,10$;
$H(11)\simeq H(12)\simeq Z_{q^4-1}$;
$H(13)\simeq H(14)\simeq Z_{q-1}\times Z_{q^3+1}$;
$H(i)\simeq Z_{q+1}\times Z_{q^3-1}$, where $i=15,\dots,22$;
$H(23)\simeq Z_{q^4+1}$;
$H(24)\simeq Z_{q^4-q^2+1}$;
$H(25)\simeq Z_{q^2-q+1}\times Z_{q^2-q+1}$.

Since $(q-1, q^3+1) = (q+1, q^3-1) = 1$, the tori $H(13)$ and $H(15)$ are cyclic. Thus, the maximal orders (by divisibility) of $2'$-elements of G are q^4-1, q^4+1, $(q-1)(q^3+1)$, $(q+1)(q^3-1)$, and q^4-q^2+1.

The other elements. Using Table IV in [17] and information on the structure of the centralizers of 2-elements and $2'$-elements, we calculate the maximal orders of the composite elements in G. They are $8(q-1)$, $8(q+1)$, $4(q^2-1)$, $4(q^2+1)$, $4(q^2-q+1)$, $4(q^2+q+1)$, $2(q^3-1)$, $2(q^3+1)$, $2(q+1)(q^2+1)$, $2(q-1)(q^2+1)$. So the lemma is proved.

§ 2. Proof of the Theorem

Let $G = F_4(q)$, $q = 2^m$, $m \in \mathbb{N}$, $m > 1$, and let H be a finite group with $\omega(H) = \omega(G)$.

Lemmas 1.1 and 1.2 imply that $G \leq \overline{H} = H/K \leq \mathrm{Aut}(G)$ for some nilpotent normal $\pi_1(H)$-subgroup K of H and the group $\overline{H}/S$ is a $\pi_1(H)$-subgroup. We complete the proof in three steps.

Proposition 1. *K is an elementary abelian p-group with $p = 2$ or $p = 3$.*

Proof. Using induction on the order of H we may assume that K is an elementary abelian p-group for some prime p. Let $p \neq 2, 3$.

There exists a subgroup A in G isomorphic to the simple linear group $A_3(q)$. Therefore, G contains a Frobenius subgroup with kernel F of order q^3 and complement C of order q^3-1 (see the proof of Lemma 3 in [18]). Since $p \neq 2$ and G is simple, we have $(|F|, |K|) = 1$ and $C_H(K) = K$. Thus, Lemma 1.3 implies that H contains an element of order $p(q^3-1)$. Using Lemma 1.6, we find that p divides $q+1$.

At the same time G includes a subgroup B isomorphic to $G_2(q)$. By Lemma 1.4 the group $G_2(q)$ includes a Frobenius subgroup with kernel F of order q^2 and complement C of order q^2-1. By Lemma 1.3 the group H contains an element of order $p(q^2-1)$. Using Lemma 1.6, we find that p divides either q^2+1 or q^2+q+1, or q^2-q+1.

Since $q = 2^m$, we have $(q+1, q^2+1) = (q+1, q^2+q+1) = 1$. Furthermore, $(q+1, q^2-q+1) = 1$, if $q \equiv 1 (\mathrm{mod}\, 3)$, and $(q+1, q^2-q+1) = 3$, if $q \equiv -1 (\mathrm{mod}\, 3)$. The proposition is proved.

Proposition 2. $K = 1$.

Proof. By Proposition 1, we may assume that K is an elementary abelian p-group, where $p = 2$ or $p = 3$.

If $p = 2$ then Lemma 1.5 implies that there exists an element of order $2(q^4+1)$ in H, which contradicts Lemma 1.6.

Let $p = 3$. The group G/K includes a subgroup D that is isomorphic to ${}^2F_4(2)$ and acts on K by conjugation in G. Inspection of the table of the Brauer 3-characters for the group ${}^2F_4(2)$ in [19] shows that the element $x \in D$ of order 16 has a fixed point in every absolutely irreducible module over a field of characteristic 3. Thus, x centralizes some nontrivial element in K, and hence $48 \in \omega(H)$; a contradiction.

Proposition 3. $H = G$.

Proof. We have $G \leq H \leq \mathrm{Aut}(G)$. The group $\mathrm{Out}(G)$ is cyclic of order $2m$, and there exists a graph automorphism σ whose image in $\mathrm{Out}(G)$ generates that group. Furthermore, $\langle \sigma^2 \rangle$ is the group of field automorphisms of G. This group centralizes in G a subgroup F isomorphic to $F_4(2)$. If σ lies in H then H includes a subgroup $F\langle \sigma \rangle$ containing an element of order 32 (see [20, p. 169]), which contradicts Lemma 1.6. So we may suppose the factor group $\widetilde{H} = H/G$ to include only field automorphisms. Since the centralizer C of each field automorphism contains a subgroup isomorphic to $F_4(2)$, we have $16 \in \omega(C)$. If some odd prime p divides $|\widetilde{H}|$ then $16p \in \omega(H)$; a contradiction. So $\widetilde{H}$ is a cyclic 2-group generated by some field automorphism of G.

