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RECOGNITION BY SPECTRUM FOR FINITE SIMPLE LINEAR GROUPS OF SMALL DIMENSIONS OVER FIELDS OF CHARACTERISTIC 2

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Two groups are said to be isospectral if they share the same set of element orders. For every finite simple linear group L of dimension n over an arbitrary field of characteristic 2, we prove that any finite group G isospectral to L is isomorphic to an automorphic extension of L. An explicit formula is derived for the number of isomorphism classes of finite groups that are isospectral to L. This account is a continuation of the second author's previous paper where a similar result was established for finite simple linear groups L in a sufficiently large dimension ($n > 26$), and so here we confine ourselves to groups of dimension at most 26.

INTRODUCTION

The *spectrum* $\omega(G)$ of a group G is the set of its elements orders. Two groups are said to be *isospectral* if their spectra coincide. A finite group L is said to be *recognizable by spectrum* if every finite group G with $\omega(G) = \omega(L)$ is isomorphic to L. If we denote by $h(L)$ the number of pairwise nonisomorphic finite groups isospectral to L, then the property that L is recognizable is written as the equality $h(L) = 1$. A group L is *almost recognizable* if $1 < h(L) < \infty$, and is *irrecognizable* if $h(L) = \infty$. The problem of being recognizable by spectrum for a group L reduces to determining whether L is recognizable, almost recognizable, or irrecognizable, and in a stronger setting, to finding the value of $h(L)$. The latest survey on this subject can be found in [1, 2].

For simple linear groups $L_n(2^k)$, the recognizability problem is solved with $n = 2$ [3], $n = 3$ [4, 5], $n = 4$ [6], $11 \leqslant n \leqslant 17$ [7, 8], $n \geqslant 26$ [8, 9], and also for $k = 1$ [10, 11]. The goal of the present paper is to solve the problem for all the remaining groups $L_n(2^k)$, thus settling the question of whether finite simple linear groups over fields of characteristic 2 are recognizable by spectrum.

THEOREM. Let $L = L_n(q)$, where $n \geqslant 2$ and $q = 2^k$, and let $d = (n, q - 1)$.

(1) If $n = 2^m + 1$ for some natural number m then $h(L) = 1$.

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(2) If $n \neq 2^m + 1$ for any natural number m then $h(L)$ is equal the number of positive integers dividing the d-share of $(\frac{q-1}{d}, k)$. Moreover, a finite group G satisfies $\omega(G) = \omega(L)$ if and only if G is isomorphic to a natural extension of L by a field automorphism of order dividing the d-share of $(\frac{q-1}{d}, k)$.

In particular, L is recognizable iff n is of the form $2^m + 1$ or $(d, \frac{q-1}{d}, k) = 1$.

Research on the recognizability problem for a finite simple group involves studying properties of quasirecognizability and recognizability among covers, as well as spectra of automorphic extensions. A simple group L is said to be *quasirecognizable* if every finite group G that is isospectral to L has a unique non-Abelian composition factor and this factor is isomorphic to L. A group L is said to be *recognizable among its covers* if every finite group that contains L as a homomorphic image is isospectral to L iff it is isomorphic to L. For a simple group L which is quasirecognizable and recognizable among covers, the number $h(L)$ is equal to the number of pairwise nonisomorphic automorphic extensions of L whose spectra do not differ from $\omega(L)$.

As follows from [12], all simple groups $L_n(2^k)$ are recognizable by spectrum among their covers. Isospectral automorphic extensions of $L_n(2^k)$ are described in [8]. Thus, to solve the problem posed, it is sufficient to state that $L_n(2^k)$ is quasirecognizable for $5 \leqslant n \leqslant 26$ and $q > 2$.

1. PRELIMINARIES

We denote by $[x]$ the integer part of a number x and by $\pi(m)$ the set of prime divisors of a natural number m. For a finite group G, put $\pi(G) = \pi(|G|)$. By $[m_1, m_2, \dots, m_s]$ and $(m_1, m_2, \dots, m_s)$ we denote, respectively the least common multiple and the greatest common divisor of numbers $m_1, m_2, \dots, m_s$. For a natural number r, the r-*share* of a natural number m is the greatest divisor t of m with $\pi(t) \subseteq \pi(r)$. We write m_r for the r-share of m and write $m_{r'}$ for the quotient m/m_r.

