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MINIMAL PERMUTATION REPRESENTATIONS OF FINITE SIMPLE EXCEPTIONAL GROUPS OF TYPES G_2 AND F_4

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A minimal permutation representation of a group is a faithful permutation representation of least degree. Well-studied to date are the minimal permutation representations of finite sporadic and classical groups for which degrees, point stabilizers, as well as ranks, subdegrees, and double stabilizers, have been found. Here we attempt to provide a similar account for finite simple exceptional groups of types G_2 and F_4.

A minimal permutation representation of a group is its faithful permutation representation of least degree. Let G be a finite group. Denote by n the degree of its minimal permutation representation. If G is simple, that representation is transitive and, consequently, similar to a representation by permutations of the set Ω of right cosets with respect to some proper subgroup P, in which the element g of G agrees with a permutation sending each coset Px to Pxg. The subgroup P is a point stabilizer in the given representation, and each point stabilizer is conjugate to P and has index n in G. Consider the induced action of G on $\Omega \times \Omega$ given by the rule $(\alpha, \beta)g = (\alpha g, \beta g)$ for $\alpha, \beta \in \Omega$, $g \in G$, and with each orbit Δ, associate the graph Γ_Δ whose vertices are elements of Ω and directed edges are elements in Δ. It is then easy to see that G acts transitively on vertices and edges of the graph, and the stabilizer of an edge $(\alpha, \beta) = (P, Px)$ is the subgroup $G_\alpha \cap G_\beta = P \cap x^{-1}Px$. The set of orbits of G on $\Omega \times \Omega$ is in one-to-one correspondence with the set of orbits $P = G_\alpha$ on Ω. Namely, if we put $\Delta(P) = \{\delta \in \Omega |\ (\alpha, \delta) \in \Delta\}$, then $\Delta(P)$ is an orbit of P on Ω, and $\Delta \to \Delta(P)$ gives the desired correspondence. The number r of orbits of G on $\Omega \times \Omega$, or (which is the same) of P on Ω, is called the *rank* of representation. The length n_i of an arbitrary orbit $\Delta_i(P)$ of the group P on Ω equal to the index of a *double stabilizer* $M_i = G_{\alpha\beta} = G_\alpha \cap G_\beta$ in P, where $(\alpha, \beta) \in \Delta_i$, is referred to as the *subdegree* of representation. The trivial orbit $\{(\alpha, \alpha) |\ \alpha \in \Omega\}$ is denoted Δ_1; accordingly, we have $n_1 = 1$ and $M_1 = P$.

Thus far we have studied minimal permutation representations of finite simple sporadic groups (see the summary table in [1]) and of finite simple classical groups (cf. [2] and [3]), where the degrees and point stabilizers of these groups are found and their ranks, subdegrees, and double stabilizers are specified. The purpose of the present article is to take up a similar research on finite simple exceptional groups of types G_2 and F_4.

It should be observed that, in all the groups mentioned except $G_2(3)$ and $G_2(4)$, a subgroup of least index is parabolic. This fact follows from the basic result obtained in [4].

In the next section, we give a brief account of the necessary data on root systems, simple Lie algebras, and Chevalley groups (for more details, see [5] and [6]).

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1. DEFINITIONS, NOTATION, AND PRELIMINARY LEMMAS

Let $\mathcal{L}$ be a simple Lie algebra over a field K and $\mathcal{L} = \mathcal{K} \oplus \mathcal{L}_{r_1} \oplus \ldots \oplus \mathcal{L}_{r_k}$ be its Cartan decomposition. Here $\mathcal{K}$ is a Cartan subalgebra and $\mathcal{L}_{r_1}, \ldots, \mathcal{L}_{r_k}$ are one-dimensional root subspaces. Denote by Φ a root system in $\mathcal{K}$ corresponding to the given decomposition. Choose some subsystem $\Pi = \{p_1, \ldots, p_l\}$ in Φ, a basis of the vector space $\mathcal{K}$, which we call a *system of simple roots*; $p_1, \ldots, p_l$ are then referred to as *simple roots*. The number l equal to $\dim \mathcal{K}$ is called the *rank* of $\mathcal{L}$. By Φ^+ (resp., Φ^-) we denote a system of positive (negative) roots corresponding to the system Π. The Killing form $(\,,\,)\colon \mathcal{K} \to K$ is a nondegenerate symmetric bilinear form on $\mathcal{K}$. For two roots $r, s \in \Phi$, denote by A_{rs} the number $2(r,s)/(r,r)$, and by A_{ij} the number $A_{p_i p_j}$, where $p_i, p_j \in \Pi$. Let r and s be linearly independent. Since Φ is finite, there exist integers $p, q \geq 0$ such that $ir + s \in \Phi$ for any i satisfying the inequalities $-p \leq i \leq q$, but $-(p+1)r + s$ and $(q+1)r + s$ do not lie in Φ. It turns out that A_{rs} is an image of the integer $p - q$ in the prime subfield of K. The sequence of roots $-pr + s, \ldots, s, \ldots, qr + s$ is called the *r-series of roots* containing s.

A Weyl group $W = W(\Phi)$ of Φ is one generated by the set $\{w_r \mid r \in \Phi\}$, where $w_r(x) = x - 2(r,x)/(r,r)\cdot r$ is the reflection of the space $\mathcal{K}$ in a hyperplane orthogonal to r. As a generating set of W, as is known, we can take the set $\{w_r \mid r \in \Pi\}$.

Let J be some subset of the system Π of simple roots in Φ. Denote by $\mathcal{K}_J$ the linear span of vectors in J, by Φ_J the set $\Phi \cap \mathcal{K}_J$, and by W_J the subgroup of W generated by $\{w_r \mid r \in J\}$. Then Φ_J is a root system in K_J, J is a system of simple roots in Φ_J, and $W_J = W(\Phi_J)$. Suppose $J_i = \{p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_l\}$ is a maximal subsystem of Π. Write Φ_i for a subsystem Φ_{J_i}, and W_i for a subgroup W_{J_i}.

For convenience, below we give a table listing simple Lie algebras (up to isomorphism) over an arbitrary field, with the parameters $l = \operatorname{rank} \mathcal{L}$, $N = |\Phi^+|$, $\dim \mathcal{L}$, and with respective Dynkin diagrams.

A *Chevalley group of type* $\mathcal{L}$ over a field K is an automorphism group of a Lie algebra $\mathcal{L}$ generated by elements $x_r(t)$ for all $r \in \Phi$, $t \in K$ (for definition of $x_r(t)$, see [6, 4.3]). The group is denoted by $\mathcal{L}(K)$, or by $\mathcal{L}(q)$ if the order of the field equals q. In what follows we assume that $|K| = q = p^s$, where p is the characteristic of K.

Let $G = \mathcal{L}(K)$ be a Chevalley group of type $\mathcal{L}$ over K. Denote by X_r the set $\{x_r(t) \mid t \in K\}$. Since $x_r(t_1) \cdot x_r(t_2) = x_r(t_1 + t_2)$, X_r is an Abelian group isomorphic to the additive group of K, which we call a *root subgroup* of G.

Let $U = \langle X_r \mid r \in \Phi^+\rangle$, $V = \langle X_r \mid r \in \Phi^-\rangle$. Then $G = \langle U, V\rangle$, $U \cap V = 1$, and U and V are unipotent p-subgroups, where p is the characteristic of K. Below we give the Chevalley commutator formula which is important for our further reasoning.

