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# MINIMAL PERMUTATION REPRESENTATIONS OF FINITE SIMPLE EXCEPTIONAL TWISTED GROUPS

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*A minimal permutation representation of a group is its faithful permutation representation of least degree. Here the minimal permutation representations of finite simple exceptional twisted groups are studied: their degrees and point stabilizers, as well as ranks, subdegrees, and double stabilizers, are found. We can thus assert that, modulo the classification of finite simple groups, the aforesaid parameters are known for all finite simple groups.*

A *minimal permutation representation* of a group is a faithful permutation representation of least degree. Well studied to date are the minimal permutation representations of finite simple sporadic groups (see summary table in [1]), of finite simple classical groups (cf. [2, 3]), and of finite simple exceptional Chevalley groups (cf. [4, 5]), for which degrees, point stabilizers, as well as ranks, subdegrees, and double stabilizers, have been found. Here we provide a similar account for finite simple exceptional twisted groups, that is, for the groups ${}^2B_2(q)$, ${}^2G_2(q)$, ${}^2F_4(q)$, ${}^3D_4(q)$, and ${}^2E_6(q)$. Modulo the classification of finite simple groups, the following result is thus valid.

MAIN THEOREM. Let $G$ be a finite simple group. Then a degree, rank, subdegrees, a point stabilizer, and double stabilizers of the minimal permutation representation(s) of $G$ are known.

We follow the notation and terminology of [4, 5].

## 1. PRELIMINARY DATA ON TWISTED GROUPS

Let $\rho$ be a nontrivial symmetry of the Dynkin diagram for a simple Lie algebra $\mathcal{L}$, $\Pi$ a system of simple roots in $\mathcal{L}$, and $\bar{r} = \rho(r)$ an image of $r \in \Pi$ under the action of $\rho$. Then there exists the only isometry $\tau$ of the Euclidean space $\mathcal{V} = \mathcal{K}_{\mathrm{R}}$ ($\mathcal{K}$ is the Cartan subalgebra of $\mathcal{L}$) such that $\tau(r) = c\bar{r}$, where $c$ is a positive real number depending on the type of $\mathcal{L}$. Denote by $\mathcal{V}^1$ a subspace of fixed vectors of $\mathcal{V}$ w.r.t. $\tau$, and by $v^1$ the projection of $v \in \mathcal{V}$ on $\mathcal{V}^1$. If $|\tau| = t$, then $v^1 = \frac{1}{t} \cdot \sum_{k=0}^{t-1} \tau^k(v)$. Let $W$ be a Weyl group of $\mathcal{L}$. For every $r \in \Pi$, we have $\tau w_r \tau^{-1} = w_{\bar{r}}$. So $\tau$ normalizes $W$ in the group of all isometries of $\mathcal{V}$. Denote by $W^1$ a centralizer of the automorphism $\tau$ in $W$. Then $W^1 = \langle w_{r^1} \mid r \in \Pi \rangle$ (cf. Cor. 13.1.4 in [6]). Let $\Phi$ be a root system of $\mathcal{L}$ spanned by $\Pi$ and $\Phi^+$ be a system of positive roots corresponding to $\Pi$. Denote by $\Phi^1$, ${\Phi^+}^1$, and $\Pi^1$ the projections of $\Phi$, $\Phi^+$, and $\Pi$ on $\mathcal{V}^1$. Then $\mathcal{V}^1$ is a linear span of $\Phi^1$, every element in ${\Phi^+}^1$ is a linear combination of vectors in $\Pi^1$, with nonnegative coefficients, and for every $r^1, s^1 \in \Phi^1$, $w_{r^1}(s^1)$ is an element of $\Phi^1$. Thus, the set $\Phi^1$ behaves itself as a root system for $W^1$ and $\Pi^1$ behaves itself as a system

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of simple roots for $\Phi^1$. However, it may so happen that there exist vectors in $\Phi^1$ and $\Pi^1$ such that one is a positive multiple of the other. The following result allows us to meet this difficulty.

LEMMA 1. Let $w$ run through elements of $W^1$ and $I$ run through $\rho$-orbits of $\Pi$. Then the sets $w(\Phi_I^+)$ form a partition of $\Phi$ into equivalence classes. The roots $r$ and $s$ are in the same set if and only if $r^1 = cs^1$, where $c \in \mathbb{R}^+$. For such $r$ and $s$, the reflections $w_{r^1}$ and $w_{s^1}$ coincide.

Proof. See Lemma 13.2.1 in [6]. Denote by $\overline{\Phi}^1$ the set of equivalence classes from Lemma 1, by $\overline{\Pi}^1$ a subset of $\overline{\Phi}^1$ consisting of classes containing elements of $\Pi$, and by $w_S$ a reflection $w_{r^1}$, where $r \in S$. If $\{I_1, I_2, \dots, I_k\}$ is the set of all $\rho$-orbits of $\Pi$, then $\overline{\Pi}^1 = \{\Phi_{I_i}^+ \mid i = 1, \dots, k\}$ and the set $\{r_1^1, \dots, r_k^1 \mid r_i \in \Phi_{I_i}^+\}$ forms a basis for $\mathcal{V}^1$.

Let $G = \mathcal{L}(K)$ be a finite Chevalley group of type $\mathcal{L}$ over a finite field $K$ of order $q = p^s$ ($p$ is the characteristic of $K$) such that its Dynkin diagram has a nontrivial symmetry $\rho$ of order $t$. Suppose that $p = 2$, if $\mathcal{L} = B_2$ or $F_4$, $p = 3$, if $\mathcal{L} = G_2$, and $p$ in the other cases. Then there exists a graph isomorphism $\overline{g}$ of $G$ such that $g(X_r) = X_{\overline{r}}$ for every $r \in \Pi$. If there exists a field isomorphism $f$ of $G$ such that the automorphism $\sigma = gf$ has order $t$, then the subgroups $U$, $V$, $H$, and $N$ of $G$ are $\sigma$-invariant and $\sigma$ acts on $N/H \simeq W$ by the rule $\sigma(w_r) = w_{\overline{r}}$ for every $r \in \Pi$. The automorphism $f$ exists if the following conditions on the field $K$ hold:

(a) if all roots in $\mathcal{L}$ have the same length, then the order of $K$ is a square of prime power;

(b) if $\mathcal{L} = B_2$ or $F_4$, then $|K| = 2^{2m+1}$;

(c) if $\mathcal{L} = G_2$, then $|K| = 3^{2m+1}$.

Let $U^1 = \langle x \in U \mid \sigma(x) = x \rangle$, $V^1 = \langle x \in V \mid \sigma(x) = x \rangle$, $G^1 = \langle U^1, V^1 \rangle$, $H^1 = G^1 \cap H$, $N^1 = G^1 \cap N$, and $B^1 = G^1 \cap B$. The group $G^1 = {}^t\mathcal{L}(K)$ is called a *twisted group of type ${}^t\mathcal{L}$ over $K$*.