Let G be a finite group and $\omega(G)$ its spectrum. The divisibility relation endows $\omega(G)$ with a partial order, and the subset of elements that are maximal under this order is denoted by $\mu(G)$. For a prime r, we refer to the maximal degree of r in $\omega(G)$ as the r-*period* of G.

The *Gruenberg–Kegel graph* (or *prime graph*) of G is a graph $GK(G)$ whose vertex set is $\pi(G)$ and two vertices p and r are connected by an edge if and only if $pr \in \omega(G)$. The number of connected components of $GK(G)$ is denoted by $s(G)$; the maximal cardinality of independent sets of vertices (or the *independence* number), by $t(G)$; the maximal cardinality of independent sets containing vertex 2, by $t(2, G)$. The last-mentioned quantity, by analogy with an ordinary independence number, is called the 2-*independence* number of $GK(G)$. The *neighborhood* of a vertex is a set consisting of the vertex itself and vertices adjacent to that vertex.

LEMMA 1 [13, 14]. Let L be a finite non-Abelian simple group satisfying $t(L) \geqslant 3$ and $t(2, L) \geqslant 2$ and G be a finite group with $\omega(G) = \omega(L)$. Then the following statements hold:

(1) there exists a non-Abelian simple group S such that $S \leq \overline{G} = G/K \leq \operatorname{Aut} S$, where K is a maximal normal soluble subgroup of G;

(2) for every independent set ρ of vertices in $GK(G)$ with $|\rho| > 2$, at most one prime from ρ lies in $\pi(K) \cup \pi(\overline{G}/S)$; in particular, $t(S) \geqslant t(G) - 1$;

(3) every prime $r \in \pi(G)$ nonadjacent to 2 in $GK(G)$ does not divide the product $|K| \cdot |\overline{G}/S|$; in particular, $t(2, S) \geqslant t(2, G)$.

LEMMA 2 [15, Lemma 1]. Let G be a finite group, K be a normal subgroup of G, and G/K be a Frobenius group with kernel F and cyclic complement C. If $(|F|, |K|) = 1$, and F is not contained in $KC_G(K)/K$, then $r|C| \in \omega(G)$ for some prime divisor r of $|K|$.

LEMMA 3. Let G be a finite group, K be a normal soluble subgroup of G, and $S \le \overline{G} = G/K \le \operatorname{Aut} S$ for a simple group S. Suppose $\pi(S) \setminus \pi(K)$ contains numbers t and s whose neighborhoods in $GK(G)$ are disjoint. If $r \in \pi(K)$ is adjacent neither to t nor to s in $GK(G)$, and S includes a Frobenius subgroup with cyclic complement C and kernel F for which $(|F|, r) = 1$, then $r|C| \in \omega(G)$.

Proof. Put $\widetilde{G} = G/O_{r'}(K)$ and $\widetilde{K} = K/O_{r'}(K)$. Then $R = O_r(\widetilde{K}) \neq 1$. Suppose $\widetilde{K} \neq R$. Then there exists a prime u such that $U = O_u(\widetilde{K}/R)$ is not trivial. Since $O_{r'}(\widetilde{K}) = 1$, it follows that $U \cap RC_{\widetilde{K}}(R)/R = 1$. By assumption, at least one of the numbers t or s is not adjacent to u in $GK(G)$. Denote this number by v. Let x be an element of order v in $\widetilde{G}/R$. Then $H = U \leftthreetimes \langle x \rangle$ is a Frobenius subgroup of $\widetilde{G}/R$. The preimage of H in $\widetilde{G}$ satisfies the conditions of Lemma 2; therefore G contains an element of order rv. Contradiction.