LEMMA 1. Let $G = \mathcal{L}(K)$ be a Chevalley group over a field K, r and s linearly independent roots in $\mathcal{L}$, and t and u elements of K. Then

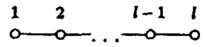
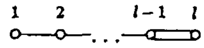
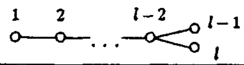
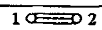
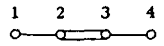
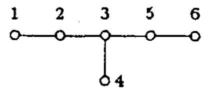
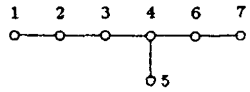
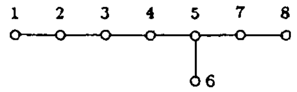
$$[x_s(u), x_r(t)] = \prod_{i,j>0} x_{ir+js}(C_{ijrs} \cdot (-t)^i u^j),$$

where the product is taken over all pairs i, j of positive integers such that $ir + js \in \Phi$, in order of increasing $i + j$. Constants C_{ijrs} are defined thus:

$$C_{1jrs} = (-1)^j M_{s,r,j}, \;\; C_{i1rs} = M_{r,s,i},$$

$$C_{32rs} = \frac{1}{3} \cdot M_{r+s,r,2}, \;\; C_{23rs} = -\frac{2}{3} M_{r+s,s,2},$$

TABLE 1

L	rank $\mathcal{L}$	N	dim $\mathcal{L}$	Dynkin diagram
A_l $(l \geq 1)$	l	$\frac{1}{2}l(l+1)$	$l(l+2)$	
B_l $(l \geq 2)$	l	l^2	$l(2l+1)$	
C_l $(l \geq 3)$	l	l^2	$l(2l+1)$	
D_l $(l \geq 4)$	l	$l(l-1)$	$l(2l-1)$	
G_2	2	6	14	
F_4	4	24	52	
E_6	6	36	78	
E_7	7	63	133	
E_8	8	120	248	

where $M_{r,s,0} = 1$, and $M_{r,s,i} = \pm\frac{(p+1)(p+2)\dots(p+i)}{i!}$ for $i \geq 1$. Here p is the greatest natural number such that $pr + s \in \Phi$, and the sign of $M_{r,s,i}$ depends on the choice of a basis in $\mathcal{L}$. Each constant is equal to ± 1, ± 2, or ± 3.

Proof. See Thm. 5.2.2 in [6].

Define elements $h_r(t)$ and $n_r(t)$ of the group $G = \mathcal{L}(K)$ by setting $n_r(t) = x_r(t) \cdot x_{-r}(-t^{-1})x_r(t)$ and $h_r(t) = n_r(t) \cdot n_r(-1)$. Write n_r for the element $n_r(-1)$. The group $H = \langle h_r(t) | \; r \in \Phi, t \in K^* \rangle$ is Abelian. Each element of H can be represented as $h = \prod_{i=1}^{l} h_{p_i}(t_i)$, where $\prod = \{p_1, \dots, p_l\}$ is the system of simple roots in $\mathcal{L}$. Moreover, $h = \prod_{i=1}^{l} h_{p_i}(t_i) = 1$ iff $\prod_{i=1}^{l} t_i^{A_{ij}} = 1$ for all $j = 1, \dots, l$. So the order of H is equal to $\frac{1}{d}(q-1)^l$, and the value d for Chevalley groups is given below in Table 2.

Direct computations show that $h_s(\lambda) \cdot x_r(t) \cdot h_s(\lambda)^{-1} = x_r(\lambda^{A_{rs}} \cdot t)$, from which we have $h_s(\lambda) \cdot X_r \cdot h_s(\lambda)^{-1} = X_r$. Therefore, H normalizes the subgroups U and V. Moreover, $U \cap H = V \cap H = 1$. The subgroup UH of G is denoted by B.

TABLE 2

$\mathcal{L}(q)$	$A_l(q)$	$B_l(q)$	$C_l(q)$	$D_l(q)$
d	$(l+1, q-1)$	$(2, q-1)$	$(2, q-1)$	$(4, q^l - 1)$

$\mathcal{L}(q)$	$G_2(q)$	$F_4(q)$	$E_6(q)$	$E_7(q)$	$E_8(q)$
d	1	1	$(3, q-1)$	$(2, q-1)$	1

Let N be a subgroup of G generated by the subgroup H and elements n_r for all $r \in \Phi$. We have $n_r x_s(t) n_r^{-1} = x_{w_r(s)}(\eta_{r,s} \cdot t)$, whence $n_r X_s n_r^{-1} = X_{w_r(s)}$.

LEMMA 2. There exists a homomorphism φ from N onto W such that $\varphi(n_r) = w_r$. Its kernel is H. In this way $H \trianglelefteq N$ and $N/H \simeq W$.

Proof. See Thm. 7.2.2 in [6].

For subgroups U, H, and N, the equality $UH \cap N = H$ holds.

LEMMA 3. Let $\mathcal{L}$ be a simple Lie algebra ($\mathcal{L} \neq A_1$) and K a field. For each root r in $\mathcal{L}$ and each element t of K, write the symbol $\bar{x}_r(t)$. Suppose $\bar{G}$ is an abstract group, generated by elements $\bar{x}_r(t)$ which satisfy the following relations:

$$\bar{x}_r(t_1) \cdot \bar{x}_r(t_2) = \bar{x}_r(t_1 + t_2),$$

$$[\bar{x}_s(u), \bar{x}_r(t)] = \prod_{i,j>0} \bar{x}_{ir+js}(C_{ijrs} \cdot (-t)^i u^j),$$

$$\bar{h}_r(t_1) \cdot \bar{h}_r(t_2) = \bar{h}_r(t_1 \cdot t_2), \ \ t_1 \cdot t_2 \neq 0,$$

where $\bar{h}_r(t) = \bar{n}_r(t) \cdot \bar{n}_r(-1)$ and $\bar{n}_r(t) = \bar{x}_r(t) \cdot \bar{x}_{-r}(-t^{-1}) \cdot \bar{x}_r(t)$. Let $\bar{Z}$ be the center of G. Then $\bar{Z} = \{\prod_{i=1}^{l} \bar{h}_i(t_i) | \prod_{i=1}^{l} t_i^{A_{ij}} = 1 \text{ for } j = 1, \ldots, l\}$ and the quotient group $\bar{G}/\bar{Z}$ is isomorphic to the Chevalley group $G = \mathcal{L}(K)$.

Proof. See Thm. 12.1.1 in [6].

We call $\bar{G}$ a *universal* Chevalley group. Sometimes $G = \mathcal{L}(K)$ is referred to as an *adjoint* Chevalley group, not to confuse it with $\bar{G}$. A preimage of an arbitrary subgroup S of G under the natural homomorphism from $\bar{G}$ onto G is denoted by $\bar{S}$.

LEMMA 4 (Bruhat). (1) $G = BNB$.

(2) $G = \bigcup_{w \in W} Bn_w B$. If $Bn_w B = Bn_{w'} B$, then $w = w'$, where n_w and $n_{w'}$ are preimages of elements w and w' in W under the natural homomorphism $\varphi \colon N \to W$.

(3) Suppose that for each subset J in the system of simple roots Π, a subgroup W_J is generated by reflections w_r, $r \in J$, and N_J is the preimage of W_J in N under φ. Then $P_J = BN_J B$ is a subgroup in G.

Proof. See Thm. 4 in [5] and Prop. 8.2.2 in [6].

A subgroup which is conjugate in G to the subgroup P_J, defined in Lemma 4, is called *parabolic*. We know that P_J is generated by the subgroup H and by root subgroups X_r, $r \in \Phi^+ \cup \Phi_J$.

If J is empty, then $P_J = B$. The subgroup B admits a decomposition into a semidirect product of subgroups U and H. And it turns out that every parabolic subgroup is decomposable in a similar way.

Let $\bar{\Phi}_J = \Phi \backslash \Phi_J$. For every subset J in the system Π, define a subgroup U_J of G generated by root subgroups X_r with $r \in \Phi^+ \cap \bar{\Phi}_J$. Obviously, $U_J = \prod_{r \in \Phi^+ \cap \bar{\Phi}_J} X_r$, with factors in arbitrary order. Let L_J be a subgroup of G generated by H and by root subgroups X_r, $r \in \Phi_J$.

LEMMA 5 (Levi decomposition). (1) $U_J \trianglelefteq P_J$.

(2) $P_J = U_J L_J$ and $U_J \cap L_J = 1$.

(3) P_J is a normalizer of U_J in G.

Proof. See Thm. 8.5.2 in [6].

Note that for a universal group $\bar{G}$, the Levi decomposition has the form $\bar{P}_J = U_J \cdot \bar{L}_J$.

Let $J_i = \Pi \backslash \{p_i\}$. For brevity, we then write P_i for P_{J_i}, U_i for U_{J_i}, and L_i for L_{J_i}. Similarly, denote by H_J a subgroup of H generated by the set $\{h_r(t) | \ r \in J, \ t \in K^*\}$, and by H_i the subgroup H_{J_i}.

