Siberian Mathematical Journal, Vol. 46, *No.* 3, *pp.* 396–404, 2005
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ON CONNECTION BETWEEN THE STRUCTURE OF A FINITE GROUP AND THE PROPERTIES OF ITS PRIME GRAPH

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UDC 512.542

Abstract: It is shown that the condition of nonadjacency of 2 and at least one odd prime in the Gruenberg–Kegel graph of a finite group G under some natural additional conditions suffices to describe the structure of G; in particular, to prove that G has a unique nonabelian composition factor. Applications of this result to the problem of recognition of finite groups by spectrum are also considered.

Keywords: finite group, finite simple group, prime graph of a finite group, spectrum of a group, recognition by spectrum

Introduction

Let G be a finite group, let $\pi(G)$ be the set of prime divisors of the order of G, and let $\omega(G)$ be the spectrum of G; i.e., the set of element orders of G. The Gruenberg–Kegel graph (or prime graph) $GK(G)$ is defined as follows: The vertex set of this graph is $\pi(G)$. The primes p and q, considered as vertices of $GK(G)$, are adjacent by edge (briefly, adjacent) if and only if G contains an element of order pq. Denote by $s(G)$ the number of connected components of $GK(G)$ and by $\pi_i(G)$, $i = 1, \dots, s(G)$, its ith connected component. If G has even order then put $2 \in \pi_1(G)$. Denote by $\omega_i(G)$ the set of $n \in \omega(G)$ such that every prime divisor of n belongs to $\pi_i(G)$. Obviously, the graph $GK(G)$ is uniquely determined from $\omega(G)$.

Gruenberg and Kegel gave the following description for finite groups with disconnected prime graph.

Theorem (Gruenberg–Kegel). *If a finite group G has the disconnected prime graph $GK(G)$ then one of the following statements holds:*

(a) *$s(G) = 2$ and G is a Frobenius group;*

(b) *$s(G) = 2$ and G is a 2-Frobenius group, i.e., $G = ABC$, where A and AB are normal subgroups of G; AB and BC are Frobenius groups with kernels A, B and complements B, C respectively;*

(c) *there exists a nonabelian simple group S such that $S \le \overline{G} = G/K \le \mathrm{Aut}(S)$, where K is the maximal normal soluble subgroup of G; furthermore, K and $\overline{G}/S$ are $\pi_1(G)$-groups, the graph $GK(S)$ is disconnected, $s(S) \ge s(G)$, and for every i, $2 \le i \le s(G)$, there is j, $2 \le j \le s(S)$, such that $\omega_i(G) = \omega_j(S)$.*

This deep result was firstly published in [1] (we give here a refined statement of the theorem as in [2]). Together with the full classification of finite simple groups with disconnected prime graph which was obtained by Williams and Kondrat′ev (see [1, 3]), this result gave rise to a series of important corollaries (for example, see [1, Theorems 3–6; 3, Theorems 2 and 3]). In the recent years this theorem has been permanently used for recognition of finite groups by spectrum (this will be discussed in detail in § 2 of the present article).

The author was supported by the Russian Foundation for Basic Research (Grant 05–01–00797), the State Maintenance Program for the Leading Scientific Schools of the Russian Federation (Grant NSh–2069.2003.1), the Program "Development of the Scientific Potential of Higher School" of the Ministry for Education of the Russian Federation (Grant 8294), the Program "Universities of Russia" (Grant UR.04.01.202), and a grant of the Presidium of the Siberian Branch of the Russian Academy of Sciences (No. 86–197).

Novosibirsk. Translated from *Sibirskiĭ Matematicheskiĭ Zhurnal*, Vol. 46, No. 3, pp. 511–522, May–June, 2005.
Original article submitted November 29, 2004.

The proof of the Gruenberg–Kegel Theorem relies substantially upon the fact that G contains an element of odd prime order which is disconnected with 2 in $GK(G)$. It turns out that disconnectedness could be successfully replaced in most cases by a weaker condition for the prime 2 to be nonadjacent to at least one odd prime.

Denote by $t(G)$ the maximal number of primes in $\pi(G)$ pairwise nonadjacent in $GK(G)$. In other words, if $\rho(G)$ is some independent set with the maximal number of vertices in $GK(G)$ (the subset of vertices of a graph is called an *independent set* if its vertices are pairwise nonadjacent) then $t(G) = |\rho(G)|$. In graph theory this number is called the *independence number* of a graph. By analogy we denote by $t(2, G)$ the maximal number of vertices in the independent sets of $GK(G)$ containing 2. We call this number the 2-*independence number*. The main purpose of this article is to prove the following statement which can be applied to a wide class of finite groups including the groups with connected Gruenberg–Kegel graph.