Let $S$ be an equivalence class on $\Phi$ defined as in Lemma 1. Then $X_S = \langle X_r \mid r \in S \rangle = \prod_{r \in S} X_r$. Denote by $X_S^1$ a subgroup $C_{X_S}(\sigma)$ of $X_S$. For every element $w \in W^1$, there exists an element $n_w \in N^1$ corresponding to $w$ under the natural homomorphism from $N$ onto $W$. Hence, the factor group $N^1/H^1$ is isomorphic to $W^1$.

Let $J$ be a subset of $\overline{\Pi}^1$, $W_J^1 = \langle w_S \mid S \in J \rangle$, and $N_J^1$ be the preimage of $W_J^1$ in $N^1$. Then $P_J^1 = B^1 N_J^1 B^1$ is a subgroup of $G$. By analogy with Chevalley groups, a subgroup conjugated in $G^1$ with $P_J^1$ is called *parabolic*. All basic properties of parabolic subgroups of Chevalley groups are likewise shareable by parabolic subgroups of twisted groups. We specify, for instance, the Levi decomposition for $P_J^1$. Let $\overline{\Phi}_J^0 = \overline{\Phi^+}^1 \cap (\overline{\Phi}^1 \setminus \overline{\Phi}_J^1)$, $U_J^1 = \langle X_S^1 \mid S \in \overline{\Phi}_J^0 \rangle = \prod_{S \in \overline{\Phi}_J^0} X_S^1$, and $L_J^1 = \langle H^1, X_S^1 \mid S \in \overline{\Phi}_J^1 \rangle$.

LEMMA 2 (Levi decomposition for twisted groups). (1) $U_J^1 \trianglelefteq P_J^1$; (2) $P_J^1 = U_J^1 L_J^1$, $U_J^1 \cap L_J^1 = 1$; (3) $P_J^1$ is a normalizer of $U_J^1$ in $G^1$.

Proof. The subgroups $B^1$ and $N^1$ form the so-called $(B, N)$-pair for the group $G^1$ (cf. Thm. 13.5.4 in [6]). In [7], the Levi decomposition lemma (cf. Prop. 47.4) is proven to hold for every group with a $(B, N)$-pair.

The group $\overline{G}^1 = C_{\overline{G}}(\sigma)$ is universal for $G^1$. Groups $\overline{U}^1$, $\overline{V}^1$, $\overline{H}^1$, $\overline{N}^1$, and $\overline{P}_J^1$ are defined similarly. For the subgroup $\overline{P}_J^1$ of $\overline{G}^1$, we also have the Levi decomposition: $\overline{P}_J^1 = \overline{U}_J^1 \overline{L}_J^1$.

We shall use the same brief notation for maximal subsets and subgroups as was used in [4, 5]. For example, if $\overline{p}_i$ is a class in $\overline{\Pi}^1$, $J = \overline{\Pi}^1 \setminus \{\overline{p}_i\}$, then $\overline{\Phi}_i^1$ denotes a maximal subset $\overline{\Phi}_J^1$ of $\overline{\Phi}^1$ and $P_i^1$ denotes a maximal parabolic subgroup $P_J^1$ of $G^1$.

Finally, we give orders of twisted groups.

LEMMA 3. The orders of twisted groups are given in the following list:

$|^2A_l(q)| = \frac{1}{(l+1,q+1)} \cdot q^{l(l+1)/2} \prod_{k=1}^{l} (q^{k+1} + (-1)^k)$,
$|^2D_l(q)| = \frac{1}{(4,q^l+1)} \cdot q^{l(l-1)}(q^l + 1) \prod_{k=1}^{l-1} (q^{2k} - 1)$,
$|^2E_6(q)| = \frac{1}{(3,q+1)} \cdot q^{36}(q^2 - 1)(q^5 + 1)(q^6 - 1)(q^8 - 1)(q^9 + 1)(q^{12} - 1)$,
$|^3D_4(q)| = q^{12}(q^2 - 1)(q^6 - 1)(q^8 + q^4 + 1)$,
$|^2B_2(q)| = q^2(q - 1)(q^2 + 1)$,
$|^2G_2(q)| = q^3(q - 1)(q^3 + 1)$,
$|^2F_4(q)| = q^{12}(q - 1)(q^3 + 1)(q^4 - 1)(q^6 + 1)$.

**Proof.** See Thm. 14.3.2 in [6].

## 2. GROUPS $^2B_2(q)$ AND $^2G_2(q)$

Notice that $q = 2^{2m+1}$ for a Suzuki group $^2B_2(q)$ and $q = 3^{2m+1}$ for a Ree group $^2G_2(q)$. The group $^2B_2(q) \simeq 5 : 4$ is not simple. The commutator subgroup of $^2G_2(q)$ is isomorphic to $L_2(8) : 3$. The minimal permutation representation of $L_2(8)$ was described in [2]. We can therefore assume that $m > 0$.

From the main result stated in [8], it follows that subgroups of least index in $^2B_2(q)$ and $^2G_2(q)$ are parabolic. For the given groups, this means that the subgroup of least index is a group $B^1 = U^1H^1$. For $B_2$ and for $G_2$, the subgroup $W^1$ is a cyclic group of order two. Therefore, the rank of the minimal permutation representation is equal to 2 in both cases. It is easy to verify that the double stabilizer of that representation is a group $H^1$. The structure of subgroups $U^1$ and $H^1$ for the groups $^2B_2(q)$ and $^2G_2(q)$ was described by Suzuki and Ree in [9] and [10]. It is thus to them that we should give credit for being the first to furnish descriptions of the minimal permutation representations for $^2B_2(q)$ and $^2G_2(q)$. Here the parameters of those representations are given only for the sake of completeness.

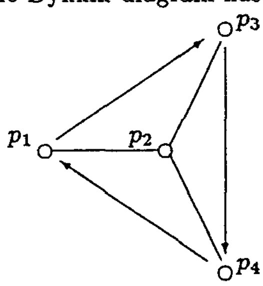
**THEOREM 1.** For simple non-Abelian groups $G =^2 B_2(q)$, $q = 2^s$, $s$ is an odd integer greater than 1, the parameters $n$, $n_2$, $P$, and $M_2$ of minimal permutation representations are given in the following list:
$n = q^2 + 1$, $n_2 = q^2$, $P = (2^s \cdot 2^s) : (q - 1)$, $M_2 = (q - 1)$.
The rank of the representation is equal to 2.

**THEOREM 2.** For simple non-Abelian groups $G =^2 G_2(q)$, $q = 3^s$, $s$ is an odd integer greater than 1, the parameters $n$, $n_2$, $P$, and $M_2$ of minimal permutation representations are given in the following list:
$n = q^3 + 1$, $n_2 = q^3$, $P = (3^s \cdot 3^s \cdot 3^s) : (q - 1)$, $M_2 = (q - 1)$.
The rank of the representation is equal to 2.

## 3. GROUP $^3D_4(q)$

### A. Structure of $W^1$ and $\overline{\Phi}^1$

The order of symmetry $\rho$ equals 3. The Dynkin diagram has the form



As representatives of equivalence classes of $\overline{\Pi}^1$ we can take the vectors $r_1^1 = p_2$ and $r_2^1 = \frac{1}{3}(p_1+p_3+p_4)$. Thus $\overline{\Pi}^1$ is of type $G_2$. Hence $W^1 \simeq W(G_2)$.