Hence, $\widetilde{K} = R$. The group S, treated as a subgroup of $\widetilde{G}/\widetilde{K}$, has a trivial intersection with $\widetilde{K}C_{\widetilde{G}}(\widetilde{K})/\widetilde{K}$. Otherwise, S, being simple, would be in $\widetilde{K}C_{\widetilde{G}}(\widetilde{K})/\widetilde{K}$, and so G would contain an element of order tr. Applying Lemma 2, we infer that $r|C| \in \omega(G)$. The lemma is proved.

LEMMA 4 [16, Cor. 3]. Let $L = L_n(q)$, where $n \geqslant 2$ and q is a power of an odd prime p, and $d = (n, q-1)$. Then $\omega(L)$ consists of all divisors of the following numbers:

(1) $\frac{q^n-1}{d(q-1)}$;

(2) $\frac{[q^{n_1}-1, q^{n_2}-1]}{(n/(n_1,n_2), q-1)}$, where $n_1, n_2 > 0$ and $n_1 + n_2 = n$;

(3) $[q^{n_1}-1, q^{n_2}-1, \ldots, q^{n_s}-1]$, where $s \geqslant 3$, $n_1, n_2, \ldots, n_s > 0$, and $n_1 + n_2 + \ldots + n_s = n$;

(4) $p^m \frac{q^{n_1}-1}{d}$, where $m, n_1 > 0$ and $p^{m-1} + 1 + n_1 = n$;

(5) $p^m [q^{n_1}-1, \ldots, q^{n_s}-1]$, where $s \geqslant 2$, $m, n_1, \ldots, n_s > 0$, and $p^{m-1} + 1 + n_1 + \ldots + n_s = n$;

(6) p^m if $p^{m-1} + 1 = n$ for $m > 0$.

If q is a natural number, r is an odd prime, and $(q, r) = 1$, then $e(r, q)$ denotes the multiplicative order of q modulo r, that is, a minimal natural number m with $q^m \equiv 1 \pmod r$. For an odd q, we put $e(2, q) = 1$ if $q \equiv 1 \pmod 4$, and $e(2, q) = 2$ otherwise.

LEMMA 5 (Zsigmondy's theorem [17]). Let q be a natural number greater than 1. For every natural m, there then exists a prime r with $e(r, q) = m$ but for the cases where $q = 2$ and $m = 1$, $q = 3$ and $m = 1$, and $q = 2$ and $m = 6$.

A prime r with $e(r, q) = m$ is called a *primitive prime divisor* of $q^m - 1$. A divisor t of $q^m - 1$ is a *greatest primitive divisor* if $\pi(t)$ consists of primitive prime divisors and t is the greatest divisor with this property. A formula for expressing greatest primitive divisors in terms of cyclotomic polynomials $\phi_n(x)$ is given by the following:

LEMMA 6. Let q and m be natural numbers, $q > 1$, $m \geqslant 3$, and let k be the greatest primitive divisor of $q^m - 1$. Then

$$k = \frac{\phi_m(q)}{\prod\limits_{r \in \pi(m)} (\phi_{m_{r'}}(q), r)}.$$

Proof. Let r be a primitive prime divisor of $q^m - 1$. Since $m \geqslant 3$, it follows that r is odd. It is well known that $q^m - 1$ can be factored into a product of values $\phi_d(q)$, where d runs over the set of divisors of m. In this product, by definition, the number r can divide only the factor $\phi_m(q)$. Hence, k divides $\phi_m(q)$. On the other hand, the set $\pi(\phi_m(q))$ may contain nonprimitive prime divisors.

Let r be an odd prime divisor of $\phi_m(q)$. By [18, Chap. IX, Lemma 8.1(1)], this is possible only if $m = e(r, q)$, or else if $m = e(r, q) r^i$ for $i > 0$ with the r-share of $\phi_m(q)$ equal to r. Primitive prime divisors of $q^m - 1$ are exactly those r for which $m = e(r, q)$. Thus, we have to divide $\phi_m(q)$ by odd primes r such that

$m = e(r,q)r^i$ for some $i > 0$. If $m = e(r,q)r^i$ for $i > 0$ then r is in $\pi(m)$ and divides $\phi_{m_{r'}}(q)$. Conversely, if r divides $\phi_{m_{r'}}(q)$ and belongs to $\pi(m)$ then $m_{r'} = e(r,q)$ and $m = e(r,q)r^i$ for $i > 0$.