Let τ be the automorphism of $\mathbf{F}_q$ of order 2, and let t be an element of $\mathbf{F}_q$ such that $t \neq t^\tau$. We identify τ with the field automorphism of G that it induces. Obviously, τ is of order 2, as an element of H, and its image $\tilde{\tau}$ is the unique involution in $\widetilde{H}$. Let $\Pi = \{\alpha_i \mid i = 1, \dots, 4\}$ be the system of fundamental roots of the Lie algebra F_4 and let $x_{\alpha_i}(t)$, $i = 1, \dots, 4$, be the corresponding root elements of G. Consider the element $g = x_{\alpha_1}(t)x_{\alpha_2}(t)x_{\alpha_3}(t)x_{\alpha_4}(t)$ of G and the element $h = g\tau$ of H. Using the Chevalley commutator formula, we can check that the element g is of order 16, and that the canonical form of unipotent element h^2 of G involves $x_{\alpha_1}(t+t^\tau)$, $x_{\alpha_2}(t+t^\tau)$, $x_{\alpha_3}(t+t^\tau)$, and $x_{\alpha_4}(t+t^\tau)$. Since $t \neq 0$ and $t+t^\tau \neq 0$; therefore, g and h^2 are regular unipotent elements by [21, Proposition 5.1.3], and [21, Proposition 5.1.2] implies that these elements have the same order. Hence h is of order 32, and so $32 \in \omega(H)$; a contradiction. Thus, $H = G$ and the theorem is proved.

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References

1. *Mazurov V. D.*, "Recognition of finite groups by a set of orders of their elements," Algebra i Logika, **37**, No. 6, 371–379 (1998).
2. *Shi W.*, "A characteristic property of $PSL_2(7)$," J. Austral. Math. Soc. Ser. A., **36**, No. 3, 354–356 (1984).
3. *Shi W.*, "A characteristic property of A_5," J. Southwest-China Teach. Univ., **3**, 11–14 (1986).
4. *Shi W.*, "A characteristic property of J_1 and $PSL_2(2^n)$," Adv. Math., **16**, 397–401 (1987).
5. *Brandl R. and Shi W.*, "The characterization of $PSL(2, q)$ by its element orders," J. Algebra, **163**, No. 1, 109–114 (1994).
6. *Mazurov V. D.*, "Characterizations of groups by arithmetic properties," Algebra Colloq., **11**, No. 1, 129–140 (2004).
7. *Mazurov V. D.*, "Recognition of finite simple groups $S_4(q)$ by their element orders," Algebra i Logika, **41**, No. 2, 93–110 (2002).
8. *Kondrat′ev A. S.*, "On prime graph components for finite simple groups," Mat. Sb., **180**, No. 6, 787–797 (1989).
9. *Williams J. S.*, "Prime graph components of finite groups," J. Algebra, **69**, No. 2, 487–513 (1981).
10. *Alekseeva O. A. and Kondrat′ev A. S.*, "Quasirecognition of one class of finite simple groups by the set of element orders," Sibirsk. Mat. Zh., **44**, No. 2, 195–207 (2003).
11. *Mazurov V. D.*, "Characterization of finite groups by sets of all orders of the elements," Algebra i Logika, **36**, No. 1, 23–32 (1997).
12. *Vasilyev A. V.*, "Minimal permutation representations of finite simple exceptional groups of types G_2 and F_4," Algebra i Logika, **35**, No. 6, 371–383 (1996).
13. *Carter R. W.*, Simple Groups of Lie Type, John Wiley & Sons, London (1972).
14. *Vasilyev A. V.*, "Recognizing groups $G_2(3^n)$ by their element orders," Algebra i Logika, **41**, No. 2, 74–80 (2002).
15. *Steinberg R.*, Lectures on Chevalley Groups. Notes Prepared by John Faulkner and Robert Wilson, Yale University, New Haven, Conn. (1968).
16. *Veldkamp F. D.*, "Representations of algebraic groups of type F_4 in characteristic 2," J. Algebra, **16**, No. 2, 326–339 (1970).
17. *Shinoda K.*, "The conjugacy classes of Chevalley groups of type (F_4) over finite fields of characteristic 2," J. Fac. Sci. Univ. Tokyo Sect. IA Math., **21**, 133–159 (1974).
18. *Zavarnitsin A. V.*, Element Orders in Coverings of the Groups $L_n(q)$ and Recognition of the Alternating Group A_{16} [in Russian] [Preprint, No. 48], NII Diskret. Mat. Inform., Novosibirsk (2000).
19. *Jansen C., Lux K., Parker R. A., and Wilson R. A.*, An Atlas of Brauer Characters, Clarendon Press, Oxford (1995).
20. *Conway J. H., Curtis R. T., Norton S. P., Parker R. A., and Wilson R. A.*, Atlas of Finite Groups, Clarendon Press, Oxford (1985).
21. *Carter R. W.*, Finite Groups of Lie Type, John Wiley & Sons, New York (1985).