Consider the partition of $\Phi$, and more precisely, of $\Phi^+$, specified in Lemma 1. Denote the equivalence classes that obtain by $\alpha\overline{p}_1 + \beta\overline{p}_2$, in the notation for the algebra $G_2$ adopted in [4]. We have

$\overline{p}_1 = \{p_1, p_3, p_4\}$, $\overline{p}_2 = \{p_2\}$, $\overline{p}_1 + \overline{p}_2 = \{p_1 + p_2, p_2 + p_3, p_2 + p_4\}$,
$2\overline{p}_1 + \overline{p}_2 = \{p_1 + p_2 + p_3, p_1 + p_2 + p_4, p_2 + p_3 + p_4\}$, $3\overline{p}_1 + \overline{p}_2 = \{p_1 + p_2 + p_3 + p_4\}$,
$3\overline{p}_1 + 2\overline{p}_2 = \{p_1 + 2p_2 + p_3 + p_4\}$.

## B. Group ${}^3D_4(q)$ and its parabolic subgroups of least index

The group $G^1 = {}^3D_4(q)$ is a subgroup of $G = D_4(q^3)$ generated by subgroups $U^1$ and $V^1$. $G^1$ coincides with its universal group. From the main result of [8] (see also [11]), it follows that a subgroup of least index in $G^1$ has to be parabolic. There are, up to conjugation, two maximal parabolic subgroups in $G$: $P_1^1$ and $P_2^1$. We shall find orders of these groups using Lemma 2.

First note that, in the partition of $\Phi$, there are classes of types $S = \{r\}$ and $S = \{r, \overline{r}, \overline{\overline{r}}\}$. Proposition 13.6.4 in [6] implies that $X_S^1 = \{x_r(t) \,|\, t = t^f, t \in K\} \simeq p^s$ in the first case and $X_S^1 = \{x_r(t)x_{\overline{r}}(t^f)x_{\overline{\overline{r}}}(t^{f^2}) \,|\, t \in K\} \simeq p^{3s}$ in the second. Thus, the order of

$$U_1^1 = \prod_{S \in \overline{\Phi}_1^0} X_S^1 = X_{\overline{p}_1}^1 \cdot X_{\overline{p}_1+\overline{p}_2}^1 \cdot X_{2\overline{p}_1+\overline{p}_2}^1 \cdot X_{3\overline{p}_1+\overline{p}_2}^1 \cdot X_{3\overline{p}_1+2\overline{p}_2}^1$$

is equal to $q^{11}$. The order of

$$U_2^1 = X_{\overline{p}_2}^1 \cdot X_{\overline{p}_1+\overline{p}_2}^1 \cdot X_{2\overline{p}_1+\overline{p}_2}^1 \cdot X_{3\overline{p}_1+\overline{p}_2}^1 \cdot X_{3\overline{p}_1+2\overline{p}_2}^1$$

is equal to $q^9$. The group $L_1^1 = \langle H^1, X_S^1 \,|\, S \in \overline{\Phi}_1^1\rangle$ is isomorphic to an extension of $\overline{A}_1(q)$ by a cyclic subgroup of order $q-1$. Hence $|L_1^1| = q(q^2-1)(q-1)$. Similarly, the group $L_2^1 = \langle H^1, X_S^1 \,|\, S \in \overline{\Phi}_2^1\rangle$ is an extension of $\overline{A}_1(q^3)$ by the group $(q-1)$. Hence $|L_2^1| = q^3(q^6-1)(q-1)$. Using Lemma 3, we obtain $|G^1 : P_2^1| = (q^8+q^4+1)(q+1) < |G^1 : P_1^1| = (q^8+q^4+1)(q^6-1)/(q-1)$. Therefore, the subgroup of least index in $G^1$ is $P_2^1$, which we denote hereafter by $P^1$. Now we describe in more detail the structure of groups $U_1^1$, $U_2^1$, and $L_2^1$.

The commutator relations in subgroups $U_1^1$ and $U_2^1$ of $G^1$ are similar to those in $U_1$ and $U_2$ of $G_2(q)$ (cf. [4]). However, since all roots in $D_4$ have the same length, the nontriviality of commutators in these relations does not any longer depend on the characteristic of a field $K$. For the group $U_1^1$, we have

$$1 \neq [X_{\overline{p}_1+\overline{p}_2}^1, X_{\overline{p}_1}^1] \leq X_{2\overline{p}_1+\overline{p}_2}^1 \cdot X_{3\overline{p}_1+\overline{p}_2}^1 \cdot X_{3\overline{p}_1+2\overline{p}_2}^1,$$

$$1 \neq [X_{2\overline{p}_1+\overline{p}_2}^1, X_{\overline{p}_1}^1] \leq X_{3\overline{p}_1+\overline{p}_2}^1, \quad 1 \neq [X_{2\overline{p}_1+\overline{p}_2}^1, X_{\overline{p}_1+\overline{p}_2}^1] \leq X_{3\overline{p}_1+2\overline{p}_2}^1,$$

$$1 = [X_{S_1}^1, X_{S_2}^1] \text{ in all other cases.}$$

For the group $U_2^1$,

$$1 \neq [X_{3\overline{p}_1+\overline{p}_2}^1, X_{\overline{p}_2}^1] \leq X_{3\overline{p}_1+2\overline{p}_2}^1, \quad 1 \neq [X_{2\overline{p}_1+\overline{p}_2}^1, X_{\overline{p}_1+\overline{p}_2}^1] \leq X_{3\overline{p}_1+2\overline{p}_2}^1,$$

$$1 = [X_{S_1}^1, X_{S_2}^1] \text{ in all other cases.}$$

Thus $U_1^1 \simeq p^{2s} \cdot (p^{3s} \cdot p^{6s})$ and $U_2^1 \simeq p^s \cdot p^{8s}$.

Denote by $L_2'$ a group $\langle X^1_{\overline{p}_1}, X^1_{-\overline{p}_1}\rangle \simeq \overline{A}_1(q^3)$. Its center is

$$Z = \{h_{p_1}(\mu) \cdot h_{p_3}(\mu^q) \cdot h_{p_4}(\mu^{q^2}) \mid \mu^2 = 1\}.$$

Let $h_0 = h_{p_1}(\lambda) \cdot h_{p_2}(\lambda^2) \cdot h_{p_3}(\lambda) \cdot h_{p_4}(\lambda)$, where $\lambda^q = \lambda$. Then $h_0$ centralizes $L_2'$. On the other hand, $\langle h_0\rangle \cap L_2' = Z$. Therefore,

$$P^1 = P^1_2 = U^1_2 L^1_2 \simeq (p^s \cdot p^{8s}) : (d \cdot (A_1(q^3) \times (q-1)/d) \cdot d),$$

where $d = (q-1, 2)$.