Orders of Chevalley groups are listed in Table 3 below.

TABLE 3

$\mathcal{L}$	$d_1, d_2, \ldots, d_l$
A_l	$2, 3, \ldots, l+1$
B_l	$2, 4, 6, \ldots, 2l$
C_l	$2, 4, 6, \ldots, 2l$
D_l	$2, 4, 6, \ldots, 2l-2, l$
G_2	$2, 6$
F_4	$2, 6, 8, 12$
E_6	$2, 5, 6, 8, 9, 12$
E_7	$2, 6, 8, 10, 12, 14, 18$
E_8	$2, 8, 12, 14, 18, 20, 24, 30$

LEMMA 6. Let $G = \mathcal{L}(q)$ be a finite Chevalley group and $\bar{G}$ the corresponding universal group. Then $|G| = \frac{1}{d} \cdot |\bar{G}|$, where d is from Table 2. We have $|\bar{G}| = q^N \cdot (q^{d_1} - 1)(q^{d_2} - 1) \ldots (q^{d_l} - 1)$, where $N = |\Phi^+|$, and numbers $d_1, d_2, \ldots, d_l$ are as in Table 3.

Proof. See Thm. 9.4.10 and Prop. 10.2.5 in [6].

We will determine the order of a parabolic subgroup P_J of $G = \mathcal{L}(K)$. Let J be a proper subsystem of Π. We call a subset of simple roots, $I \subseteq J$, a *connected component* if a part of the Dynkin diagram corresponding to roots in I is a connected graph, and for any two roots $r \in I$ and $s \in J \backslash I$, the equality $(r, s) = 0$ holds. Obviously, J can be uniquely represented as the union $J = I_1 \cup I_2 \cup \ldots \cup I_t$ of mutually disjoint connected components. Let $|I_m| = l_m$ and $|J| = \sum_{m=1}^{t} l_m = l_0$. Denote by $\mathcal{L}_m$ a simple Lie algebra over the same field K, having the same Dynkin diagrams as I_m; $d_{m,i}$ is the invariant d_i of $\mathcal{L}_m$ (see Table 3).

Proposition 1. If $J \subset \Pi$ is represented as the union $J = I_1 \cup I_2 \cup \ldots \cup I_t$ of disjoint connected components, then the order of the subgroup P_J in $G = \mathcal{L}(K)$ is equal to

$$\frac{1}{d} \cdot q^N \cdot (q-1)^{l-l_0} \prod_{m=1}^{t} (q^{d_{m,1}} - 1)(q^{d_{m,2}} - 1) \ldots (q^{d_{m,l_m}} - 1),$$

where $d = |\bar{G}|/|G|$, $N = |\Phi^+|$, $l = |\Pi|$, and the other values are defined as above.

Proof. Note that if $r \in \Phi_{I_m}$, $s \in \Phi_{I_k}$, $m \neq k$, then, for any positive i and j, $ir + js$ fails to belong to Φ.

Consider a subgroup $\bar{P}_J$ of the universal group $\bar{G}$, which is the preimage of the parabolic subgroup P_J under the natural homomorphism from $\bar{G}$ onto G. The subgroup $\bar{P}_J$ contains H, and so it contains the center of $\bar{G}$, which is the kernel of a homomorphism from $\bar{G}$ onto G. By Lemma 5, we have $\bar{P}_J = U_J \bar{L}_J$. For any $m = 1, \ldots, t$, the group $\bar{L}_{I_m} = \langle \bar{x}_r(t) | \, r \in \Phi_{I_m}, \, t \in K \rangle$ is isomorphic to $\bar{\mathcal{L}}_m(K)$ and lies in $\bar{L}_J$. By the remark made above and Lemma 1, it follows that, for $m \neq k$, the subgroups $\bar{L}_{I_m}$ and $\bar{L}_{I_k}$ normalize each other. In addition, $\bar{L}_{I_m} \cap \bar{L}_{I_k} = 1$. Indeed, if $g \in \bar{L}_{I_m} \cap \bar{L}_{I_k}$, then $g \in \bar{H}$. But $(\bar{L}_{I_m} \cap \bar{L}_{I_k}) \cap \bar{H} = \bar{H}_{I_m} \cap \bar{H}_{I_k} = 1$. Therefore, $\bar{L}_J = (\bar{L}_{I_1} \times \bar{L}_{I_2} \times \ldots \times \bar{L}_{I_t}) \cdot H_{\bar{J}}$, where $\bar{J} = \Pi \backslash J$. Thus, the order $|\bar{L}_J|$ equals $(q-1)^{l-l_0} \cdot \prod_{m=1}^{t} q^{N_m} \cdot (q^{d_{m,1}} - 1)(q^{d_{m,2}} - 1) \ldots (q^{d_{m,l_m}} - 1)$.

Let $|U_J| = q^{N_0}$. Since $|\bar{G}| = q^N \cdot r$, where $(r, q) = 1$, and the order of the group U contained in $\bar{P}_J$ equals q^N, we have $N_0 + N_1 + N_2 + \ldots + N_t = N$, proving the proposition.

Finally, we give some standard notation which will be used throughout.

By $A \cdot B$ (resp., $A : B$), denote a (split) extension of the group A by a group B, and by A^m a direct product of m groups, each of which is isomorphic to A. In the subgroup notation, m denotes a cyclic group of order m.

2. GROUP $G_2(q)$

A. Algebra G_2. The rank of G_2 equals 2. Therefore, the system of simple roots, Π, equals $\{p_1, p_2\}$, and the system of positive roots $\Phi^+ = \{p_1, p_2, p_1 + p_2, 2p_1 + p_2, 3p_1 + p_2, 3p_1 + 2p_2\}$. We have the matrix $A = (A_{ij}) = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$. The Dynkin diagram has the form $p_1 \Lleftarrow p_2$, with the arrow pointing from a longer root to a shorter one. Note that $(p_2, p_2) = 3(p_1, p_1)$.

B. Group $G_2(q)$ and its parabolic subgroups of least index. If the field K has order 2, then the group $G_2(2)$ is not simple. Its commutator subgroup is a simple group isomorphic to the unitary group $U_3(3)$. Information on the minimal permutation representation of $U_3(3)$ can be found in [2]. If the order q of K is equal to 3 or 4, then a subgroup of least index in $G_2(q)$ is not parabolic. We consider such groups later. Now assume $q > 4$. In this case, as was mentioned above, a subgroup of least index should be presented by the maximal parabolic subgroup of $G = G_2(q)$. There are, up to conjugation, two such subgroups: $P_1 = P_{\{p_2\}}$ and $P_2 = P_{\{p_1\}}$. The universal group $\bar{G}_2(q)$ coincides with the adjoint group. From Lemma 6 and Proposition 1, therefore, it follows that $|G| = q^6(q^6-1)(q^2-1)$ and $|P_1| = |P_2| = q^6(q^2-1)(q-1)$. Thus, the degree of the minimal permutation representation equals $n = |G : P_1| = |G : P_2| = (q^6 - 1)/(q - 1)$, and we have two such representations — on cosets w.r.t. P_1 and on cosets w.r.t P_2.

We describe the structure of subgroups P_1 and P_2 using the Levi decomposition. For $i = 1, 2$, $P_i = U_i L_i$.

We have $U_1 = \langle X_r \mid r \in \Phi^+ \cap \bar{\Phi}_1 \rangle = \langle X_{p_1}, X_{p_1+p_2}, X_{2p_1+p_2}, X_{3p_1+p_2}, X_{3p_1+2p_2} \rangle$. Using Lemma 1, let us write out all nontrivial commutators of elements in U_1. We have

$[x_{p_1+p_2}(u), x_{p_1}(t)] = x_{2p_1+p_2}(\pm 2tu) x_{3p_1+p_2}(\pm 3t^2 u) x_{3p_1+2p_2}(\pm 3tu^2)$,

$[x_{2p_1+p_2}(u), x_{p_1}(t)] = x_{3p_1+p_2}(\pm 3tu)$,

$[x_{2p_1+p_2}(u), x_{p_1+p_2}(t)] = x_{3p_1+2p_2}(\pm 3tu)$.