Suppose $\phi_m(q)$ is divisible by 2. Then m is a power of 2, as follows by [18, Chap. IX, Lemma 8.1(2)]. Moreover, since $m \geqslant 3$, we conclude that $\phi_m(q)$ is not divisible by 4. Thus, we should divide $\phi_m(q)$ by 2 if q is odd and $m = 2^i$. If q is odd and $m = 2^i$, then 2 divides $\phi_{m_{2'}}(q) = q - 1$. Conversely, if 2 divides $\phi_{m_{2'}}(q)$, then $m_{2'} = 1$, and hence m is a power of 2. The lemma is proved.

2. PROOF OF THE THEOREM

Let $L = L_n(q)$, where q is even. As noted, the theorem has already been proven to hold for all $n < 5$ and for $q = 2$, and we may so assume that $n \geqslant 5$ and $q > 2$. For $3 \leqslant i \leqslant n$, denote by k_i the greatest primitive divisor of $q^i - 1$ (which is not 1 by Lemma 5). Note that 3 divides $q^2 - 1$, and hence these divisors are all coprime to 3. Furthermore, they all are in $\omega(L)$. Let $r_i \in \pi(k_i)$, $3 \leqslant i \leqslant n$. According to [19, Tables 4, 8], the independence number $t(L)$ is equal to $[\frac{n+1}{2}]$ and the 2-independence number $t(2,L)$ is equal to 3; $\{2, r_n, r_{n-1}\}$ and $\{r_n, r_{n-1}, \ldots, r_{[n+1/2]}\}$ are independent sets of vertices in $GK(L)$.

Let G be a finite group and $\omega(G) = \omega(L)$. By Lemma 1, G has a unique non-Abelian composition factor S. Denote the soluble radical of G by K. Then $S \leq \overline{G} = G/K \leq \operatorname{Aut} S$. Furthermore, S satisfies $t(S) \geqslant t(G) - 1$, and any number in $\pi(k_{n-1}) \cup \pi(k_n)$ does not divide the product $|K| \cdot |\overline{G}/S|$. The last-mentioned fact entails $k_n, k_{n-1} \in \omega(S)$.

In [8, Props. 1-4], it was stated that the factor S is isomorphic either to L or to one of the groups $L_2(u)$, $G_2(u)$, ${}^2G_2(u)$, or $E_8(u)$, where u is odd.

PROPOSITION 1. A group S is not isomorphic to $L_2(u)$, where u is odd.

Proof. Suppose $S \simeq L_2(u)$ and $u = v^l$, where v is an odd prime. Then $\mu(S) = \{v, (u+1)/2, (u-1)/2\}$. The numbers r_n and r_{n-1} are in $\pi(S)$ and are not adjacent to 2 in $GK(S)$. Therefore, one of the numbers is equal to v and the other is a divisor of $(u+\varepsilon)/2$, where ε is specified by $u \equiv \varepsilon \pmod 4$.

We claim that 4 is in $\omega(S)$, or in $\omega(K)$. Assume the contrary. Since $8 \in \omega(G)$, there must be an element of order 4 in $\overline{G}$. Hence, $\overline{G}/S$ should contain an element of order 2. If S admitted a field automorphism of order 2, l would be even and $(u-1)/2$ would be divisible by 4. Consequently, $\overline{G}$ admits a diagonal automorphism of S; that is, $\overline{G}$ contains a subgroup isomorphic to $PGL_2(u)$. There is a cyclic torus of order $u + \varepsilon$ in $PGL_2(u)$, and either r_n or r_{n-1} is adjacent to 2 in $GK(G)$. Contradiction.

Denote r_{n-2} by r. As noted, there are no elements of orders $r_n r$ and $r_{n-1} r$ in L. By Lemma 4, L contains no elements of order $4r$.