## C. Representation of $G^1$ on cosets w.r.t. $P^1$

Our goal is to define double stabilizers of the representation of $G^1$ on the cosets w.r.t. $P^1$, that is, groups of the form $P^1 \cap (P^1)^{n_w}$. Since $H \leq P^1$, the action of an element $n_w \in N^1$ on $P^1$ is determined by the action of its image $w \in W^1$ on $\overline{\Phi}^1$. Let $w_1 = w_{\overline{p}_2} \in W^1$. Then

$$w_1(\overline{p}_2) = -\overline{p}_2, \quad w_1(\overline{\Phi^+}^1 \setminus \{\overline{p}_2\}) = \overline{\Phi^+}^1 \setminus \{\overline{p}_2\}, \quad w_1(-\overline{p}_1) = -\overline{p}_1 - \overline{p}_2.$$

So the double stabilizer is

$$M_2 = P^1 \cap (P^1)^{w_1} = U^1_1 : H^1 \simeq (p^{2s} \cdot (p^{3s} \cdot p^{6s})) : ((q^3-1) \times (q-1)).$$

The degree is $n_2 = |P^1 : M_2| = q(q^3+1)$.

The element $w_2 = w_{3\overline{p}_1 + 2\overline{p}_2}$ acts on $\overline{\Phi^+}^1$ in the following way:

$$w_2(\overline{p}_1) = \overline{p}_1, w_2(\overline{\Phi^+}^1 \setminus \{\overline{p}_1\}) = \overline{\Phi^-}^1 \setminus \{-\overline{p}_1\}.$$

Hence $M_3 = P^1 \cap (P^1)^{w_2} = L^1_2$ and $n_3 = |P^1 : M_3| = q^9$.

Consider an element $w_3 = w_{\overline{p}_1 + \overline{p}_2}$. We have

$$w_3(\overline{p}_1) = 2\overline{p}_1 + \overline{p}_2, \quad w_3(\overline{p}_2) = -3\overline{p}_1 - \overline{p}_2, \quad w_3(\overline{p}_1 + \overline{p}_2) = -\overline{p}_1 - \overline{p}_2, \quad w_3(3\overline{p}_1 + \overline{p}_2) = 3\overline{p}_1 + \overline{p}_2.$$

Since the order of $w_3$ equals 2, these relations determine the action of $w_3$ on $\overline{\Phi}^1$. Therefore, $M_4 = P^1 \cap (P^1)^{w_1} = \langle X^1_{\overline{p}_1}, X^1_{2\overline{p}_1+\overline{p}_2}, X^1_{3\overline{p}_1+\overline{p}_2}\rangle : H^1 \simeq (p^s \cdot p^{6s}) : ((q^3-1) \times (q-1))$ and $n_4 = q^5(q^3+1)$.

Taking the sum of the three subdegrees that obtain and the trivial one yields $\sum_{i=1}^{4} n_i = n$. Hence the rank of the representation equals 4.

THEOREM 3. For simple non-Abelian groups $G = {}^3D_4(q)$, the parameters $n$, $n_2$, $n_3$, $n_4$, $P$, $M_2$, $M_3$, and $M_4$ of minimal permutation representations are given in the following list:

$n = (q^8 + q^4 + 1)(q+1)$, $n_2 = q(q^3+1)$, $n_3 = q^9$, $n_4 = q^5(q^3+1)$,

$P = (p^s \cdot p^{8s}) : (d \cdot (A_1(q^3) \times (q-1)/d) \cdot d)$,

$M_2 = (p^{2s} \cdot (p^{3s} \cdot p^{6s})) : ((q^3-1) \times (q-1))$,

$M_3 = d \cdot (A_1(q^3) \times (q-1)/d) \cdot d$,

$M_4 = (p^s \cdot p^{6s}) : ((q^3-1) \times (q-1))$,

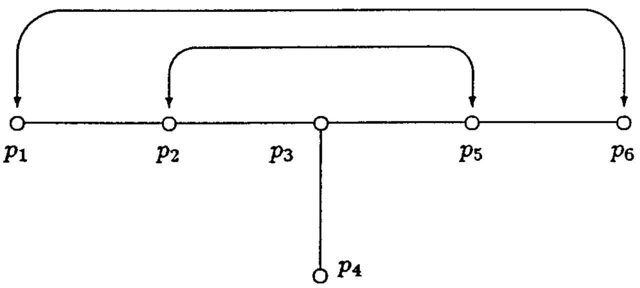
where $d = (2, q-1)$.

The rank of the representation is equal to 4.

## 4. GROUP $^2E_6(q)$

### A. Structure of $W^1$ and $\overline{\Phi}^1$

The order of symmetry $\rho$ equals 2. The Dynkin diagram has the form




As representatives of equivalence classes of $\overline{\Pi}^1$ we can take the vectors

$$r_1^1 = p_4, \quad r_2^1 = p_3, \quad r_3^1 = \frac{1}{2}(p_2 + p_5), \quad r_4^1 = \frac{1}{2}(p_1 + p_6).$$

Thus $\overline{\Pi}^1$ is of type $F_4$. Hence $W^1 \simeq W(F_4)$.

The partition $\overline{\Phi^+}^1$ of $\Phi^+$ contains 12 classes of type $S = \{r\}$ and 12 classes of type $S = \{r, \overline{r}\}$. Put

$$\overline{p}_1 = \{p_4\}, \quad \overline{p}_2 = \{p_3\}, \quad \overline{p}_3 = \{p_2, p_5\}, \quad \overline{p}_4 = \{p_1, p_6\}.$$

The notation for other equivalence classes is the same as for the system of positive roots $\Phi^+$ of algebra $F_4$ in [4].

### B. Group $^2E_6(q)$ and its parabolic subgroups of least index

The group $G^1 = {}^2E_6(q)$ is a subgroup of $G = E_6(q^2)$ generated by subgroups $U^1$ and $V^1$. If 3 does not divide $q+1$, then $G^1$ coincides with its universal group $\overline{G}^1$. Otherwise, it is isomorphic to the factor group $\overline{G}^1/\overline{Z}^1$ by the center $\overline{Z}^1$ of $\overline{G}^1$, which in turn coincides with the center of $\overline{G}$. Hence,

$$\overline{Z}^1 = \{\overline{h}_{p_1}(\mu) \cdot \overline{h}_{p_2}(\mu^2) \cdot \overline{h}_{p_5}(\mu) \cdot \overline{h}_{p_6}(\mu^2) \mid \mu \in K^*, \ \mu^{q+1} = \mu^3 = 1\}.$$

From the main result stated in [8], it follows that a subgroup of least index in $G^1$ is parabolic. There are, up to conjugation, four maximal parabolic subgroups in $G^1$. Computing their orders in the same way as was done in the previous section for $^3D_4(q)$, we see that $P^1 = P_1^1$ has least index in $G^1$. The degree $n$ of the representation of $G^1$ on the cosets w.r.t. $P^1$ is equal to $(q^{12}-1)(q^6-q^3+1)(q^4+1)/(q-1)$.