Hence,

if $q = 2^s$, then $U_1 = 2^{2s} \cdot 2^{3s}$;

if $q = 3^s$, then $U_1 = 3^s \cdot 3^{2s} \times 3^{2s}$;

if $q = p^s$, $p \neq 2, 3$, then $U_1 = P^{2s} \cdot (p^s \cdot p^{2s})$.

For the group $U_2 = \langle X_{p_2}, X_{p_1+p_2}, X_{2p_1+p_2}, X_{3p_1+p_2}, X_{3p_1+2p_2} \rangle$, we have $[x_{3p_1+p_2}(u), x_{p_2}(t)] = x_{3p_1+2p_2}(\pm tu)$, $[x_{2p_1+p_2}(u), x_{p_1+p_2}(t)] = x_{3p_1+2p_2}(\pm 3tu)$.

Therefore,

if $q = 2^s$, then $U_2 = 2^s \cdot 2^{4s}$;

if $q = 3^s$, then $U_2 = 3^s \cdot 3^{2s} \times 3^{2s}$;

if $q = p^s$, $p \neq 2, 3$, then $U_2 = p^s \cdot p^{4s}$.

Now consider the group $L_1 = \langle H, X_r \mid r \in \Phi_1 \rangle$. Write $H_1 = \langle h_{p_2}(\lambda) \mid \lambda \in K^* \rangle$ and distinguish the subgroup $L_1' = \langle H_1, X_{p_2}, X_{-p_2} \rangle$ in L_1. Obviously, L_1' is isomorphic to the universal Chevalley group $\bar{A}_1(q) \simeq SL_2(q)$, and $L_1 = L_1' \cdot \langle h \rangle$, where h is an element of H not lying in H_1. The element $h_{2p_1+p_2}(\lambda)$, $\lambda \in K^*$, centralizes L_1' since $(p_2, 2p_1+p_2) = 0$. Consequently, $h_{2p_1+p_2}(\lambda) x_{p_2}(t) h^{-1}_{2p_1+p_2}(\lambda) = x_{p_2}(\lambda^{A_{p_2,2p_1+p_2}} \cdot t) = x_{p_2}(t)$. To make it explicit under which conditions $L_1 = L_1' \times \langle h_{2p_1+p_2}(\lambda) \rangle$, we represent $h_{2p_1+p_2}(\lambda)$ as a product of $h_{p_1}(\lambda)$ and $h_{p_2}(\lambda)$. Since $A_{2p_1+p_2,s} = \frac{2(2p_1+p_2,s)}{(2p_1+p_2,2p_1+p_2)} = 2 \cdot \frac{2(p_1,s)}{(p_1,p_1)} + 3 \cdot \frac{2(p_2,s)}{(p_2,p_2)} = 2A_{p_1,s} + 3A_{p_2,s}$, we have $h_{2p_1+p_2}(\lambda) = h^2_{p_1}(\lambda) \cdot h^3_{p_2}(\lambda)$. Thus $h_{2p_1+p_2}(\lambda) \in H_1$, and so therefore $L_1 \neq L_1' \times \langle h_{2p_1+p_2}(\lambda) \rangle$

iff $\lambda^2 = 1$ in the field K. In addition, $h_{2p_1+p_2}(\lambda) = h_{p_2}(\lambda)$ lies in the center of $L_1' \simeq \bar{A}_1(q)$. Therefore, $L_1 \simeq d \cdot (A_1(q) \times (q-1)/d) \cdot q$, where $d = (2, q-1)$.

Likewise, $L_2 = L_2' \cdot \langle h \rangle$, where $h \notin H_2 = \langle h_{p_1}(\lambda) | \lambda \in K^* \rangle$ and $L_2' = \langle H_2, X_{p_1}, X_{-p_1} \rangle \simeq \bar{A}_1(q)$. Consider an element $h_{3p_1+2p_2}(\lambda)$ in H. Since $(p_1, 3p_1+2p_2) = 0$, the element centralizes L_2' for any $\lambda \in K^*$. Moreover, $h_{3p_1+2p_2}(\lambda) = h_{p_1}(\lambda) \cdot h_{p_2}^2(\lambda)$ because $A_{3p_1+2p_2} = A_{p_1,s} + 2A_{p_2,s}$. Therefore, $L_2 \simeq d \cdot (A_1(q) \times (q-1)/d) \cdot d$, where $d = (2, q-1)$.

C. Representation of G on cosets w.r.t. P_1. We need to define double stabilizers, that is, groups of the form $P_1 \cap P_1^x$. To do this, we have to choose appropriate x. Use elements in N which do not map into W_1 under the natural homomorphism φ: $N \to W$. Since $H \leq P_1$, an element n can be identified with its image w. Recall that if $r, s \in \Phi$, then $X_s^{w_r} = X_{w_r(s)}$.

We have $P_1 = \langle H, X_{p_2}, X_{-p_2}, X_{p_1}, X_{p_1+p_2}, X_{2p_1+p_2}, X_{3p_1+p_2}, X_{3p_1+2p_2} \rangle$. Then $P_1^{w_{p_1}} = \langle H,\ X_{3p_1+p_2},\ X_{-3p_1-p_2},\ X_{-p_1},\ X_{2p_1+p_2},\ X_{p_1+p_2}, X_{p_2},\ X_{3p_1+2p_2} \rangle$. Therefore, $M_2 = P_1 \cap P_1^{w_{p_1}} = U_2 : H$ and $|M_2| = q^5 \cdot (q-1)^2$. Consequently, $|P_1 : M_2| = q(q+1)$.

Now, consider the action of $w_2 = w_{2p_1+p_2}$ on P_1. As was already mentioned, $(p_2, 2p_1+p_2) = 0$. Therefore $L_1^{w_2} = L_1$, and it is easy to show that $U_1^{w_2} = V_1$. We have $M_3 = P_1 \cap P_1^{w_2} = L_1$. The order of M_3 is equal to that of L_1, which is $q(q^2-1)(q-1)$, whence $|P_1 : M_3| = q^5$.

Specify a subgroup $P_1^{w_3}$, where $w_3 = w_{3p_1+p_2}$. We have $P_1^{w_3} = \langle H,\ X_{3p_1+2p_2},\ X_{-3p_1-p_2},\ X_{-2p_1-p_2},\ X_{p_1+p_2},\ X_{-p_1},\ X_{-3p_1-p_2},\ X_{p_2} \rangle$. This means that $M_4 = P_1 \cap P_1^{w_3} = \langle X_{p_2}, X_{p_1+p_2}, X_{3p_1+p_2} \rangle : H$. By Lemma 6, $\langle X_{p_2}, X_{p_1+p_2}, X_{3p_1+p_2} \rangle \simeq p^{3s}$. It follows that $|M_4| = q^3(q-1)^2$ and $|P_1 : M_4| = q^3(q+1)$.

Adding lengths of the three above-specified orbits and the length of the trivial orbit yields $1 + |P_1 : M_2| + |P_1 : M_3| + |P_1 : M_4| = |G : P_1|$. Hence, the rank of the representation of G on the cosets w.r.t. P_1 is equal to 4.

D. Representation of G on cosets w.r.t. P_2. We have $P_2 = \langle H,\ X_{p_1},\ X_{-p_1},\ X_{p_2},\ X_{p_1+p_2},\ X_{2p_1+p_2},\ X_{3p_1+p_2},\ X_{3p_1+2p_2} \rangle$. Conjugate P_2 by w_{p_2}. We obtain $P_2^{w_{p_2}} = \langle H,\ X_{p_1+p_2},\ X_{-p_1-p_2},\ X_{-p_2},\ X_{p_1},\ X_{2p_1+p_2},\ X_{3p_1+p_2},\ X_{3p_1+2p_2} \rangle$. It follows that $M_2' = P_2 \cap P_2^{w_{p_2}} = U_1 : H$. Therefore, $|P_1 : M_2| = |P_2 : M_2'|$. Since $(p_1, 3p_1+p_2) = 0$, we have $L_2^{w_{3p_1+2p_2}} = L_2$. Furthermore, $U_2^{w_{3p_1+2p_2}} = V_2$. Consequently, $M_3' = P_2 \cap P_2^{w_{3p_1+p_2}} = L_2$ and $|P_2 : M_3'| = |P_1 : M_3|$.