Theorem. *Let G be a finite group satisfying the two conditions:*

(a) *there exist three primes in $\pi(G)$ pairwise nonadjacent in $GK(G)$; i.e., $t(G) \geq 3$;*

(b) *there exists an odd prime in $\pi(G)$ nonadjacent in $GK(G)$ to the prime 2; i.e., $t(2, G) \geq 2$.*

Then there is a finite nonabelian simple group S such that $S \leq \overline{G} = G/K \leq \mathrm{Aut}(S)$ for the maximal normal soluble subgroup K of G. Furthermore, $t(S) \geq t(G) - 1$, and one of the following statements holds:

(1) *$S \simeq A_7$ or $L_2(q)$ for some odd q, and $t(S) = t(2, S) = 3$.*

(2) *For every prime $p \in \pi(G)$ nonadjacent to 2 in $GK(G)$ a Sylow p-subgroup of G is isomorphic to a Sylow p-subgroup of S. In particular, $t(2, S) \geq t(2, G)$.*

Note that condition (a) implies an insolubility of G (see Proposition 1 below), and so by the Feit–Thompson Theorem it is not necessary to assume in the hypotheses of the theorem that G is of even order. Moreover, it turns out that condition (a) can be replaced by a weaker condition of insolubility of G without any modification in the claim of the theorem (see Propositions 2 and 3 below).

The paper consists of two sections. The first presents a proof of the main theorem. In the second we consider some applications of the obtained result to the problem of recognition of finite groups by spectrum.

§ 1. Proof of the Theorem

We start with a proof of a simple lemma to be repeatedly used in the course of this section.

Lemma 1.1. *Let a finite group G have a normal series of subgroups $1 \leq K \leq M \leq G$, and the primes p, q and r are such that p divides $|K|$, q divides $|M/K|$, and r divides $|G/M|$. Then p, q, and r cannot be pairwise nonadjacent in $GK(G)$.*

PROOF. Let G be a minimal counterexample (by order) to the claim of the lemma.

Consider a Sylow p-subgroup P of the group K and its normalizer $N = N_G(P)$ in G. By the Frattini argument $G = KN$ and $N/(N \cap K) \simeq G/K$. Therefore, N has the normal series of subgroups $1 \leq P \leq N \cap M \leq N$, and the factors of this series satisfy the conditions of the lemma. Since the primes p, q, and r are pairwise nonadjacent in $GK(G)$, they are pairwise nonadjacent in $GK(N)$. Since G is a minimal counterexample, we may assume that $N = G$ and $P \lhd G$.

We now consider the factor group $\overline{G} = G/P$. Put $\overline{M} = M/P$, let $\overline{Q}$ be a Sylow q-subgroup in $\overline{M}$, and let $\overline{N} = N_{\overline{G}}(\overline{Q})$ be the normalizer of $\overline{Q}$ in $\overline{G}$. By the Frattini argument we again obtain $\overline{N}/(\overline{N} \cap \overline{M}) \simeq \overline{G}/\overline{M}$ and so $|\overline{N}/\overline{Q}|$ is divided by r. Denote by Q and N the preimages of $\overline{Q}$ and $\overline{N}$ in G. Then N has the normal series of subgroups $1 \leq P \leq Q \leq N$, and the factors of this series again satisfy the conditions of the lemma. Hence $N = G$, and $Q \lhd G$.

Let x be an element of order r in G. Since G does not contain elements of order pr and qr, the element x induces a fixed-point-free automorphism of order r of the group Q. By the Thompson Theorem (see [4]) Q is nilpotent. Therefore, it contains an element of order pq; a contradiction. The lemma is proved.

The theorem is a direct corollary of the three propositions below.

Proposition 1. *Let G be a finite group, $t(G) \geq 3$, and let K be the maximal normal soluble subgroup of G. Then for every subset ρ of primes in $\pi(G)$ such that $|\rho| \geq 3$ and all primes in ρ are pairwise nonadjacent in $GK(G)$, the intersection $\pi(K) \cap \rho$ contains at most one number. In particular, G is insoluble.*

Proof. Let p, q, and r be three pairwise nonadjacent primes in ρ. Assume that p and q divide the order of K. Consider a subgroup R of G generated by the subgroups K and S_r, where S_r is a Sylow r-subgroup of G. Note that it is immaterial whether K includes S_r or not. Anyway, R is a soluble group. Then S_r obviously has a normal series with factors satisfying the conditions of Lemma 1.1; a contradiction. Thus, at most one prime in ρ can divide the order of K. In particular, K is a proper subgroup of G. Therefore, G is insoluble. The proposition is proved.