First we describe the structure of a subgroup $\overline{P}^1$ of $\overline{G}^1$. It has the Levi decomposition $\overline{P}^1 = \overline{U}_1^1 \overline{L}_1^1$, where $\overline{U}_1^1 = \langle \overline{X}_S^1 \mid S \in \overline{\Phi}_1^0 \rangle$, $\overline{L}_1^1 = \langle \overline{H}^1, \overline{X}_S^1 \mid S \in \overline{\Phi}_1^1 \rangle$. Since $H^1$ contains $\overline{Z}^1$, we have $\overline{U}^1 \simeq U^1$ and $\overline{U}_1^1 \simeq U_1^1$. If the class $S$ is of type $\{r\}$, then $X_S^1 = \{x_r(t) \mid t = t^f, \ t \in K\} \simeq p^s$. But if $S = \{r, \overline{r}\}$, then $X_S^1 = \{x_r(t) x_{\overline{r}}(t^f) \mid t \in K\} \simeq p^{2s}$. The above argument for the commutator relations in unipotent subgroups of the groups $^3D_4(q)$ and $G_2(q)$ works also for groups $^2E_6(q)$ and $F_4(q)$. This allows us to determine the structure of a subgroup $U_1^1$ (computations being too lengthy are omitted). We have $U_1^1 \simeq p^s \cdot \overline{p}^{20s}$.

Denote by $\overline{L}_1'$ a subgroup $\langle \overline{X}_S^1 \mid S \in \overline{\Phi}_1^1 \rangle$. It is isomorphic to the universal twisted group $^2\overline{A}_5(q)$. Hence,

$$Z(\overline{L}_1') = \{\overline{h}_{p_1}(\lambda) \cdot \overline{h}_{p_2}(\lambda^2) \cdot \overline{h}_{p_3}(\lambda^3) \cdot \overline{h}_{p_5}(\lambda^{2q}) \cdot \overline{h}_{p_6}(\lambda^q) \mid \lambda \in K^*, \ \lambda^{q+1} = \lambda^6 = 1\}.$$

Consider the element $h_0 = \overline{h}_{p_1}(\lambda)\cdot\overline{h}_{p_2}(\lambda^2)\cdot\overline{h}_{p_3}(\lambda^3)\cdot\overline{h}_{p_4}(\lambda^{q+1})\cdot\overline{h}_{p_5}(\lambda^{2q})\times\overline{h}_{p_6}(\lambda^q)$, where $(\lambda^{q-1})^3 = 1$. The element is fixed by an automorphism $\sigma$ and so lies in $\overline{H}^1$. Furthermore, it centralizes the subgroup $\overline{L}_1'$, and $\langle h_0\rangle \cap Z(\overline{L}_1') = Z(\overline{G}^1)$. Thus $\overline{L}_1^1 = d_+'\cdot(d_+\cdot{}^2A_5(q)\times(q-1)/d_+')\cdot d_+'$, where $d_+ = (2, q+1)$, $d_+' = (3, q+1)$. And the structure of $\overline{P}^1$ is thereby determined. Factoring out $P^1$ by the center $\overline{Z}^1$ of $\overline{G}^1$, we obtain

$$P^1 \simeq (p^s\cdot p^{20s}) : (d_+\cdot{}^2A_5(q)\times(q-1)/d_+')\cdot d_+'.$$

## C. Representation of $G^1$ on cosets w.r.t. $P^1$

The action of an element $n_w \in N^1$ on $P^1$, as we mentioned above, is determined by the action of its image $w \in W^1$ on $\overline{\Phi}^1$. In our case $W^1 \simeq W(F_4)$ and $\overline{\Phi}^1 \sim \Phi(F_4)$. Hence, the action of $w_S \in W^1$ on $\overline{\Phi}^1$ is similar to the action of a corresponding element $w_r \in W(F_4)$ (cf. subsection A) on $\Phi(F_4)$. However, to describe double stabilizers of the representation of $G^1$ on the cosets w.r.t. $P^1$, we need to take into account that the class $S \in \overline{\Phi}^1$ may well contain two elements of $\Phi(E_6)$.

Denote by $w_1$ an element $w_{\overline{p}_1} \in W^1$. Consider the action of $w_{p_1}$ on $\overline{\Phi}^1(F_4)$ (cf. Diagram 1 in [4]). If, in that diagram, we replace subsets of $\Phi$ by respective subsets of $\overline{\Phi}^1$, the resulting diagram will show us the action of $w_1$ on $\overline{\Phi}^1$. Therefore $\overline{U}_1^1 \cap (\overline{P}^1)^{w_1} \simeq U_1^1 \cap (P^1)^{w_1} \simeq p^s\cdot p^{18s}\times p^s$.

The group $\overline{L}_1^1 \cap (\overline{P}^1)^{w_1} = \overline{U}_{1,2}^1 : \overline{L}_{1,2}^1$, where $\overline{U}_{1,2}^1 \simeq U_{1,2}^1 = \langle X_S^1 \mid S \in \overline{\Phi^+}_1^1 \setminus \overline{\Phi^+}_{1,2}^1\rangle \simeq p^{9s}$ and $\overline{L}_{1,2}^1 = \langle \overline{H}^1, X_S^1 \mid S \in \overline{\Phi}_{1,2}^1\rangle$, is an extension of $\overline{L}_{1,2}' = \langle X_S^1 \mid S \in \overline{\Phi}_{1,2}^1\rangle \simeq \overline{A}_2(q^2)$ by a subgroup in $\overline{H}^1$ of order $(q-1)^2$. Consider the group $\overline{L}_{1,2}^1$ in detail. We have

$$Z(\overline{L}_{1,2}') = \{\overline{h}_{p_1}(\lambda)\cdot\overline{h}_{p_2}(\lambda^2)\cdot\overline{h}_{p_5}(\lambda^{2q})\cdot\overline{h}_{p_6}(\lambda^q) \mid \lambda \in K^*, \lambda^3 = 1\}.$$

The subgroup $\langle h_0\rangle$ (the element $h_0$ was defined in subsection B) includes $Z(\overline{L}_{1,2}')$, if $\lambda^{q+1} = \lambda^3 = 1$, and centralizes $\overline{L}_{1,2}'$. The element $\overline{h}_{p_3}(\lambda) \in \overline{H}$ lies in $\overline{H}^1$, for every $\lambda$ such that $\lambda^{q-1} = 1$, and centralizes the subgroup $\overline{L}_{1,2}'$; the intersection $\langle\overline{L}_{1,2}'\rangle \cap \langle\overline{h}_{p_3}(\lambda)\rangle$ is trivial. Hence $\overline{L}_{1,2}^1 \simeq (d_+'\cdot(d'\cdot A_2(q^2)\times(q-1)/d_+')\times(q-1))\cdot d_+'$, where $d' = (3, q-1)$, $d_+' = (3, q+1)$. Obviously, $Z(\overline{G}^1) \le Z(\overline{L}_{1,2}')$. Therefore,

$$M_2 = P^1 \cap (P^1)^{w_1} \simeq (p^s\cdot p^{18s}\times p^s) : (p^{9s} : ((d'\cdot A_2(q^2)\times(q-1)/d_+')\times(q-1))\cdot d_+').$$

The index of a subgroup $M_2$ in $P^1$ is equal to $q(q^5+1)(q^3+1)(q+1)$.

Denote by $w_2$ an element $w_{2\overline{p}_1+3\overline{p}_2+4\overline{p}_3+2\overline{p}_4} \in W^1$. Diagram 2 for the element $w_{2p_1+3p_2+4p_3+2p_4} \in W(F_4)$ (cf. [4]) implies that $w_2(\overline{\Phi}_1^1) = \overline{\Phi}_1^1$, $w_2(\overline{\Phi^+}^1 \setminus \overline{\Phi^+}_1^1) = \overline{\Phi^-}^1 \setminus \overline{\Phi^-}_1^1$. Hence $M_3 = P^1 \cap (P^1)^{w_2} = L_1^1$. The subdegree is $n_3 = |P^1 : M_3| = q^{21}$.