Consider the action of $w_{p_1+p_2}$ on P_2. We have $P_2^{w_{p_1+p_2}} = \langle H,\ X_{2p_1+p_2},\ X_{-2p_1-p_2},\ X_{-3p_1-2p_2},\ X_{-p_1-p_2},\ X_{p_1},\ X_{3p_1+p_2},\ X_{-p_2} \rangle$. Hence, $M_4' = P_2 \cap P_2^{w_{p_1+p_2}} = \langle X_{p_1},\ X_{2p_1+p_2},\ X_{3p_1+p_2} \rangle : H$. By Lemma 1, it follows that, for $q = 3^s$, we have $M_4' \simeq 3^{3s} : (q-1)^2$, and for $q = p^s$, $p \neq 3$, the group M_4' is isomorphic to $(p^s \cdot p^{2s}) : (q-1)^2$. In any case $|M_4'| = |M_4|$ and $|P_1 : M_4| = |P_2 : M_4'|$.

Thus, the rank of the representation of G on the cosets w.r.t. P_2 is equal to 4.

E. Group $G_2(4)$. Information on the group $G = G_2(4)$ is contained in [7, p. 97]. In particular, the subgroup of least index in G is a simple sporadic group J_2. The representation of G on cosets w.r.t. J_2 has degree $n = 416$, and its rank equals 3. The number of nontrivial orbits of $P = J_2$ in $\Omega = \{Px | x \in G\}$ is thus equal to 2.

In order to compute lengths of these orbits and determine forms of the corresponding double stabilizers M_2 and M_3, we make use of the data on maximal subgroups in the group, given in [7, p. 42]. We note that, up to conjugation, there are not more than three maximal subgroups in J_2 with indices lesser than $n - n_1 = 415$. Denote these subgroups by T_1, T_2, and T_3. Their indices are equal to, respectively, 100, 280, and 315. The index of any proper subgroup of T_i ($i = 1, 2, 3$) in J_2 is always greater than 415. Therefore, the double stabilizers of M_2 and M_3 belong to the set $\{T_1, T_2, T_3\}$. Comparing indices shows that $M_2 = T_1 \simeq U_3(3)$, $M_3 = T_3 \simeq (2 \cdot 2^4) : A_5$ and $n_2 = 100$, $n_3 = 315$.

F. Group $G_2(3)$. Maximal subgroups of $G_2(3)$ are listed in [7, p. 61]. There are, up to conjugation, two groups P_1 and P_2 with least index in $G_2(3)$. These are conjugate in $\mathrm{Aut}\, G$ and isomorphic to $U_3(3):2 \simeq G_2(2)$. The degree of the permutation representation of G on the cosets w.r.t. P_1 (or P_2) is equal to 351, and its rank is 3.

The group P_1 has, up to conjugation, four maximal subgroups (besides $U_3(3)$ of course); see [7, p. 14]. Denote these subgroups by T_1, T_2, T_3, and T_4. Their indices are equal to 28, 36, 63, and 63, respectively. Since the number $n_2 + n_3 = n - n_1 = 350$ is not divisible by 3, one of the two double stabilizers, say M_2, should lie in T_1. Then $n_2 = 28 \cdot k_2$, where k_2 is some natural number which is still undetermined. On the other hand, $n_2 + n_3$ is not divisible by 4, and so also n_3 is not. Therefore, M_3 lies either in T_3 or in T_4. Moreover, n_3, as well as n_2 and $n_2 + n_3$, will be divisible by 2. Hence, $n_3 = 63 \cdot 2 \cdot k_3$, where k_3 is some natural number. After cancellation, the equality $350 = 2 \cdot 5^2 \cdot 7 = 28 \cdot k_2 + 63 \cdot 2 \cdot k_3$ gives $25 = 2k_2 + 9k_3$. It follows that $k_3 = 1$ and $k_2 = 8$. Therefore, $n_2 = 2^5 \cdot 7 = 224$ and $n_3 = 2 \cdot 3^2 \cdot 7 = 126$. The structure of T_1 yields $M_2 \simeq (3 \cdot 3^2) : 2$. The tables of characters for groups $G_2(3)$ and $U_3(3):2$ allow us to determine orders of all elements in the group M_3, which in turn makes it possible to describe its structure, and namely $M_3 \simeq 2 \cdot (2^2 \times A_4) \simeq Q_8 *_{Z_2} SU_2(3)$.

THEOREM 1. For simple non-Abelian groups $G = G_2(q)$, $q > 2$, the parameters n, n_2, n_3, n_4, P, M_2, M_3, M_4 of minimal permutation representations are given in the following list.

in the case $q = 3$,

$n = 351$, $n_2 = 224$, $n_3 = 126$;

$P = U_3(3):2$, $M_2 = (3 \cdot 3^2):2$, $M_3 = 2 \cdot (2^2 \times A_4)$;

in the case $q = 4$,

$n = 416$, $n_2 = 100$, $n_3 = 315$;

$P = J_2$, $M_2 = U_3(3)$, $M_3 = (2 \cdot 2^4) : A_5$;

in the case $q > 4$,

$n = (q^6 - 1)/(q - 1)$, $n_2 = q(q + 1)$, $n_3 = q^5$, $n_4 = q^3(q + 1)$;

in addition, if $q = 2^s$, $s \geq 3$, then

$P = (2 \cdot 2^{4s}) : (A_1(q) \times (q - 1))$,

$M_2 = (2^{2s} \times 2^{3s}) : (q - 1)^2$,

$M_3 = A_1(q) \times (q - 1)$,

$M_4 = (2^s \cdot 2^{2s}) : (q - 1)^2$

or

$P = (2^{2s} \cdot 2^{3s}) : (A_1(q) \times (q - 1))$,

$M_2 = (2^s \cdot 2^{4s}) : (q - 1)^2$,

$M_3 = A_1(q) \times (q - 1)$,

$M_4 = 2^{3s} : (q - 1)^2$;

if $q = 3^s$, $s \geq 2$, then

$P = (3^s \cdot 3^{2s} \times 3^{2s}) : (2 \cdot (A_1(q) \times (q - 1)/2) \cdot 2)$,

$M_2 = (3^s \cdot 3^{2s} \times 3^{2s}) : (q - 1)^2$,

$M_3 = 2 \cdot (A_1(q) \times (q - 1)/2) \cdot 2$,

$M_4 = 3^{3s} : (q - 1)^2$;

if $q = p^s$, p is a prime, $p > 3$, then

$P = (p^{2s} \cdot (p^s \cdot p^{2s})) : (2 \cdot (A_1(q) \times (q - 1)/2) \cdot 2)$,

$M_2 = (p^s \cdot p^{4s}) : (q - 1)^2$,

$M_3 = 2 \cdot (A_1(q) \times (q-1)/2) \cdot 2,$
$M_4 = p^{3s} : (q-1)^2$

or

$P = (p^s \cdot p^{4s}) : (2 \cdot (A_1(q) \times (q-1)/2) \cdot 2),$
$M_2 = (p^s \cdot (p^s \cdot p^{2s})) : (q-1)^2,$
$M_3 = 2 \cdot (A_1(q) \times (q-1)/2) \cdot 2,$
$M_4 = (p^s \cdot p^{2s}) : (q-1)^2.$

For q equal to 2 or 3, the rank of the representation of $G_2(3)$ is equal to 3, and in all other cases to 4.

Remark. In the case where $q = 3$, there exists a graph automorphism of G sending P_1 to P_2, and so representations of G on the cosets w.r.t. P_1 and P_2 will be similar.