Suppose $r \in \pi|\overline{G}/S|$. Then $\overline{G}$ admits a field automorphism φ of order r. The centralizer $C_S(\varphi)$ is isomorphic to $L_2(u^{1/r})$, which contains an element of order v; hence $vr \in \omega(G)$. Contradiction.

Assume $r \in \pi(S)$. Then r divides $(q-\varepsilon)/2$. If $4 \in \omega(S)$, then $(q-\varepsilon)/2$ is divisible by 4, and so there is an element of order $4r$ in G, which is impossible. If $4 \notin \omega(S)$ then $2 \in \pi(K)$. A Borel subgroup B of S is a Frobenius group with kernel of order u and cyclic complement of order $(u-1)/2$. Applying Lemma 3 with $t = r_n$ and $s = r_{n-1}$, we infer that $(u-1) \in \omega(G)$. Thus, if $\varepsilon = 1$ then $4r \in \omega(G)$, and if $\varepsilon = -1$ then one of r_n, r_{n-1} divides $(u-1)/2$; so r is adjacent to one of these numbers in $GK(G)$. We arrive at a contradiction in any case.

Suppose $r \in \pi(K)$. If again we apply Lemma 3 with the Frobenius group B where $t = r_n$ and $s = r_{n-1}$ we see that $r(u-1)/2 \in \omega(G)$. If $\varepsilon = -1$ then one of the numbers $r_n r$ or $r_{n-1} r$ is in $\omega(G)$, a contradiction. If $(u-1)/2$ is divisible by 4, then $4r \in \omega(G)$, a contradiction. Hence, $u \equiv 1 \pmod 4$, and there are no elements of order 4 in S. We have $4 \in \omega(K)$.

Let H be a Hall $\{2,r\}$-subgroup of K and $N = N_G(H)$. By the Frattini argument, $G = NK$, and so $N/(N \cap K) \simeq G/K$. An element of order r_n in N acts fixed-point-freely on H; therefore H is nilpotent by Thompson's theorem. This means that $4r$ is in $\omega(H)$, a contradiction. The proposition is proved.

If S is a group of type E_8, G_2, or ${}^2G_2$ over a field of odd characteristic, then 2 is adjacent to the characteristic in $GK(S)$. Since k_n and k_{n-1} are in $\omega(S)$ and have no divisors adjacent to 2 in $GK(S)$, each of these numbers divides the order of some maximal torus of S. Orders of maximal tori for the groups under consideration are stated in [19, Lemma 1.3].

PROPOSITION 2. A group S is not isomorphic to $E_8(u)$, where u is odd.

Proof. In [19], based on the adjacency criterion outlined in [19, Props. 2.5, 3.2, and 4.5], an independent vertex set of $GK(E_8(u))$ consisting of 11 vertices was constructed and the conclusion was made that the independence number of this graph is equal to 11. But [19, Prop. 3.2] shows that there is no loss of independency in adding a primitive prime divisor w of u^5-1 to the set constructed. On the other hand, $GK(E_8(u))$ lacks in thirteen pairwise nonadjacent vertices. Thus, we need to introduce the following amendments into [19, Table 9]: (i) enlarge the maximal independent set of vertices in the graph $GK(E_8(u))$ by adding w, and (ii) change the value of $t(E_8(u))$ from 11 to 12.*

Suppose $S \simeq E_8(u)$ and u is odd. Since $t(S) = 12$ and $[\frac{n+1}{2}] = t(L) \leqslant t(S)+1$, it follows that $n \leqslant 26$. Orders of maximal tori in S whose divisors may be nonadjacent to 2 are u^8-u^4+1, $u^8-u^6+u^4-u^2+1$, $u^8+u^7-u^5-u^4-u^3+u+1$, and $u^8-u^7+u^5-u^4+u^3-u+1$. Each of the orders does not exceed $2u^8$; hence $k_n, k_{n-1} \leqslant 2u^8$.