Consider an element $w_3 = w_{\overline{p}_1+2\overline{p}_2+3\overline{p}_3+2\overline{p}_4} \in W^1$. Its action on $\overline{\Phi}^1$ is similar to the action of $w_{p_1+2p_2+3p_3+2p_4} \in W(F_4)$ on $\Phi(F_4)$ (cf. Diagram 3 in [4]). So we have $\overline{U}_1^1 \cap (\overline{P}^1)^{w_3} \simeq U_1^1 \cap (P^1)^{w_3} \simeq p^s\cdot p^{8s}\times p^{6s}$.

The group $\overline{L}_1^1 \cap (\overline{P}^1)^{w_3} = \overline{U}_{1,4}^1 : \overline{L}_{1,4}^1$, where $\overline{U}_{1,4}^1 \simeq U_{1,4}^1 = \langle X_S^1 \mid S \in \overline{\Phi^+}_1^1 \setminus \overline{\Phi^+}_{1,4}^1\rangle \simeq p^s\cdot p^{8s}$ and $\overline{L}_{1,4}^1 = \langle\overline{H}^1, X_S^1 \mid S \in \overline{\Phi}_{1,4}^1\rangle$ is an extension of $\overline{L}_{1,4}' = \langle X_S^1 \mid S \in \overline{\Phi}_{1,4}^1\rangle \in {}^2\overline{A}_3(q)$ by a subgroup in $\overline{H}^1$ of order $(q^2-1)(q-1)$. Consider the group $\overline{L}^1{}_{1,4}$ in detail. We have

$$Z(\overline{L}_{1,4}') = \{\overline{h}_{p_2}(\lambda)\cdot\overline{h}_{p_3}(\lambda^2)\cdot\overline{h}_{p_5}(\lambda^q) \mid \lambda \in K^*, \lambda^{q+1} = \lambda^4 = 1\}.$$

Then $Z(\overline{L}_{1,4}') \cap Z(\overline{G}^1) = 1$. If $\lambda$ generates $K^*$, then the element $h_1 = \overline{h}_{p_1}(\lambda^2)\cdot\overline{h}_{p_2}(\lambda)\cdot\overline{h}_{p_5}(\lambda^q)\cdot\overline{h}_{p_6}(\lambda^{2q})$ has order $q^2-1$ and centralizes $\overline{L}_{1,4}'$. It is clear that $\overline{L}_{1,4}' \cap \langle h_1\rangle$ is a subgroup of order $d_+ = (2, q+1)$. Since

$Z(\overline{G}^1) \leq \langle h_1 \rangle$, we obtain $M_4 = P^1 \cap (P^1)^{w_3} \simeq (p^s \cdot p^{8s} \times p^{6s}) : ((p^s \cdot p^{8s}) : (d_+ \cdot (e'_+ \cdot {}^2\overline{A}_3(q) \times (q^2-1)/c_+)) \cdot (q-1))$, where $c_+ = d_+ \cdot d'_+$, $e'_+ = (4, q+1)/d_+$. The index of $M_4$ in $P^1$ is equal to $q^6(q^5+1)(q^4+q^2+1)$.

Consider an element $w_4 = w_2 \cdot w_1 \in W^1$. Its action on $\overline{\Phi}^1$ can be determined from Diagram 4 (cf. [4]). We have $\overline{U}_1^1 \cap (\overline{P}^1)^{w_4} \simeq U_1^1 \cap (P^1)^{w_4} \simeq p^{10s}$. Furthermore, $L_1^1 \cap (P^1)^{w_4} = L_1^1 \cap (P^1)^{w_1}$. Thus,

$$M_5 = P^1 \cap (P^1)^{w_4} \simeq p^{10s} : (p^{9s} : ((d' \cdot A_2(q^2) \times (q-1)/d'_+) \times (q-1)) \cdot d'_+).$$

The index of $M_5$ in $P^1$ is equal to $q^{11}(q^5+1)(q^3+1)(q+1)$. Taking the sum of the four subdegrees that obtain and the trivial one yields $\sum_{i=1}^{5} n_i = n$. Hence the rank of the representation equals 5.

THEOREM 4. For simple non-Abelian groups $G = {}^2E_6(q)$, the parameters $n$, $n_2$, $n_3$, $n_4$, $n_5$, $P$, $M_2$, $M_3$, $M_4$, and $M_5$ of minimal permutation representations are given in the following list:

$n = (q^{12}-1)(q^6-q^3+1)(q^4+1)/(q-1)$, $n_2 = q(q^5+1)(q^3+1)(q+1)$, $n_3 = q^{21}$, $n_4 = q^6(q^5+1)(q^4+q^2+1)$, $n_5 = q^{11}(q^5+1)(q^3+1)(q+1)$,

$P = (p^s \cdot p^{20s}) : (d_+ \cdot {}^2A_5(q) \times (q-1)/d'_+) \cdot d'_+$,

$M_2 = (p^s \cdot p^{18s} \times p^s) : (p^{9s} : ((d' \cdot A_2(q^2) \times (q-1)/d'_+) \times (q-1)) \cdot d'_+)$,

$M_3 = (d_+ \cdot {}^2A_5(q) \times (q-1)/d'_+) \cdot d'_+$,

$M_4 = (p^s \cdot p^{8s} \times p^{6s}) : ((p^s \cdot p^{8s}) : (d_+ \cdot (e'_+ \cdot {}^2\overline{A}_3(q) \times (q^2-1)/c_+)) \cdot (q-1))$,

$M_5 = p^{10s} : (p^{9s} : ((d' \cdot A_2(q^2) \times (q-1)/d'_+) \times (q-1)) \cdot d'_+)$,

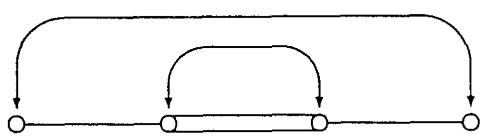
where $d_+ = (2, q+1)$, $d'_+ = (3, q+1)$, $d' = (3, q-1)$, $c_+ = d_+ \cdot d'_+$, $e'_+ = (4, q+1)/d_+$.

The rank of the representation is equal to 5.

## 5. GROUP ${}^2F_4(q)$

### A. Structure of $W^1$ and $\overline{\Phi}^1$

The order of symmetry $\rho$ equals 2. The Dynkin diagram has the form




The representatives of equivalence classes of $\overline{\Pi}^1$ are

$$r_1^1 = \frac{1}{2}(p_1 + \sqrt{2}p_4), \quad r_2^1 = \frac{1}{2}(p_2 + \sqrt{2}p_3).$$

Thus the group $W^1 = \langle w_{r_1^1}, w_{r_2^1} \rangle$ is isomorphic to a dihedral group of order 16. Below we use the notation for elements of $W^1$ introduced by Ree in [12], in which the group ${}^2F_4(q)$ was originally constructed.