3. GROUP $F_4(q)$

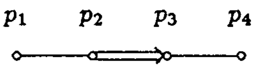
A. Algebra F_4. The rank of F_4 is 4. If e_1, e_2, e_3, e_4 form an orthonormal basis of the Euclidean space $\mathcal{K}_{\mathbb{R}}$, the system Π of simple roots for F_4 is defined as follows: $p_1 = e_1 - e_2$, $p_2 = e_2 - e_3$, $p_3 = e_3$, and $p_4 = \frac{1}{2}(e_4 - e_1 - e_2 - e_3)$. The system of positive roots is

$$\Phi^+ = \left\{ \begin{array}{l} e_i \pm e_j, \ i < j, \ i = 1,2; \ j = 2,3; \\ e_4 \pm e_j, \ j = 1,2,3; \\ e_i, \ i = 1,2,3,4; \\ \frac{1}{2}(e_4 \pm e_1 \pm e_2 \pm e_3). \end{array} \right\}, \quad |\Phi^+| = 24.$$

The matrix A has the form

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

The Dynkin diagram is



B. Group $F_4(q)$ and its parabolic subgroups of least index. From the main result stated in [4], it follows that a subgroup of least index in $G = F_4(q)$ is parabolic. There are, up to conjugation, four maximal parabolic subgroups in G: P_1, P_2, P_3, and P_4. Since the universal group $\tilde{F}_4(q)$ coincides with the adjoint group, that is, with G, By Lemma 6 and Proposition 1 we have

$|G| = q^{24} \cdot (q^{12}-1)(q^8-1)(q^6-1)(q^2-1),$
$|P_1| = |P_4| = q^{24} \cdot (q^6-1)(q^4-1)(q^2-1)(q-1),$
$|P_2| = |P_3| = q^{24} \cdot (q^3-1)(q^2-1)(q^2-1)(q-1).$

Obviously, $|G : P_1| = |G : P_4| = (q^{12}-1)(q^4+1)/(q-1) > |G : P_2| = |G : P_3|$. Thus, the minimal permutation representation of G is isomorphic to its representation on the cosets either w.r.t. P_1 or w.r.t. P_4.

We describe the structure of subgroups P_1 and P_4. By Lemma 5, $P_i = U_i : L_i$, $i = 1, 4$. As in the case of $G_2(q)$, apply the Chevalley commutator formula (Lemma 1) to determine the structure of U_1 and U_4. (Computations are omitted.) We have:

If $q = 2^s$, then $U_1 \simeq U_4 \simeq 2^s \cdot 2^{8s} \times 2^{6s}$.

If $q = p^s$, $p \neq 2$, then $U_1 \simeq p^s \cdot p^{14s}$, $U_4 \simeq p^{7s} : p^{8s}$.

Now consider the group $L_1 = \langle H, X_r \mid r \in \Phi_1 \rangle$. Denote by H_1 a subgroup $\langle h_{p_i}(\lambda) \mid i = 2,3,4;\ \lambda \in K^* \rangle$ and consider the subgroup $L_1' = \langle H_1, X_r \mid r \in \Phi_1 \rangle$ in L_1. Obviously, L_1' is isomorphic to the universal Chevalley group $\bar{C}_3(q)$, and $L_1 = L_1' \cdot \langle h \rangle$, where $h \in H \backslash H_1$. The element $r_0 = e_1 + e_4 = 2p_1 + 3p_2 + 4p_3 + 2p_4$ satisfies the condition $(r_0, p_i) = 0$ for $i = 2,3,4$. Therefore, $h_{r_0}(\lambda)$ centralizes L_1' for any $\lambda \in K^*$. On the other hand, $h_{r_0}(\lambda) = h_{p_1}^2(\lambda) \cdot h_{p_2}^3(\lambda) \cdot h_{p_3}^2(\lambda) h_{p_4}(\lambda)$, and if $\lambda^2 = 1$, then $h_{r_0}(\lambda)$ lies in the center of the subgroup L_1'. Thus, $L_1 \simeq d \cdot (C_3(q) \times (q-1)/d) \cdot d$, where $d = (2, q-1)$.

Similarly, $L_4 = L_4' \cdot \langle h \rangle$, where $h \in H \backslash H_4$, $H_4 = \langle h_{p_i}(\lambda) \mid i = 1,2,3;\ \lambda \in K^* \rangle$, $L_4' = \langle H_4, X_r \mid r \in \Phi_4 \rangle \simeq \bar{B}_3(q)$. The element $s_0 = e_4 = p_1 + 2p_2 + 3p_3 + 2p_4$ satisfies $(s_0, p_i) = 0$ for $i = 1,2,3$. Therefore, $h_{s_0}(\lambda)$ centralizes L_4' for any $\lambda \in K^*$. In addition, if $\lambda^2 = 1$, the element $h_{s_0}(\lambda) = h_{p_1}^2(\lambda) h_{p_2}^4(\lambda) h_{p_3}^3(\lambda) h_{p_4}^2(\lambda)$ lies in the center of the subgroup L_4'. Eventually, $L_4 \simeq d \cdot (B_3(q) \times (q-1)/d) \cdot d$, where $d = (2, q-1)$.

C. Representation of G on cosets w.r.t. P_1. As was already noted (see 'C' in Sec. 2), the action of an element $n \in N$ on P_1 is determined by the action of its image $w \in W$ on Φ. We need to adopt the following notation: $\Phi_{1,2} = \Phi_1 \cap \Phi_2 = \Phi_{\{p_3, p_4\}}$ and $\Phi_{1,4} = \Phi_1 \cap \Phi_4 = \Phi_{\{p_2, p_3\}}$.

The action of w_{p_1} on Φ is shown in the following:

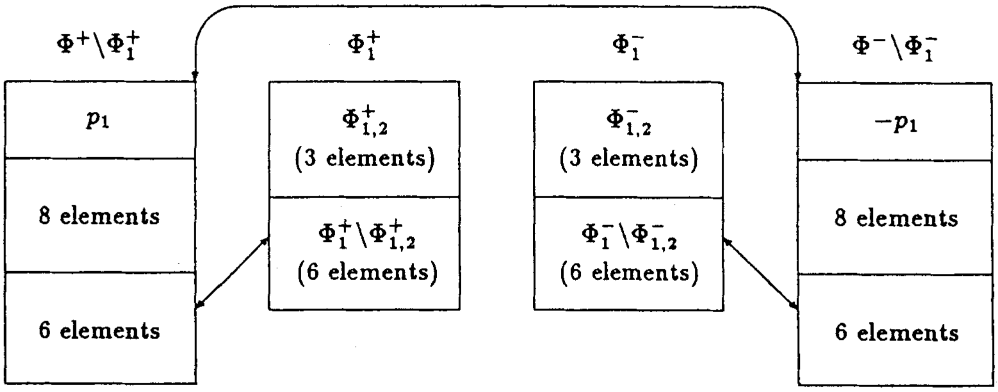


Diagram 1

The arrow pointing from the set X to the set Y says that $w_{p_1}(X) = Y$; the absence of arrow outgoing from Z indicates that $w_{p_1}(Z) = Z$.

Now we are in a position to determine the structure of the double stabilizer $M_2 = P_1 \cap P_1^{w_{p_1}} = (U_1 \cap P_1^{w_{p_1}}) : (L_1 \cap P_1^{w_{p_1}})$. From Diagram 1, we can see that the group $U_1 \cap P_1^{w_{p_1}}$ has order q^{14}. Applying Lemma 1, for this group we derive the following:

If $q = 2^s$, then $U_1 \cap P_1^{w_{p_1}} \simeq 2^s \cdot 2^{6s} \times 2^{7s}$.

If $q = p^s$, $p \neq 2$, then $U_1 \cap P_1^{w_{p_1}} \simeq p^s \cdot p^{12s} \times p^s$.

Using the Chevalley formula from Lemma 1 shows that the subgroup $U_{1,2} = \langle X_r \mid r \in \Phi_1^+ \backslash \Phi_{1,2}^+ \rangle$ of L_1 is isomorphic to p^{6s}. Let $H_{1,2} = \langle h_{p_i}(\lambda) \mid i = 3,4;\ \lambda \in K^* \rangle$, $L_{1,2}' = \langle H_{1,2}, X_r \mid r \in \Phi_{1,2} \rangle \simeq \bar{A}_2(q) \simeq SL_3(q)$. Then $L_1 \cap P_1^{w_{p_1}} = U_{1,2} : L_{1,2}' \cdot H'$, where the order of H' is $(q-1)^2$. The elements $h_{p_1}(\lambda)$ and $h_{r_0}(\lambda)$ centralize $L_{1,2}'$, and so $h_{p_1}^{-2}(\lambda) \cdot h_{r_0}(\lambda)$ also does. On the other hand, if $\lambda^3 = 1$, this element lies in the center of $L_{1,2}'$. Hence, $L_1 \cap P_1^{w_{p_1}} \simeq p^{6s} : (d' \cdot (A_2(q) \times (q-1)/d') \times (q-1)) \cdot d'$, where $d' = (3, q-1)$.