On the other hand, $E_8(u)$ includes a cyclic torus of order u^8-1; so $u^8-1 \in \omega(L)$. In particular, $32 \in \omega(L) \setminus \mu(L)$. By Lemma 4, multiples of 32 can arise in $\omega(L)$ only if they divide expressions of the form $2^m[q^{n_1}-1, \dots, q^{n_s}-1]$, where $m \geqslant 5$ and $2^{m-1}+1+n_1+\dots+n_s = n$. Thus, if $n \leqslant 17$ then either there are no elements of order 32 in L, or $32 \in \mu(L)$; for larger n, every element of $\omega(L)$, which is a multiple of 32, does not exceed $32(q^{n-17}-1)$. Hence, $n \geqslant 18$ and $u^8 \leqslant 32q^{n-17}$. Substituting the last estimate into the inequality in the previous paragraph, we conclude that $k_n, k_{n-1} \leqslant 64q^{n-17}$.

At the moment we show that at least one of the inequalities above leads to a contradiction, by examining every n from 18 to 26 separately. In each case we make use of the formula for greatest primitive divisors given in Lemma 6.

If p is an odd prime then

$$k_{p^t} = \frac{q^{p^t}-1}{(q^{p^{t-1}}-1)(q-1,p)} \geqslant \frac{q^{p^{t-1}(p-1)}}{(q-1,p)}.$$

Thus, for $n = 18$, the condition that $k_{17} \leqslant 64q$ implies that $q^{16} \leqslant 64q(q-1,17)$. In a similar way, we derive $q^{18} \leqslant 64q^3(q-1,19)$ for $n = 19, 20$, $q^{22} \leqslant 64q^7(q-1,23)$ for $n = 23, 24$, and $q^{20} \leqslant 64q^9(q-1,5)$ for $n = 25, 26$. The resulting inequalities are impossible in all cases.

Using estimates $k_{20} = \frac{q^{10}+1}{(q^2+1)(q^2+1,5)} \geqslant \frac{q^8}{2}$ and $k_{22} = \frac{q^{11}+1}{(q+1)(q+1,11)} \geqslant \frac{q^{10}}{2(q+1,11)}$, we infer that $q^8 \leqslant 128q^4(q^2+1,5)$, for $n = 21$, and $q^{11} \leqslant 64q^5(q+1,11)$ for $n = 22$. These inequalities are false for $q > 2$. The proposition is proved.

PROPOSITION 3. A group S is not isomorphic to $G_2(u)$, where u is odd.

Proof. Let $S \simeq G_2(u)$ and u be odd. Then $t(S) = 3$; so $t(L) \leqslant 4$ and $n \leqslant 8$. Orders of maximal tori of S that have prime divisors nonadjacent to 2 in $GK(S)$ are equal to u^2+u+1 and to u^2-u+1 and, consequently, do not exceed $2u^2$. Thus, $k_n, k_{n-1} \leqslant 2u^2$.

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*We are grateful to W. Shi and H. He who drew our attention to this fact.

There is a cyclic torus of order u^2-1 in S. Therefore, $u^2-1\in\omega(L)$. Notice that u^2-1 is divisible by 8. Multiples of 8 can arise in $\omega(L)$ only if they divide expressions of the form $2^m[q^{n_1}-1,\dots,q^{n_s}-1]$, where $m\geqslant 3$ and $2^{m-1}+1+n_1+\ldots+n_s=n$. Hence, $u^2\leqslant 8q^{n-5}$. Thus, $k_n,k_{n-1}\leqslant 16q^{n-5}$. From these inequalities we conclude that $q^4\leqslant 16q(q-1,5)$, for $n=5,6$, and $q^6\leqslant 16q^3(q-1,7)$ for $n=7,8$. The resulting inequalities are false for $q>2$. The proposition is proved.

PROPOSITION 4. A group S is not isomorphic to ${}^2G_2(u)$.

Proof. Suppose $S\simeq {}^2G_2(u)$, where $u=3^{2l+1}>3$. Then $t(S)=5$, and so $n\leqslant 12$. Orders of maximal tori in S are equal to $u-1$, $u+1$, $u-\sqrt{3u}+1$, and $u+\sqrt{3u}+1$ and, consequently, do not exceed $2u$. Thus, $k_n,k_{n-1}\leqslant 2u$.