Let $e_1$, $e_2$, $e_3$, and $e_4$ form an orthonormal basis for the space $\mathcal{K}_{\mathbf{R}}$ ($\mathcal{K}$ is the Cartan subalgebra of $F_4$) associated — as in Sec. 3 of [4] — with the system of simple roots $\Pi$ in $F_4$. Then $W^1$ contains elements of types (1) $w(\varepsilon, \delta, \infty)$, (2) $w(\infty, \varepsilon, \delta)$, and (3) $w(\varepsilon, \delta, \eta)$, which act on the basis vectors in the following way:

(1) $e_1 \to \varepsilon e_1$, $e_2 \to \varepsilon e_2$, $e_3 \to \delta e_3$, $e_4 \to \delta e_4$;

(2) $e_1 \to \varepsilon e_3$, $e_2 \to \varepsilon e_4$, $e_3 \to \delta e_1$, $e_4 \to \delta e_2$;

(3) $e_1 \to \frac{1}{2}(\varepsilon(e_1+e_2)+\delta(e_3+e_4))$, $e_2 \to \frac{1}{2}(\varepsilon(e_1-e_2)+\delta(e_3-e_4))$, $e_3 \to \frac{1}{2}\eta(\varepsilon(e_1+e_2)-\delta(e_3+e_4))$, $e_4 \to \frac{1}{2}\eta(\varepsilon(e_1-e_2)-\delta(e_3-e_4))$;
here $\varepsilon, \delta, \eta = \pm 1$.

The system of positive roots in $F_4$ has a partition into eight equivalence classes forming $\overline{\Phi^+}^1$. Each class is of type $\{r, \bar{r}\}$ or $\{r, \bar{r}, r+\bar{r}, 2r+\bar{r}\}$. Following [12], for brevity, we denote roots of the algebra $F_4$ by the sequences of numbers of vectors in the basis of $\mathcal{K}_{\mathbf{R}}$, with respective signs. For example, the root $p_1 = e_2 - e_3$ is denoted by $2-3$; $p_2 = e_3 - e_4$ by $3-4$; $p_3 = e_4$ by $4$; $p_4 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4)$ by $1-2-3-4$; and $2p_1 + 3p_2 + 4p_3 + 2p_4 = e_1 + e_4$ by $1+4$. Then the partition of $\Phi^+$ into equivalence classes will be written in the following way:

$$\begin{array}{ll} S_1 = \{2, 1-2, 1, 1+2\}, & S_5 = \{1+4, 1+2+3-4\}, \\ S_2 = \{4, 3-4, 3, 3+4\}, & S_6 = \{1-4, 1+2-3+4\}, \\ S_3 = \{1-2+3-4, 2+4, 1+2+3+4, 1+3\}, & S_7 = \{2+3, 1-2+3+4\}, \\ S_4 = \{1-2-3+4, 2-4, 1+2-3-4, 1-3\}, & S_8 = \{2-3, 1-2-3-4\}. \end{array}$$

Clearly, $\overline{\Pi}^1 = \{S_2, S_8\}$.

## B. Group ${}^2F_4(q)$ and its parabolic subgroups of least index

The group $G^1 = {}^2F_4(q)$ coincides with its universal group. By Lemma 64 from [13], therefore, it may be defined as a subgroup of $G = F_4(q)$, consisting of all the elements of $G$ fixed by $\sigma$. Recall that $q$ has to be equal to $2^{2m+1}$. The group ${}^2F_4(2)$ is not simple. Its commutator subgroup ${}^2F_4(2)'$ is a simple Tits group. We shall put off the treatment of its minimal permutation representation till the end of this section, assuming for the moment that $m \geq 1$. That a subgroup of least index in $G^1$ is parabolic follows from the main result of [8] (see also [14]). There are, up to conjugation, four maximal parabolic subgroups in $G^1$. Our computations show that of least index in $G^1$ is a subgroup $P^1 = P^1_{\bar{1}} = P^1_{S_2}$. Therefore $n = |G^1 : P^1| = (q^6+1)(q^3+1)(q+1)$. We describe the structure of $P^1$ using the Levi decomposition: $P^1 = U^1_{\bar{1}} L^1_{\bar{1}}$.

Let $\theta = 2^m$. To determine the structure of root subgroups $X^1_S$, consider the following 12 $\sigma$-invariant elements, introduced in [12]:

$\alpha_1(t) = x_4(t^\theta)x_{3-4}(t)x_3(t^{\theta+1})$,
$\alpha_2(t) = x_3(t^\theta)x_{3+4}(t)$,
$\alpha_3(t) = x_{1-2-3-4}(t^\theta)x_{2-3}(t)$,
$\alpha_4(t) = x_{1-2-3+4}(t^\theta)x_{2-4}(t)x_{1+2-3-4}(t^{\theta+1})$,
$\alpha_5(t) = x_2(t^\theta)x_{1-2}(t)x_1(t^{\theta+1})$,
$\alpha_6(t) = x_{1-2+3-4}(t^\theta)x_{2+4}(t)x_{1+2+3+4}(t^{\theta+1})$,
$\alpha_7(t) = x_{1-2+3+4}(t^\theta)x_{2+3}(t)$,
$\alpha_8(t) = x_{1+2-3-4}(t^\theta)x_{1-3}(t)$,
$\alpha_9(t) = x_{1+2-3+4}(t^\theta)x_{1-4}(t)$,
$\alpha_{10}(t) = x_{1+2+3-4}(t^\theta)x_{1+4}(t)$,
$\alpha_{11}(t) = x_{1+2+3+4}(t^\theta)x_{1+3}(t)$,
$\alpha_{12}(t) = x_1(t^\theta)x_{1+2}(t)$.

Note that $[\alpha_i(t), \alpha_i(u)] = \alpha_j(t^{2\theta}u + tu^{2\theta})$, where $i = 1, 4, 5, 6$ and $j = 2, 8, 12, 11$, respectively.

Let $s = 2m+1$. We have

$X^1_{S_1} = \langle \alpha_5(t), \alpha_{12}(t) \mid t \in K \rangle \simeq 2^s \cdot 2^s,$
$X^1_{S_2} = \langle \alpha_1(t), \alpha_2(t) \mid t \in K \rangle \simeq 2^s \cdot 2^s,$
$X^1_{S_3} = \langle \alpha_6(t), \alpha_{11}(t) \mid t \in K \rangle \simeq 2^s \cdot 2^s,$
$X^1_{S_4} = \langle \alpha_4(t), \alpha_8(t) \mid t \in K \rangle \simeq 2^s \cdot 2^s,$
$X^1_{S_5} = \langle \alpha_{10}(t) \mid t \in K \rangle \simeq 2^s,$
$X^1_{S_6} = \langle \alpha_9(t) \mid t \in K \rangle \simeq 2^s,$
$X^1_{S_7} = \langle \alpha_7(t) \mid t \in K \rangle \simeq 2^s,$
$X^1_{S_8} = \langle \alpha_3(t) \mid t \in K \rangle \simeq 2^s.$

Thus the elements $\alpha_1(t), \dots, \alpha_{12}(t)$ generate $U^1$. Commutator relations for these elements were amply listed in [15]. Using the relations which do not contain $\alpha_1(t)$ and $\alpha_2(t)$, we obtain $U^1_1 \simeq 2^s \cdot 2^{4s} \cdot 2^{5s}$.