For $q = 2^s$, we have $M_2 = (2^s \cdot 2^{6s} \times 2^{7s}) : (2^{6s} : (d' \cdot (A_2(q) \times (q-1)/d') \times (q-1)) \cdot d')$.

For $q = p^s$, $p \neq 2$, we have $M_2 = (p^s \cdot p^{12s} \times p^s) : (p^{6s} : (d' \cdot (A_2(q) \times (q-1)/d') \times (q-1)) \cdot d')$.

In any case $|P_1 : M_2| = q \cdot (q^4 - 1)(q^3 + 1)/(q - 1)$.

For $r_0 = e_1 + e_4$, as mentioned above, the equalities $(r_0, p_i) = 0$ hold with $i = 2, 3, 4$. Therefore, a diagram for the reflection w_{r_0} has the form

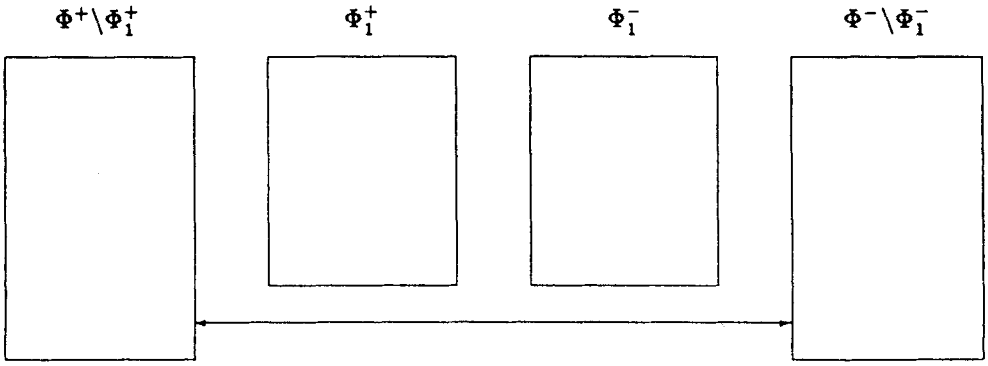


Diagram 2

In this way $M_3 = P_1 \cap P_1^{w_{r_0}} = L_1$. Consequently, $|P_1 : M_3| = q^{15}$.

Now consider the reflection w_{s_0}, where $s_0 = e_4$. We know that $(s_0, p_i) = 0$ for $i = 1, 2, 3$. Therefore, $w_{s_0}(\Phi_4) = \Phi_4$ and $w_{s_0}(\Phi^+ \backslash \Phi_4^+) = \Phi^- \backslash \Phi_4^-$. In addition, $|\Phi_1 \cap \Phi_4| = |\Phi_{1,4}| = 4$. The diagram for w_{s_0} is this:

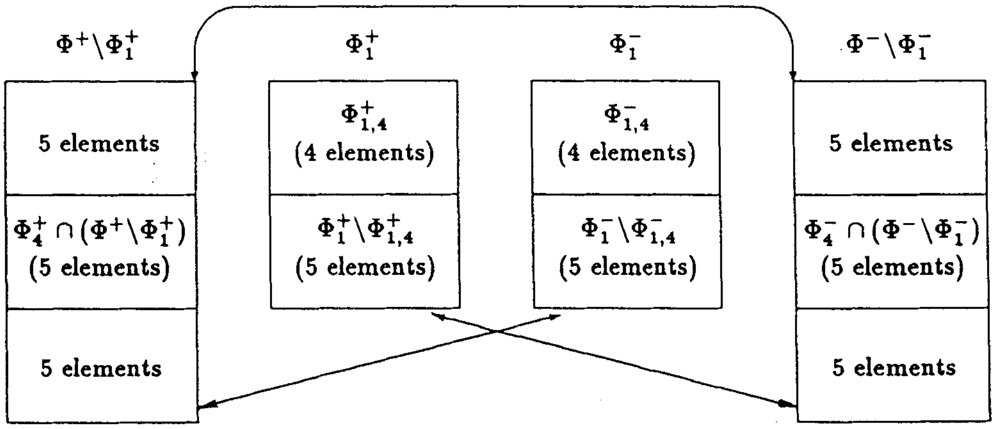


Diagram 3

From Diagram 3, we see that $|U_1 \cap P_1^{w_{s_0}}| = q^{10}$. For $q = 2^s$, we have $U_1 \cap P_1^{w_{s_0}} \simeq 2^{10s}$, and for $q = p^s$, $p \neq 2$, $U_1 \cap P_1^{w_{s_0}} \simeq p^s \cdot p^{4s} \times p^{5s}$. The subgroup $V_{1,4} = \langle X_r |\ r \in \Phi_1^- \backslash \Phi_{1,4}^- \rangle$ of $L_1 \cap P_1^{w_{s_0}}$ is isomorphic to 2^{5s} for $q = 2^s$, and to $p^s \cdot p^{4s}$ for $q = p^s$, $p \neq 2$. Let $H_{1,4} = \langle h_{p_i}(\lambda) |\ i = 2, 3;\ \lambda \in K^* \rangle$ and $L'_{1,4} = \langle H_{1,4}, X_r |\ r \in \Phi_{1,4} \rangle \simeq \bar{B}_2(q) \simeq \bar{C}_2(q) \simeq \mathrm{Sp}_4(q)$. The elements $h_{s_0}(\lambda) = h_{p_1}^2(\lambda) \cdot h_{p_4}^4(\lambda) \cdot h_{p_3}^3(\lambda) h_{p_4}(\lambda)$ and $h_{e_4 - e_1}(\lambda) = h_{p_2}(\lambda) h_{p_3}(\lambda) h_{p_4}(\lambda)$ centralize $L'_{1,4}$. For $\lambda^2 = 1$, in addition, $h_{s_0}(\lambda)$ lies in the center of $L'_{1,4}$. For $q = 2^s$, we therefore have $M_4 \simeq 2^{10s} : (2^{5s} : ((B_2(q) \times (q-1)) \times (q-1)))$. For $q = p^s$, $p \neq 2$, $M_4 \simeq (p^s \cdot p^{4s} \times p^{5s}) : ((p^s \cdot p^{4s}) : (2 \cdot (B_2(q) \times (q-1)/2) \times (q-1)/2) \times (q_2 - 1)) \cdot 2)$. Hence, $|P_1 : M_4| = q^5(q^6 - 1)/(q - 1)$.

Consider the element $w_0 = w_{r_0} \cdot w_{p_1}$. We know how the reflections w_{p_1} and w_{r_0} act on Φ. Using Diagrams 1 and 2, it is therefore not hard to understand how their composition acts on Φ. The diagram is as follows:

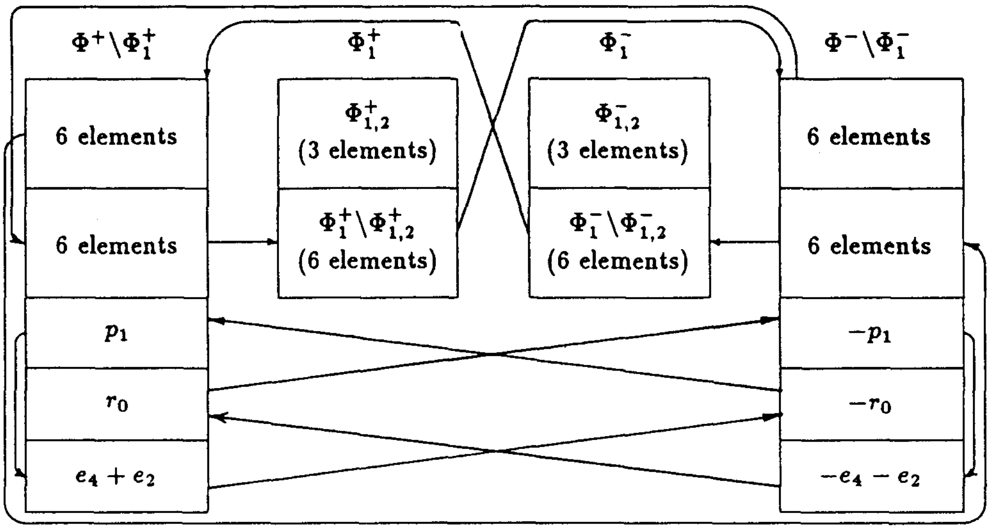


Diagram 4

For $q = p^s$, we have the double stabilizer $M_5 = P_1 \cap P_1^{w_0} \simeq p^{7s} : (p^{6s} : (d' \cdot (A_2(q) \times (q-1)/d') \times (q-1)) \cdot d')$, where $d' = (3, q-1)$. The index of M_5 in P_1 is equal to $q^8(q^4-1)(q^3+1)/(q-1)$.