The number $u+1$ is in $\omega(L)$ and is a multiple of 4; so it does not exceed $4(q^{n-3}-1)$. Hence, $u\leqslant 4q^{n-3}$. Thus, $k_n,k_{n-1}\leqslant 8q^{n-3}$.

Suppose $n=5$. Then $k_5\leqslant 8q^2$, and consequently $q^4\leqslant 8q^2(q-1,5)$. If $n=7$, then $k_7\leqslant 8q^4$ entails $q^6\leqslant 8q^4(q-1,5)$. In both cases we arrive at a contradiction with the fact that $q>2$.

Let $n=6$. It follows from $k_5\leqslant 8q^3$ that $q^4\leqslant 8q^3(q-1,5)$, whence $q\in\{4,8,16\}$. Suppose $q=4$ or $q=16$. Then k_5 is a multiple of 11. This means that 11 is in $\omega(S)$ and should therefore divide the order of a maximal torus in S. The order of every maximal torus in S divides u^6-1; hence 11 divides u^6-1. On the other hand, 11 divides $u^{10}-1$. Consequently, 11 divides u^2-1 and is therefore adjacent to 2 in $GK(S)$, a contradiction. If $q=8$, then k_5 is a multiple of 151, and hence u^6-1 is divisible by 151. Since $3^{50}-1$ is a multiple of 151, u^2-1 is divisible by 151, and so 151 is adjacent to 2 in $GK(S)$. Contradiction.

Let $n=8$. From $k_7\leqslant 8q^5$, it follows that $q^6\leqslant 8q^5(q-1,7)$, which yields $q\in\{4,8\}$. For $q=4,8$, the number k_7 is a multiple of 127; therefore 127 must divide u^6-1. The multiplicative order of 3 modulo 127 is 126, which implies $u\geqslant 3^{21}$. Thus, $3^{21}\leqslant u\leqslant 4q^5\leqslant 4\cdot 8^5\leqslant 3^{12}$. Contradiction.

Let $n\geqslant 9$. Then $16\in\omega(L)$. Since the 2-period of S is equal to 4 and the order of $\operatorname{Out}S$ is odd, K contains an element of order 4. A Borel subgroup of S is a Frobenius group with kernel of order u^3 and cyclic complement of order $u-1$. Applying Lemma 3 with $t=r_n$ and $s=r_{n-1}$, we see that $2(u-1)\in\omega(G)$.

Denote r_{n-2} by r. Suppose $r\in\pi(\overline{G}/S)$. Then $\overline{G}$ admits a field automorphism φ of S of order r. The centralizer $C_S(\varphi)$ is isomorphic to ${}^2G_2(u^{1/r})$, which contains an element of order 4; hence $4r\in\omega(G)$. Contradiction.

Suppose $r\in\pi(S)$. The fact that r is not equal to 3 implies that r divides the order of one of the maximal tori. Since $u+\sqrt{3u}+1$ is divisible by one of the numbers r_n or r_{n-1} and $u-\sqrt{3u}+1$ is divisible by the other, while $u+1$ is a multiple of 4, it follows that r divides $u-1$. Consequently, $4r$ divides $2(u-1)$, which belongs to $\omega(G)$. Contradiction.

Suppose $r\in\pi(K)$. Let H be a Hall $\{2,r\}$-subgroup of K and $N=N_G(H)$. By the Frattini argument, $G=NK$, and so $N/(N\cap K)\simeq G/K$. An element of N of order r_n acts fixed-point-freely on H, and hence H is nilpotent by Thompson's theorem. This means that $4r$ is in $\omega(H)$, a contradiction. The proposition is proved.

Thus, $S\simeq L$ and $L\leq G/K$. The preimage of L in G is isospectral to L, and therefore K is trivial by [12, Cor. 1]. To complete the proof of the theorem, it remains to apply [8, Thm. 2].

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