Note that under the conditions of Proposition 1 the insolubility of G is easy to prove using the Gruenberg–Kegel Theorem.

Proposition 2. *Let G be a finite insoluble group and $t(2, G) \geq 2$. Then there is a finite nonabelian simple group S such that $S \leq \overline{G} = G/K \leq \mathrm{Aut}(S)$ for the maximal normal soluble subgroup K of G. Furthermore, $t(S) \geq t(G) - 1$, and one of the following statements holds:*

(1) *$S \simeq A_7$ or $L_2(q)$ for some odd q, and $t(S) = t(2, S) = 3$.*

(2) *For every prime $p \in \pi(G)$ nonadjacent to 2 in $GK(G)$ a Sylow p-subgroup of G is isomorphic to a Sylow p-subgroup of S. In particular, $t(2, S) \geq t(2, G)$.*

Proof. Since G is insoluble, its maximal normal soluble subgroup K does not coincide with G. Let p be an odd prime nonadjacent to 2 in $GK(G)$.

Suppose that p divides the order of K. Consider the factor group $\widehat{G} = G/O_{p'}(K)$ and its subgroup $\widehat{K} = K/O_{p'}(K)$. Since $O_{p'}(K) \neq K$, the subgroup $O_p(\widehat{K})$ is nontrivial. Furthermore, the insolubility of G implies that a Sylow 2-subgroup S_2 of $\widehat{G}$ is nontrivial. Since p is nonadjacent to 2, the action of S_2 by conjugation on $O_p(\widehat{K})$ is faithful and free; i.e., every nontrivial element of S_2 fixes only the identity element of $O_p(\widehat{K})$. Hence the subgroup $\langle O_p(\widehat{K}), S_2\rangle$ of $\widehat{G}$ is a Frobenius group with kernel $O_p(\widehat{K})$ and complement S_2. Then S_2 is either a cyclic group or generalized quaternion group (for example, see [5, Theorem 10.3.1(iv)]). However, $\widehat{G}$ is insoluble. Therefore, S_2 is a generalized quaternion group. The Brauer–Suzuki Theorem (see [6]) implies that a Sylow 2-subgroup of $\overline{G} = G/K \simeq \widehat{G}/\widehat{K}$ is a dihedral group. All finite groups with dihedral Sylow 2-subgroups were described by Gorenstein and Walter (see [7, 8]). By this description the group $\overline{G}$ has a unique nonabelian composition factor S isomorphic to A_7 or $L_2(q)$ with q odd. It easy to check that $t(S) = t(2, S) = 3$. Thus, if p divides the order of K then item (1) of the proposition holds true.

Let p divide the order of $\overline{G}$, and do not divide the order of K. If we denote by $L = S_1 \times \dots \times S_m$ the socle of $\overline{G}$, where S_i are nonabelian simple groups, then $\overline{G} \leq \mathrm{Aut}(L)$.

Suppose $m \geq 2$. If p divides $|L|$ then there exists k such that $p \in \omega(S_k)$. On the other hand, the order of every group S_i, $i = 1, \dots, m$, is even. Since $m \geq 2$, L contains an element of order $2p$; a contradiction. Thus, we may assume that p divides the order of $\overline{G}/L$, and does not divide the order of L. Let $\varphi \in \overline{G}$ be an automorphism of L of order p and $P = S_1^{\varphi}$. Since P is simple, its every natural projection P_i to S_i, $i = 1, \dots, m$, is either trivial or isomorphic to S_1. On the other hand, since P is normal in L, every subgroup P_i, $i = 1, \dots, m$, is also normal. Hence $P_i = 1$ or $P_i = S_i$. Therefore, there is a unique $j \in \{1, \dots, m\}$ such that $S_1^{\varphi} = S_j$. If $j \neq 1$ then there arises a φ-orbit Δ of length p which consists of subgroups isomorphic to S_1. Without loss of generality we may assume that $\Delta = \{S_1, S_2, \dots, S_p\}$. Let a_1 be an involution in S_1 and $a_i = a_1^{\varphi^i}$. Let g be the element of L whose projections g_i to S_i are defined as follows: $g_i = a_i$ for $i = 1, \dots, p$ and $g_i = 1$ otherwise. Then g is an involution and the element $g\varphi \in \overline{G}$ is of order $2p$; a contradiction. Thus, $S_1^{\varphi} = S_1$, and the same is true for every S_i, $i = 1, \dots, m$. Since $\varphi \neq 1$, φ acts nontrivially on some S_k. Then φ induces an outer automorphism of S_k of order p.