We have $\sum_{i=1}^{5} |P_1 : M_i| = |G : P_1|$. Therefore, the rank of the permutation representation of G on the cosets w.r.t. P_1 is equal to 5.

D. Representation of G on cosets w.r.t. P_4. A description of this representation is similar to that of G on cosets w.r.t. P_1. In this way, replacing Φ_1 by Φ_4 and $\Phi_{1,2}$ by $\Phi_{3,4}$ in Diagram 1 yields a diagram showing the action of the reflection w_{p_4} on Φ.

Let $H_{3,4} = \langle h_{p_i}(\lambda) | \; i = 1, 2; \; \lambda \in K^* \rangle$ and $L'_{3,4} = \langle H_{3,4}, X_r | \; r \in \Phi_{3,4} \rangle \simeq \bar{A}_2(q) \simeq SL_3(q)$. Then the elements $h_{s_0}(\lambda)$ and $h_{p_4}(\lambda)$ centralize $L'_{3,4}$. Therefore, $h_{s_0}(\lambda) \cdot h_{p_4}^{-2}(\lambda)$, lying in the center of $L'_{3,4}$ for $\lambda^3 = 1$, also centralizes $L'_{3,4}$. Thus, if $q = 2^s$, we have $M'_2 = P_4 \cap P_4^{w_{p_4}} \simeq (2^s \cdot 2^{6s} \times 2^{7s}) : (2^{6s} : (d' \cdot (A_2(q) \times (q-1)/d') \times (q-1)) \cdot d')$, and if $q = p^s$, $p \neq 2$, $M'_2 \simeq (p^{7s} \cdot p^{7s}) : ((p^{3s} \cdot p^{3s}) : (d' \cdot (A_2(q) \times (q-1)/d') \times (q-1)) \cdot d')$, where $d' = (3, q-1)$. It follows that $|P_4 : M'_2| = |P_1 : M_2|$.

If we take an element w_{s_0} in place of w_{r_0} we obtain $M'_3 = P_4 \cap P_4^{w_{s_0}} = L_4$ and $|P_4 : M'_3| = |P_1 : M_3|$. Replacing Φ_1 by Φ_4 and Φ_4 by Φ_1 in Diagram 3 yields a diagram which shows the action of w_{r_0} on $\Phi_4 \cup \Phi^+$. Applying Lemma 1, we see that for $q = 2^s$, $M'_4 \simeq 2^{10s} : (2^{5s} : ((B_2(q) \times (q-1)) \times (q-1)))$, and for $q = p^s$, $p \neq 2$, $M'_4 \simeq (p^s \cdot p^{4s} \times p^{5s}) : (p^{5s} : (2 \cdot (B_2(q) \times (q-1)/2) \times (q-1)) \cdot 2)$. The index $|P_4 : M'_4|$ is equal to $|P_1 : M_4|$.

The element $w'_0 = w_{s_0} w_{p_4}$ acts on Φ as is shown in Diagram 4, with Φ_1 replaced by Φ_4 and $\Phi_{1,2}$ replaced by $\Phi_{3,4}$. Therefore, for $q = 2^s$ we have $M'_5 \simeq 2^{7s} : (2^{6s} : (d' \cdot (A_2(q) \times (q-1)/d') \times (q-1)) \cdot d')$, and for $q = p^s$, $p \neq 2$, $M'_5 \simeq (p^{3s} \cdot p^{3s} \times p^s) : ((p^{3s} \cdot p^{3s}) : (d' \cdot (A_2(q) \times (q-1)/d') \times (q-1)) \cdot d')$, where $d' = (3, q-1)$.

Obviously, $|P_4 : M'_5| = |P_1 : M_5|$.

The rank of the representation of G on the cosets w.r.t. P_4 is equal to 5, as is one w.r.t. P_1,.

THEOREM 2. For simple non-Abelian groups $G = F_4(q)$, the parameters n, n_2, n_3, n_4, n_5, P, M_2, M_3, M_4, M_5 of minimal permutation representations are contained in the following list:

$$n = \frac{(q^{12}-1)(q^4+1)}{q-1}, \quad n_2 = \frac{(q^4-1)(q^3+1)}{q-1}, \quad n_3 = q^{15},$$

$$n_4 = q^5 \cdot \frac{q^6-1}{q-1}, \quad n_5 = q^8 \cdot \frac{(q^4-1)(q^3+1)}{q-1};$$

in addition, if $q = 2^s$, then

$P = (2^s \cdot 2^{8s} \times 2^{6s}) : (C_3(q) \times (q-1))$,

$M_2 = (2^s \cdot 2^{6s} \times 2^{7s}) : (2^{6s} : (d' \cdot (A_2(q) \times (q-1)/d') \times (q-1)) \cdot d')$,

$M_3 = C_3(q) \times (q-1)$,

$M_4 = 2^{10s} : (2^{5s} : ((B_2(q) \times (q-1)) \times (q-1)))$,

$M_5 = 2^{7s} : (2^{6s} : (d' \cdot (A_2(q) \times (q-1)/d') \times (q-1)) \cdot d')$;

if $q = p^s$, $p \neq 2$, p is a prime, then

$P = (p^s \cdot p^{14s}) : (2 \cdot (C_3(q) \times (q-1)/2) \cdot 2)$,

$M_2 = (p^s \cdot p^{12s} \times p^s) : (p^{6s} \cdot (d' \cdot (A_2(q) \times (q-1)/d') \times (q-1)) \cdot d')$,

$M_3 = 2 \cdot (C_3(q) \times (q-1)/2) \cdot 2$,

$M_4 = (p^s \cdot p^{4s} \times p^{5s}) : ((p^s \cdot p^{4s}) : (2 \cdot (B_2(q) \times (q-1)/2) \times (q-1)) \cdot 2)$,

$M_5 = p^{7s} : (p^{6s} : (d' \cdot (A_2(q) \times (q-1)/d') \times (q-1)) \cdot d')$

or

$P = (p^{7s} \cdot p^{8s}) : (2 \cdot (B_3(q) \times (q-1)/2) \cdot 2)$,

$M_2 = (p^{7s} \cdot p^{7s}) : ((p^{3s} \cdot p^{3s}) : (d' \cdot (A_2(q) \times (q-1)/d') \times (q-1)) \cdot d')$,

$M_3 = 2 \cdot (B_3(q) \times (q-1)/2) \cdot 2$,

$M_4 = (p^s \cdot p^{4s} \times p^{5s}) : (p^{5s} : (2 \cdot (B_2(q) \times (q-1)/2) \times (q-1)) \cdot 2)$,

$M_5 = (p^{3s} \cdot p^{3s} \times p^s) : ((p^{3s} \cdot p^{3s}) : (d' \cdot (A_2(q) \times (q-1)/d') \times (q-1)) \cdot d')$, where $d' = (3, q-1)$.

The rank of representation is equal to 5 in all cases.

Remark. In the statement of the theorem, we made no distinction between the representations of G w.r.t. subgroups P_1 and P_4 for $q = 2^s$: such are similar because P_1 and P_4 are conjugate in $\operatorname{Aut} G$.

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