The group $L^1_1$ is an extension of the group $L'_1 = \langle X^1_S \mid S \in \overline{\Phi}^1_1 \rangle$ by a cyclic group of order $q-1$. By Proposition 2.4 in [15], there exists an element $h$ of order $q-1$ in $H^1$ which centralizes $L'_1$ and is such that $\langle h \rangle \cap L'_1 = 1$. Therefore, the group $L^1_1$ is isomorphic to the direct product of ${}^2B_2(q)$ and $(q-1)$.

## C. Representation of $G^1$ on cosets w.r.t. $P^1$

Consider the action of an element $w_1 = w_{S_8} = w(1,1,1)$ on $\overline{\Phi}^1$. We have

$$w_1(S_8) = S_{-8}, w_1(\overline{\Phi^+}^1 \setminus \{S_8\}) = \overline{\Phi^+}^1 \setminus \{S_8\}, \quad w_1(S_{-2}) = S_{-4}.$$

Hence $U^1_1 \cap (P^1)^{w_1} \simeq 2^s \cdot 2^{2s} \cdot 2^{2s} \cdot 2^{4s} \cdot 2^{2s}$. Clearly, $L'_1 \cap (P^1)^{w_1} = H^1 \simeq (q-1)^2$. Thus, the double stabilizer has the form

$$M_2 = P^1 \cap (P^1)^{w_1} \simeq (2^s \cdot 2^{2s} \cdot 2^{2s} \cdot 2^{4s} \cdot 2^{2s}) : (q-1)^2.$$

The degree is $n_2 = |P^1 : M_2| = q(q^2+1)$.

Let $w_2 = (w_{S_8} w_{S_2})^4 = w(-1,-1,\infty)$. Then for every $i = 1, \dots, 8$ we have $w_2(S_i) = S_{-i}$. Hence $M_3 = P^1 \cap (P^1)^{w_2} = L^1_1$, $n_3 = |P^1 : M_3| = q^{10}$.

Consider an element $w_3 = (w_{S_8} w_{S_2})^2 = w(\infty, 1, -1)$. We have

$$w_3(S_1) = S_2, \quad w_3(S_2) = S_{-1}, \quad w_3(S_{-2}) = S_1,$$

$$w_3(S_3) = S_{-4}, \quad w_3(S_4) = S_3, \quad w_3(S_5) = S_{-8},$$

$$w_3(S_6) = S_7, \quad w_3(S_7) = S_{-6}, \quad w_3(S_8) = S_5.$$

Therefore $M_4 = P^1 \cap (P^1)^{w_3} \simeq (2^{3s} \cdot 2^{2s} \cdot 2^{3s}) : (q-1)^2$, $n_4 = q^4(q^2+1)$.

The element $w_4 = w(-1,1,-1)$ acts on $\overline{\Phi}^1$ in the following way:

$$w_4(S_1) = S_{-4}, \quad w_4(S_2) = S_3, \quad w_4(S_{-2}) = S_{-3},$$

$$w_4(S_3) = S_2, \quad w_4(S_4) = S_{-1}, \quad w_4(S_5) = S_{-8},$$

$$w_4(S_6) = S_{-6}, \quad w_4(S_7) = S_7, \quad w_4(S_8) = S_{-5}.$$

Hence $M_5 = P^1 \cap (P^1)^{w_4} \simeq (2^{2s} \cdot 2^{2s} \times 2^s) : (q-1)^2$, $n_5 = q^7(q^2+1)$.

Since $\sum_{i=1}^{5} n_i = n$, the rank of the representation of $G^1$ on the cosets w.r.t. $P^1$ is equal to 5.

THEOREM 5. For simple non-Abelian groups $G = {}^2F_4(q)$, $q = 2^s$, $s$ is an odd integer greater than 1, the parameters $n$, $n_2$, $n_3$, $n_4$, $n_5$, $P$, $M_2$, $M_3$, $M_4$, $M_5$ of minimal permutation representations are given in the following list:

$n = (q^6+1)(q^3+1)(q+1)$, $n_2 = q(q^2+1)$, $n_3 = q^{10}$, $n_4 = q^4(q^2+1)$, $n_5 = q^7(q^2+1)$,

$P = (2^s \cdot 2^{4s} \cdot 2^{5s}) : ({}^2B_2(q) \times (q-1))$,

$M_2 = (2^s \cdot 2^{2s} \cdot 2^{2s} \cdot 2^{4s} \cdot 2^{2s}) : (q-1)^2$,

$M_3 = {}^2B_2(q) \times (q-1)$,

$M_4 = (2^{3s} \cdot 2^{2s} \cdot 2^{3s}) : (q-1)^2$,

$M_5 = (2^{2s} \cdot 2^{2s} \times 2^s) : (q-1)^2$.

The rank of the representation is equal to 5.

## D. Tits group ${}^2F_4(2)'$

There are, up to conjugation, two subgroups of least index in $G = {}^2F_4(2)'$. They are conjugate in $\mathrm{Aut}\, G$ and isomorphic to $A_2(3):2$. Therefore, permutation representations of $G$ on the cosets w.r.t. those subgroups are similar. Denote one of them by $P$. By [16], $|G:P| = 1600$ and the rank of the representation of $G$ on the cosets w.r.t. $P$ is equal to 3. Denote by $M_2$ and $M_3$ nontrivial double stabilizers of that representation, and by $n_2$ and $n_3$ the subdegrees corresponding to them. Then $n-1 = |G:P|-1 = 1599 = 41 \cdot 3 \cdot 13 = |P:M_2| + |P:M_3| = n_2 + n_3$. So one of the subdegrees is even and the other odd. For instance, let $n_2 = 2k+1$. There are, up to conjugation, three maximal subgroups $T_1$, $T_2$, and $T_3$ in $P$. Their indices in $P$ are equal to 13, 144, and 234, respectively. We can therefore assume that $M_2 \leq T_1$. There are four possibilities: $n_2 = 13$, $13\cdot 3$, $13 \cdot 3^2$, or $13 \cdot 3^3$. Our computations show that only the last case, that is, $n_2 = 13 \cdot 3^3$, is admissible. Hence $n_3 = 13 \cdot 2^5 \cdot 3$. Thus $M_2$ and $M_3$ lie in the maximal subgroups conjugated with a subgroup $T_1 \simeq 3^2 : (2 \cdot S_4) : 2$ (cf. [16]). If we consider the structure of $T_1$ we see that $M_2 \simeq 2 \cdot D_8 : 2$ and $M_3 \simeq 3^2$.

THEOREM 6. For simple non-Abelian groups $G = {}^2F_4(2)'$, the parameters $n$, $n_2$, $n_3$, $P$, $M_2$, $M_3$ of minimal permutation representations are these:

$n = 1600$, $n_2 = 351$, $n_3 = 1248$, $P = A_2(3):2$, $M_2 = 2 \cdot D_8 : 2$, $M_3 = 3^2$.

The rank of the representation is equal to 3.

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