The next lemma, whose proof uses the classification of finite simple groups, completes the proof of the proposition.

Lemma 1.2. *Let G be a finite almost simple group; i.e., there exists a nonabelian simple group S such that $S \leq G \leq \mathrm{Aut}(S)$. Let p be an odd prime nonadjacent to 2 in $GK(G)$. Then p does not divide the order of G/S.*

PROOF OF THE LEMMA. By the classification theorem S is a sporadic or alternating or Lie type group. If S is a sporadic or alternating group then the order of $\mathrm{Out}(S)$ is not divided by an odd number. So we may assume that S is a Lie type group. The same argument implies that S is not isomorphic to the Tits group ${}^2F_4(2)'$.

Denote the group $\mathrm{Aut}(S)$ by A. The group A has the normal series of subgroups $S \leq \widehat{S} \leq \widehat{A} \leq A$, such that the factor $\widehat{S}/S$ is isomorphic to the group of diagonal automorphisms, the factor $\widehat{A}/\widehat{S}$ is isomorphic to the group of field automorphisms, and the factor $A/\widehat{A}$ is isomorphic to the group of graph automorphisms of S (see [9, Propositions 3.3–3.6]).

Let G be a minimal counterexample to the statement of the lemma. Then $|G/S| = p$ and $G = \langle S, \varphi\rangle$. Here φ is an element of A not lying in S, and the order of its image $\overline{\varphi}$ in the factor group G/S is equal to p. By [9, Proposition 3.2] each automorphism of a finite simple group of Lie type is a product of inner, diagonal, field and graph automorphisms. So without loss of generality we may assume that $\varphi = \delta\theta\gamma$ where δ is a diagonal automorphism, θ is a field automorphism, and γ is a graph automorphism of S.

Suppose $G \not\leq \widehat{A}$. Then $p = 3$ and S is of type D_4. However, a group of type D_4 over every finite field contains an element of order 6; a contradiction. Thus, $G \leq \widehat{A}$ and $\varphi = \delta\theta$.

We first consider the following possibility: $S = A_{n-1}(q) \simeq L_n(q)$ or ${}^2A_{n-1}(q) \simeq U_n(q)$, $q = r^k$ for some prime r, and $n \geq 3$. The action of δ on S by conjugation can be represented as follows: Let A be a matrix in $SL_n(q)$ ($SU_n(q)$) and let a be the natural image of A in $L_n(q)$ ($U_n(q)$). Then a^δ is the image of A^D, where

$$D = \begin{pmatrix} E & 0 \\ 0 & \lambda \end{pmatrix},$$

E is the identity matrix of size $(n-1) \times (n-1)$ and λ is an element of order $|\delta|$ from the multiplicative group of the base field.

Let

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & I \end{pmatrix},$$

where I is the matrix of size $(n-2) \times (n-2)$ such that I is the identity matrix for even q, and $I = \mathrm{diag}\{1, \dots, 1, -1\}$ for odd q. The matrix T lies in $SL_n(q)$ ($SU_n(q)$) and its image t in $L_n(q)$ ($U_n(q)$) is an involution. Obviously, the matrix D centralizes T, and so δ centralizes t. On the other hand, T lies in $SL_n(r)$ ($SU_n(r)$), and so t lies in the centralizer of every field automorphism of S. Thus, $t^\varphi = t^{\delta\theta} = t$, and the order of element $t\varphi$ of G is divided by $2p$; a contradiction.

Return now to the common situation. Suppose $\delta = 1$. Then $\varphi = \theta$ and $|\theta| = p$. Since a group ${}^3D_4(q)$ contains an element of order 6, we may assume that $p \neq 3$ if $S = {}^3D_4(q)$. Let a group S over the field of order q admit a field automorphism of order p. Then S includes a subgroup S_0 that is isomorphic to the group of the same Lie type as S, over the field of order q_0, where $q = q_0^p$. The automorphism θ centralizes S_0 (or a subgroup conjugate to it in S). Therefore, G contains an element of order $2p$, which is impossible. So we may assume that $\delta \neq 1$.

If p divides $|\widehat{S}/S|$ then S is of type E_6 or ${}^2E_6$ and $p = 3$ (S cannot be a linear or unitary group as it has been shown above). Since the groups of types E_6 and ${}^2E_6$ over every finite field contain an element of order 6, we obtain a contradiction. Thus, we may assume that p does not divide the order of $\widehat{S}/G$ and $|\widehat{A}/\widehat{S}|$ is divided by p. Therefore, φ is a product of a nontrivial diagonal automorphism δ and a field automorphism θ of order p. The group $\widehat{S}/S$ of diagonal automorphisms is cyclic for every group S

of Lie type except the case in which S is of type D_n, n is even, q is odd, and $\widehat{S}/S$ is an elementary abelian group of order 4 (see [9, Proposition 3.6]). Since for $S = D_n(q)$, $n \geq 4$, the group S contains an element of order 6, we may assume that in this case $p \neq 3$. Since $\widehat{S}/S$ is the normal subgroup of $\widehat{A}/S$, we have the equality $\overline{\delta}^{\theta} = \overline{\delta}^{k}$, where $\overline{\delta}$ is the image of δ in A/S, and k is some natural number. Hence the equalities $|\overline{\varphi}| = |\theta| = p$ and $(|\delta|, p) = 1$ imply that p divides $\phi(|\delta|)$, where ϕ is the Euler function sending each natural number n to the number of invertible elements in $\mathbb{Z}_n$. Since $p \geq 3$, we have $|\delta| \geq 5$. It is possible only if S is of type A_n or ${}^2A_n$ and $n \geq 4$; a contradiction. The lemma and Proposition 2 are proved.

Proposition 3. *Let G be a group satisfying the conditions of Proposition* 2, *and let the groups K, S, and $\overline{G}$ be as in the claim of Proposition* 2. *Then $t(S) \geq t(G) - 1$. Moreover, for every subset ρ of $\pi(G)$ such that $|\rho| \geq 3$ and all primes in ρ are pairwise nonadjacent in $GK(G)$, at most one prime in ρ divides the product $|K| \cdot |\overline{G}/S|$.*

Proof. Since G satisfies the condition of Proposition 2, there exists a finite nonabelian simple group S such that $S \leq \overline{G} = G/K \leq \mathrm{Aut}(S)$, where K is the maximal normal soluble subgroup of G. Note that if $t(G) = 2$ the inequality $t(S) \geq t(G) - 1 = 1$ is obvious. So we may assume that $t(G) \geq 3$.

Let ρ be the set of prime divisors of order of G pairwise nonadjacent in $GK(G)$, and $|\rho| \geq 3$. If we prove that at most one of these primes divides $|K| \cdot |\overline{G}/S|$ then we obtain that at least $|\rho| - 1$ primes in ρ divide the order of S. In particular, it means that $t(S) \geq t(G) - 1$.

Denote by L the preimage of S in G and by $\widehat{G}$, the factor group $G/L \simeq \overline{G}/S$. Since $\widehat{G} \leq \mathrm{Out}(S)$, the group $\widehat{G}$ is soluble. Therefore, at most two primes in ρ can divide $|\widehat{G}|$. Suppose at first that there are exactly two such primes. The solubility of $\widehat{G}$ implies that $\widehat{G}$ includes a normal subgroup $\widehat{M}$ such that one of the two primes, for definiteness q, divides $|\widehat{M}|$, and the other r divides $|\widehat{G}/\widehat{M}|$. Since $|\rho| \geq 3$, the set ρ contains a prime p, which divides the order of L and is not equal to q and r. Let M be the preimage of $\widehat{M}$ in G. Then the normal series $1 \leq L \leq M \leq G$ satisfy the conditions of Lemma 1.1; a contradiction.

Thus, the set of prime divisors of $|\widehat{G}|$ contains at most one prime in ρ. On the other hand, Proposition 1 shows that the same is true for $\pi(K)$. Hence to complete the proof we need to consider the following situation: $p, q, r \in \rho$; p divides $|K|$, q divides $|S|$, and r divides $|\widehat{G}|$. In this case the application of Lemma 1.1 to the normal series $1 \leq K \leq L \leq G$ gives us a contradiction either. The proposition and theorem are both proved.

§ 2. Applications

Recall that the *spectrum* $\omega(G)$ of a finite group G is the set of its element orders, i.e., a natural number n belongs to $\omega(G)$ if and only if G contains an element of order n. Given an arbitrary subset ω of the set of natural numbers, denote by $h(\omega)$ the number of pairwise nonisomorphic finite groups G such that $\omega(G) = \omega$. We say that the *recognition problem is solved* for a finite group G if we know the value of $h(\omega(G))$ (for brevity, $h(G)$). More precisely, G is said to be *recognizable by spectrum* (briefly, recognizable) if $h(G) = 1$, *almost recognizable* if $1 < h(G) < \infty$, and *nonrecognizable* if $h(G) = \infty$.

Since every finite group with a nontrivial normal soluble subgroup is nonrecognizable (see [10, Lemma 1]), each recognizable or almost recognizable group is an extension of the direct product M of nonabelian simple groups by some subgroup of $\mathrm{Out}(M)$. So, of prime interest is the recognition problem for simple and almost simple groups. In the middle of the 1980s Shi found the first examples of recognizable finite simple groups (see [11, 12]). In 1994 Shi and Brandl obtained an infinite series of recognizable simple linear groups $L_2(q)$, $q \neq 9$ (see [13, 14]). At present the recognition problem is solved for many finite nonabelian simple and almost simple groups, in particular, for all sporadic groups, for all simple linear groups of dimension 3, for several infinite series of exceptional groups of Lie type, and also for all finite simple groups with orders having prime divisors at most 13. The last attempt to give a full survey of results in this area were made in [15]. However, at present the list of groups with recognition problem solved is substantially wider than the same in [15, Table 1]. See the recent results in [16–20].

Nevertheless, an exhaustive solution of the recognition problem even in the class of finite simple groups seems to be sufficiently far from completion.

To explain how to use the main theorem of the present article in this area we briefly outline a scheme of proving recognizability.

Let L be a finite nonabelian simple group, and let G be an arbitrary finite group with $\omega(G) = \omega(L)$. The proof of recognizability of L consists usually of the three main steps:

1. We prove that the factor group G/K, where K is the maximal normal soluble subgroup of G, is almost simple. In other words, we prove that there exists a nonabelian simple group S such that $S \leq \overline{G} = G/K \leq \mathrm{Aut}(S)$.

2. We prove that S is isomorphic to L.

3. We prove that $\overline{G}/S = 1$ and $K = 1$.

Obviously we may sometimes "fail" at one of the steps. For example, for $L = L_3(5)$ we fail in proving that $\overline{G}/S = 1$. It turns out that $h(L) = 2$ and the group L has the same spectrum as its extension by a graph automorphism of order 2 (see [21]). Further, the group $L = L_3(3)$ has the same spectrum as a soluble Frobenius group (see [22, Proposition 3]) and so it is nonrecognizable. However, in spite of exceptions, the above scheme is implemented in most papers on the recognition problem.

Return to the first step. Suppose that L has a disconnected prime graph. Since the spectrum of G coincides with the spectrum of L, the prime graph $GK(G)$ coincides with $GK(L)$, and so it is disconnected either. Therefore, the group G satisfies the conditions of the Gruenberg–Kegel Theorem. Hence for G one of the three statements of the Gruenberg–Kegel Theorem holds. As it was shown in [23], items (a) and (b) of the theorem can be realized only for the simple groups $L_3(3)$, $U_3(3)$, and $S_4(3)$. Thus, if L has a disconnected prime graph and differs from each of the three groups mentioned above then the factor group of G by the soluble radical K is an almost simple group. The first step of the proof of recognition of L is thus complete. Note that the Gruenberg–Kegel Theorem is used in the overwhelming majority of papers devoted to the recognition problem. In particular, a sufficiently wide list of groups with the recognition problem solved contains only three groups with connected prime graph: the group A_{16} with $h(A_{16}) = 1$ (see [24]), the group $U_4(5)$ with $h(U_4(5)) = 2$ (see [20]), and the group A_{10} with $h(A_{10}) = \infty$ (see [10]). On the other hand, in most "extensive" class of finite simple groups, classical simple groups, the groups with disconnected prime graph are rather exceptions (see the full list of finite simple groups with disconnected prime graph in [15, Tables 2a–2c]). Thus, for the further study of the recognition problem it is extremely important to make a first step in the proof of recognition in the case when a finite simple group has a connected Gruenberg–Kegel graph. Recently it has been shown in [25, Theorem 2] that if a group G has the spectrum that coincident with the spectrum of a finite simple group L distinct from the groups $L_3(3)$, $U_3(3)$, $S_4(3)$, and A_{10} then G is insoluble. However, this result is difficult to use directly, since the structure of G remains undetermined. Propositions 2 and 3 of the present article and the result of [25, Theorem 2] imply the following

Proposition 4. *Let L be a finite simple group with $t(2, L) \geq 2$ distinct from the groups $L_3(3)$, $U_3(3)$, $S_4(3)$, and A_{10}; let G be a finite group satisfying the condition $\omega(G) = \omega(L)$. Then the conclusion of the main theorem holds for G. In particular, G has a unique nonabelian composition factor.*

Note that the conditions of Proposition 4 hold for an extremely wide class of finite simple groups. It contains all sporadic groups and all groups of Lie type. The condition $t(2, G) \geq 2$ is not always realized in the only class of finite simple groups, the class of the alternating groups A_n. Namely, for $n \geq 5$ the condition $t(2, A_n) \geq 2$ takes place only if $n = p, p+1, p+2, p+3$, where p is a prime. A detailed description for the groups with $t(2, G) \geq 2$ falls beyond the framework of this article. It will be obtained as a corollary in [26] whose main result consists in indicating an adjacency criterion for two prime divisors of the order of every finite simple group. In connection with the result of Proposition 4 we state the following question due to V. D. Mazurov.

Problem. *Is it true that a finite group whose spectrum coincides with the spectrum of a finite simple group has at most one nonabelian composition factor?*

Remark. It follows from [25, Lemma 9] that every finite simple group with connected Gruenberg–Kegel graph, except the group A_{10}, has three pairwise nonadjacent prime divisors. Therefore, a positive answer to Mazurov's problem for every finite simple group L satisfying $t(2, L) \geq 2$ can be offered using only the properties of the prime graph of L. In the common situation this is insufficient, as shown by the following

Example. Let $L = A_{28}$, and $G = A_5 \times A_n$, where n is an arbitrary natural number in the set $\{23, \dots, 28\}$. Then $GK(L) = GK(G)$ but $\omega(L) \neq \omega(G)$.

Another reason for popularity of the Gruenberg–Kegel Theorem among the researchers of the recognition problem is that the disconnectedness of the prime graph of a given group L translates to the nonabelian composition factor S of a group G, having the same spectrum as L. This fact expressed by the inequality $s(S) \geq s(L)$ is a useful instrument at the second step of proving recognizability; namely, in proving that $S \simeq L$. Actually, if the number of connected components of a given group L is equal to, say, s then S is contained in the list of finite simple groups with the number of connected components at least s. Moreover, if L contains an element of order n and the prime divisors of n belong to the connected component of $GK(L)$ which does not contain 2 then $n \in \omega(S)$. It turns out often that this information is sufficient to obtain by the elementary methods the isomorphism of the simple groups S and L; in other words, to prove *quasirecognition* of L. More precisely, a finite nonabelian simple group L is called *quasirecognizable*, if a finite group G with the same spectrum as L contains a unique nonabelian composition factor and this factor is isomorphic to L. This definition stems from [27].

The main theorem of this article also gives a rich information on the structure of the spectrum of a nonabelian composition factor S of G. To illustrate how this information could be used we sketch the proof of quasirecognition of the groups ${}^2D_n(2^k)$, with n a "sufficiently large" even number. We need the following

Lemma 2.1 (Zsigmondy). *Let p be a prime and let s be a natural number, $s \geq 2$. Then one of the following statements holds:*

(a) *there exists a prime r such that r divides $p^s - 1$ and r does not divide $p^t - 1$ for every natural number $t < s$;*

(b) *$s = 6$ and $p = 2$;*

(c) *$s = 2$ and $p = 2^t - 1$ for some natural number t.*

Proof. See in [28].

A prime r satisfying (a) of Lemma 2.1 is called a *primitive* prime divisor of $p^s - 1$.

Proposition 5. *Let $L = {}^2D_n(q)$, $q = 2^k$; k and n are natural numbers with n even, and $n \geq 16$. Then L is quasirecognizable.*

Sketch of the proof. We have

$$|L| = q^{n(n-1)}(q^2 - 1)(q^4 - 1)\dots(q^{2n-2} - 1)(q^n + 1).$$

If T is a maximal torus in L then the order of T is given by

$$|T| = \prod_{j=1}^{t}\left(q^{t_j} - 1\right)\prod_{i=1}^{s}\left(q^{s_i} + 1\right),$$

where the natural numbers s_i and t_j satisfy the equality

$$\sum_{j=1}^{t} t_j + \sum_{i=1}^{s} s_i = n,$$

and s is odd (for example, see [18, Lemma 2.1]).

Let $a_1, \dots, a_l$ be a full system of representatives of the conjugacy classes of involutions in L and $C_i = C_L(a_i)$, $i = 1, \dots, l$, are their centralizators in L. Then (for example, see [3, § 3, item 5])

$$\pi\left(\prod_{j=1}^{l}|C_i|\right) = \pi\left(2 \cdot \prod_{i=1}^{n-2}(q^{2i}-1)\right).$$

Denote by r_s some primitive prime divisor of $2^{ks}-1$. Note, if $s = 2j$ is even then r_s is a divisor of $2^{kj}+1$, and it does not divide $2^{ki}+1$ for each $i < j$.

Since every semisimple element of L is contained in some maximal torus of L, two primitive prime divisors r_i and r_j are adjacent if and only if L includes a torus whose order divides by $r_i r_j$.

If we know the orders of maximal tori and the set of prime divisors of centralizators of involutions, we easily evaluate the values of the independence number $t(L)$ and the 2-independence number $t(2, L)$ of the prime graph of L. For example, the independence set with a maximal number of vertices in $GK({}^2D_{16}(q))$ is the set of primes $\rho = \{r_s \mid s = 9, 11, 13, 15, 16, 18, 20, 22, 24, 26, 28, 30, 32\}$. Hence, $t({}^2D_{16}(q)) = 13$. Furthermore, for $n \geq 16$ we have $t({}^2D_n(q)) \geq t({}^2D_{16}(q))$. Therefore, $t(L) \geq 13$. The set of vertices $\rho(2, L) = \{2, r_{n-1}, r_{2n-2}, r_{2n}\}$ is independent in $GK(L)$, and each of the other independent sets of vertices τ of $GK(L)$ which contains the vertex 2 satisfies the condition $|\tau| \leq |\rho(2, L)|$. Hence $t(2, L) = 4$.

Let G be a finite group with $\omega(G) = \omega(L)$. Since $GK(G) = GK(L)$, the conclusion of the theorem holds for G. Let S be the unique nonabelian factor of G. If H is a simple sporadic or exceptional group of Lie type then $t(H) \leq t(F_1) = 11$ (see [26] for details). Since $t(S) \geq t(G) - 1 = t(L) - 1 \geq 12$, the group S can be neither a sporadic group nor exceptional group of Lie type. Therefore, S is either an alternating group or classical group of Lie type of "sufficiently large" Lie rank. In particular, S cannot be the group ${}^2D_{n'}(2^{k'})$, where n' is an even number less than 16, since the independence number for such groups is at most 11. It turns out that all indicated groups have the 2-independence number less than 4, except the case, in which S satisfies the condition of the proposition itself (see [26]).

Thus, $S \simeq {}^2D_{n'}(2^{k'})$, n' is even, and $n' \geq 16$. It remains to prove that $n = n'$ and $k = k'$. Since the primes r_{2n}, r_{2n-2}, and r_{n-1} are nonadjacent to 2 in $GK(G)$, by (2) of the theorem they lie in $\omega(S)$ and are nonadjacent to 2 in $GK(S)$. Therefore, they are primitive prime divisors of $2^{2n'k'}-1$, $2^{(2n'-2)k'}-1$, and $2^{(n'-1)k'}-1$. Furthermore, since the primes r_{2n}, r_{2n-2}, and r_{n-1} are pairwise nonadjacent, every two of them cannot divide the same of the above-indicated difference. Assume at first that r_{2n} divides $2^{2n'k'}-1$, r_{2n-2} divides $2^{(2n'-2)k'}-1$, and r_{n-1} divides $2^{(n'-1)k'}-1$. Since r_{2n} is a primitive prime divisor of $2^{2n'k'}-1$, the number $2n'k'$ is a minimal natural such that $2^{2n'k'} \equiv 1 (\operatorname{mod} r_{2nk})$. But the number $2nk$ has the same property. Therefore, $nk = n'k'$. By analogy we have $(n-1)k = (n'-1)k'$; whence $n = n'$ and $k = k'$. Now let the primes r_{2n}, r_{2n-2}, and r_{n-1} divide the same differences in another order. Consider, for example, the case in which r_{2n} is a primitive divisor of $2^{(2n'-2)k'}-1$, and r_{2n-2} is a primitive divisor of $2^{2n'k'}-1$. Using the same arguments as in the previous case we obtain the equalities $nk = (n'-1)k'$ and $(n-1)k = n'k'$. Whence $k = -k'$, which is impossible. The other cases can be treated by analogy and lead to a contradiction either. Therefore, only the first case holds, and we have $n = n'$, $k = k'$, as required.

Remark 1. As was mentioned, the argument above is only an illustration of how the main theorem can be applied. These arguments cannot be considered a full proof of Proposition 5, since we use information on the values of the independence and 2-independence numbers of the article [26] which is still unpublished. Thus, Proposition 5 will become completely proved after publication of [26].

Remark 2. Quasirecognition of the groups ${}^2D_n(2^k)$, $n = 2^m$ (the case of $m \geq 4$ is a particular case of Proposition 5) was earlier proved in [18] on using the Gruenberg–Kegel Theorem (in this case the prime graph of the group is disconnected). However if n is not a power of 2 then the group L satisfying the condition of Proposition 5 has the connected prime graph. Thus, we obtain an infinite series of quasirecognizable groups with connected prime graph for the just time.

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